NONLINEAR TAILS IN BLACK HOLE RINGDOWN

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Overview

Ringdown Tails

For years, linear BH perturbation theory has provided the foundation for modeling BH ringdowns, yielding the well-established result that, at late times,

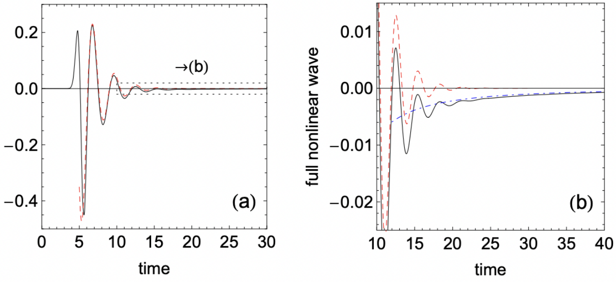


Figure: Temporal evolution of the full nonlinear field. The dashed curve is an analytical fit by the least-damped first order QNM. Figure (b) enlarges the region enclosed by the dotted box in figure (a). (From Okuzumi et al., 2009)

BHs relax through a sequence of exponentially damped oscillations, known as quasinormal modes (QNMs), followed by an inverse power-law decay with time. This is famously represented by Price's scaling: massless fields in a non-spinning BH background decay at fixed spatial locations at linear order as

$$\sim t^{-2\ell-3} \qquad \textit{Price's law} \qquad (1)$$

at very late times, where ℓ is the corresponding multipole number (Price, 1972; Leaver, 1986).

Nonlinear tails are of particular interest, as they naturally emerge from the outgoing QNM profiles present in the ringdown dynamics (Okuzumi, 2008; Lagos, 2023), with their origins closely tied to the nonlinear nature of gravity. Recent numerical advancements have identified the power-law behaviors associated with these nonlinear tails for the Weyl scalar, which are distinct from Price tails (Cardoso, 2024). For instance, it has been shown that the power-law of the $L = 4$ mode generated at second order from two $\ell = 2$ modes decays at large times as $\sim t^{-10}$, (suggesting a more general law)

$$L = 4 \text{ from two } \ell = 2 \text{ modes decay at large } t \text{ as} \quad t^{-10}, \qquad \text{Price law: } t^{-11}.$$

Moreover, recent 3+1-dimensional numerical relativity simulations of BH mergers have provided evidence for power-law tails that diverge from the Price tail (De Amicis, 2024; Ma, 2024). As we will see analytically, at second-order modes decay like

$$\sim t^{-2\ell-2}$$

violating Price's law $\sim t^{-2\ell-3}$.

Related publications:

1. M. H. Y. Cheung, V. Baibhav, E. Berti, V. Cardoso, G. Carullo, R. Cotesta, W. Del Pozzo, F. Duque, T. Helfer and E. Shukla, *et al.* “Nonlinear Effects in Black Hole Ringdown,” Phys. Rev. Lett. **130** (2023) no.8, 8
2. K. Mitman, M. Lagos, L. C. Stein, S. Ma, L. Hui, Y. Chen, N. Deppe, F. Hébert, L. E. Kidder and J. Moxon, *et al.* “Nonlinearities in Black Hole Ringdowns,” Phys. Rev. Lett. **130** (2023) no.8, 081402.
3. B. Bucciotti, L. Juliano, A. Kuntz and E. Trincherini, “Amplitudes and polarizations of quadratic quasi-normal modes for a Schwarzschild black hole,” JHEP **09** (2024), 119.
4. B. Bucciotti, V. Cardoso, A. Kuntz, D. Pereñiguez and J. Redondo-Yuste, “Ringdown nonlinearities in the eikonal regime,” [arXiv:2501.17950 [gr-qc]].
5. A. Kehagias and A. Riotto, “Nonlinear effects in black hole ringdown made simple: Quasinormal modes as adiabatic modes,” Phys. Rev. D **111** (2025) no.4, L041506
6. M. Lagos, T. Andrade, J. Rafecas-Ventosa, and L. Hui, ”Black hole spectroscopy with nonlinear quasi-normal modes,” Phys. Rev. D 111, 024018 (2025).

7. R. H. Price, “Nonspherical Perturbations of Relativistic Gravitational Collapse. I. Scalar and Gravitational Perturbations,” Phys. Rev. D 5, 2419 (1972).
8. E. W. Leaver, “Spectral decomposition of the perturbation response of the Schwarzschild geometry,” Phys. Rev. D 34, 384 (1986).
9. N. Andersson, “Evolving test fields in a BH geometry,” Phys. Rev. D 55, 468 (1997).
10. L. Barack,“Late time dynamics of scalar perturbations outside black holes. 2. Schwarzschild geometry,” Phys. Rev. D 59, 044017 (1999).
11. E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, “Wave Propagation in Gravitational Systems: Late Time Behavior,” Phys. Rev. D 52, 2118 (1995).
12. V. Cardoso, et al., “Hushing black holes: tails in dynamical spacetimes,” Phys. Rev. D 109, L121502 (2024)
13. M. De Amicis et al., Phys. Rev. Lett. 135 (2025), 171401.
14. S. Ling, S. Shah and S. S. C. Wong, “Dynamical nonlinear tails in the Schwarzschild black hole ringdown,” Phys. Rev. D 112 (2025) 024008.
15. A. Kehagias and A. Riotto, “AdS perspective on the nonlinear tails in black hole ringdowns,” Phys. Rev. D 112 (2025) 084068
16. A. Kehagias and A. Riotto, “Nonlinear tails of gravitational waves in Schwarzschild black hole ringdown,” Phys. Rev. D 112 (2025) 024023

1. One-Transparency for QNMs.

The Quasinormal modes (QNMs) describe the **damped oscillations** of perturbations around BH. They arise naturally when solving the **linearized Einstein equations** in a BH spacetime. They are solutions of the form

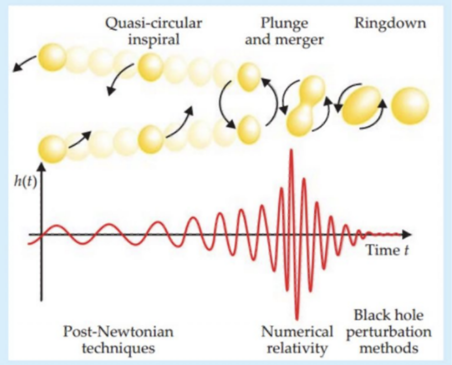
$$\Psi(t,r,\theta,\phi) = e^{-i\omega t}\psi(r)\,{}_{-2}Y_{\ell m}(\theta,\phi)$$

$\psi(r)$ satisfies the Regge-Wheeler equation

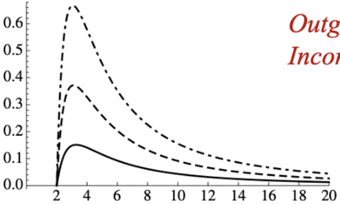
$$\frac{d^2}{dr_*^2}\psi(r) + (\omega^2 - V_{RW})\psi(r) = 0.$$

The Regee-Wheeler potential is

$$V_{RW} = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)$$



Baumgarte+Shapiro 2010



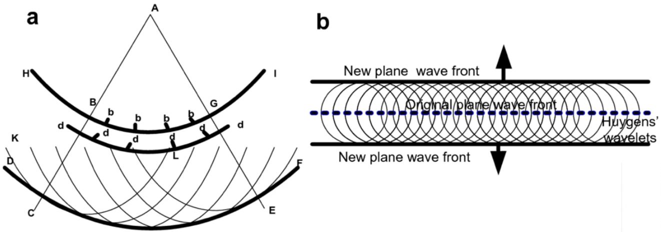
Outgoing BC at infinity
Incoming BC at horizon

Black hole spectroscopy: reconstructing the remnant black hole mass/spin from measured quasinormal modes (Detweiler 1980, Dreyer+ 2004, Berti+ 2006, Carullo+ 2019, Isi+ 2021, ...)

ω are the QNM frequencies, which are (in the eikonal limit, large ℓ)

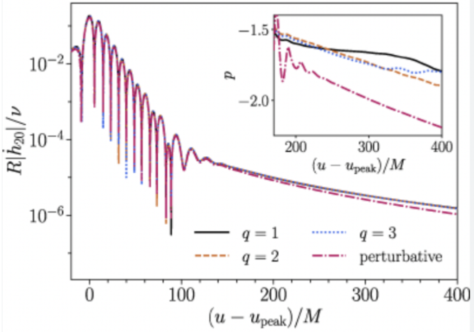
$$\boxed{\omega_{n\ell} = \frac{\ell}{3\sqrt{3}M} - i\left(n+\frac{1}{2}\right)\lambda, \qquad \lambda = \frac{1}{3\sqrt{3}M} \quad \lambda \text{ is the Lyapunov exponent}}$$

Tails and Huygens Principle



Huygens's principle states that every point on a wave front is a source of wavelets that spread out in the forward direction at the same speed as the wave itself.

Huygens' Principle as a geometrical construction. (a) A Huygens' original drawing for a spherical wave. (b) Huygens' geometrical construction presented as a plane wave.



The reason behind the existense of the tail in ringdown is the violation of the Huygens' Principle by the propagation of the perturbations in Black Hole backgrounds (curved spacetime).

From De Amicis et al., 2025

Statement of Huygens' principle. For the (massless) wave equation in $(n+1)$–dimensional Minkowski spacetime,

$$\Box\phi := \partial_t^2\phi - c^2\Delta\phi = s,$$

Huygens' principle says that sharp disturbances propagate *only on* the light cone: the retarded fundamental solution G_{ret} has support confined to $t = r/c$ (with $r = |\mathbf{x}|$). Equivalently, an initial compact pulse produces no lingering signal behind the wavefront.

Green's function viewpoint. Let $\phi(t, \mathbf{x}) = (G_{\text{ret}} * s)(t, \mathbf{x})$. Whether solutions develop *tails* (a nonzero field in the region *inside* the light cone after the front has passed) is fully determined by the support of G_{ret}:

$$\operatorname{supp} G_{\text{ret}} \subseteq \{t \geq 0,\ t \geq r/c\}.$$

- If $G_{\text{ret}}(t, r) \propto \delta(t - r/c)$ (supported *only* on the cone), then the signal is sharply localized on the front and there are **no tails**. Huygens' principle holds.
- If G_{ret} has additional support for $t > r/c$ (timelike interior), then the front is followed by a decaying wake — the **tails**. This is a violation of Huygens' principle.

Flat space examples. In even spatial dimensions the wave equation violates Huygens' principle; in odd spatial dimensions $n \geq 3$ it holds. Explicitly, with $c = 1$:

$$\boxed{n = 3: \quad G_{\text{ret}}(t, r) = \frac{\delta(t - r)}{4\pi r}\,\Theta(t)}$$

is supported only on the cone (no tails), while

$$\boxed{n = 2: \quad G_{\text{ret}}(t, r) = \frac{\Theta(t - r)}{2\pi\sqrt{t^2 - r^2}}}$$

has support for *all* $t > r$ (algebraic tail $\sim (t^2 - r^2)^{-1/2}$). Similarly, in $n = 1$ one finds

$$\boxed{n = 1: \quad G_{\text{ret}}(t, r) = \tfrac{1}{2}\,\Theta(t - |x|)}$$

which also exhibits interior support. Thus,

Tails $\iff$ **failure of Huygens' principle**

in these cases.

Hadamard form (curved spacetimes and general operators). For a broad class of hyperbolic operators on curved backgrounds, the retarded Green function admits the local Hadamard form

$$G_{\mathrm{ret}}(x,x') \;=\; U(x,x')\,\delta\big(\sigma(x,x')\big) \;+V(x,x')\,\Theta\big(-\sigma(x,x')\big),$$

where σ is Synge's world function (half the squared geodesic distance), and U, V are smooth bi-scalars.

- The *direct* term $U\,\delta(\sigma)$ propagates *on* the null cone (Huygens part).
- The **tail** term $V\,\Theta(-\sigma)$ propagates *inside* the cone (timelike separations).

In 3+1–dimensional flat spacetime:

$$\begin{array}{rcl} \text{For the massless wave operator, } V \equiv 0 & \Longleftrightarrow & \text{Huygens' principle holds.} \\ \text{Curvature and potentials produce } V \neq 0 & \Longleftrightarrow & \text{tails are generated.} \end{array}$$

Hadamard problem

Physical mechanisms that generate tails.

1. **Geometry/backscattering:** spacetime curvature scatters waves from the front into the interior, yielding $V \neq 0$ (e.g. gravitational wave tails in curved backgrounds).
2. **Long- or short-range modifications:** adding a potential or mass term (e.g. Klein–Gordon $(\Box - \mu^2)\phi = 0$) creates timelike support of G_{ret}. In 3+1D Minkowski, the massive Green function contains Bessel functions and obeys

$$G_{\text{ret}}^{(\mu)}(t,r) \;\propto\; \Theta(t-r)\,\frac{J_1\!\left(\mu\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}},$$

which is manifestly supported inside the cone: a tail.

"Tail = violation of Huygens". A "tail" is precisely a nonvanishing field for $t > r/c$ after the front has passed. This can only occur if the retarded propagator has timelike support, i.e. if the $\Theta(-\sigma)$ (or interior) piece is present. Therefore:

$$\text{Huygens holds} \iff \operatorname{supp} G_{\text{ret}} \subseteq \text{null cone} \iff \text{no tails.}$$

Conversely, any mechanism that generates interior support (dimension, curvature, mass/potentials) creates tails and thus violates Huygens' principle. (Tails are the rule.)

Linear Tails of Schwarzschild BHs

Our starting point is a Schwarzschild BH with mass M whose spacetime is described by the metric

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_2^2, \qquad f(r) = 1 - \frac{r_s}{r},$$

where $r_s = 2G_N M$ is the Schwarzschild radius. In this BH background, a massless scalar field Ψ evolves according to

$$\Box\Psi(r, t, \theta, \phi) = 0$$

We can expand Ψ in scalar spherical harmonics $Y_{\ell m}$, as

$$\Psi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{\psi_\ell(r_*, t)}{r_*} Y_{\ell m}(\theta, \phi)$$

where $r_* = r + 2M\ln(r/2M - 1)$ is the usual tortoise coordinate.

Then the equation for $\psi_\ell(r_*, t)$ turns out to be

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V(r_*)\right]\psi_\ell = 0, \tag{2}$$

where

$$V(r_*) = \left(1 - \frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{r_s}{r^3}\right).$$

To solve this problem, one needs to construct the corresponding Green function which satisfies

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V(r)\right) G(r_*, y, t) = \delta(t)\delta(r_* - r_*'). \tag{3}$$

Once $G(r_*, y, t)$ is known, the solution is

$$\psi_\ell(r_*, t) = \int \left[G(r_*, y, t)\partial_t\psi_\ell(y, 0) + \partial_t G(r_*, y, t)\,\psi_\ell(y, 0)\right]\mathrm{d}y. \tag{4}$$

We can construct the Green function $G(r_*, y, t)$ by reducing the problem into an ordinary DE by the integral Fourier transform

$$G(r_*, y, t) = \frac{1}{2\pi} \int \hat{G}(r_*, y, \omega) e^{-i\omega t} d\omega.$$

Then $\hat{G}(r_*, y, \omega)$ can be constructed by the two linerly-independent solutions $u_\ell^{in}, u_\ell^{out}$ of

$$\left(\frac{\partial^2}{\partial r_*^2} + \omega^2 - V(r_*) \right) u_\ell(r_*, \omega) = 0.$$

These two solutions are such that, the first $u_\ell^{in}(r_*, \omega)$ is purely ingoing at the horizon

$$u_\ell^{in}(r_*, \omega) \sim \begin{cases} e^{-i\omega r_*} & r_* \longrightarrow -\infty \\ A_{out} e^{i\omega r_*} + A_{in} e^{-i\omega r_*} & r_* \longrightarrow +\infty \end{cases}$$

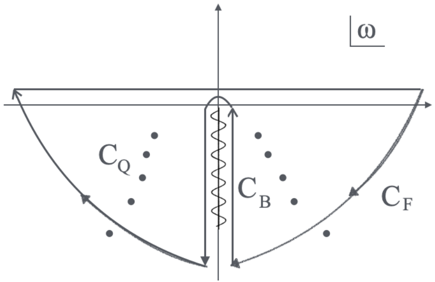
and $u_\ell^{out}(r_*, \omega)$ is outgoing at spatial infinity

$$u_\ell^{out}(r_*, \omega) \sim \begin{cases} B_{out} e^{i\omega r_*} + B_{out} e^{-i\omega r_*} & r_* \longrightarrow -\infty \\ e^{+i\omega r_*} & r_* \longrightarrow +\infty \end{cases}$$

Then the Green function is given by

$$\hat{G}(r_*, y, \omega) = -\frac{1}{W}\begin{cases} u_\ell^{in}(r_*, \omega)u_\ell^{out}(y, \omega) & r_* < y \\ \\ u_\ell^{in}(y, \omega)u_\ell^{out}(r_*, \omega) & r_* > y \end{cases},$$

where $W = 2i\omega A_{in}$ is the Wroskian. In order to find the Green function in the time domain, we need to integrate over the frequencies in the complex ω-plane.



According to a theorem, power-law potential have branch cuts (customary taken on the negative imaginary axis) and therefore a standard path to calculate

$$G(r_*, y, t) = \frac{1}{2\pi}\int \hat{G}(r_*, y, \omega)e^{-i\omega t}d\omega,$$

is the one shown next.

There are three contributions to the Green function:

$$G(r_*, y, t) = G_F(r_*, y, t) + G_Q(r_*, y, t) + G_B(r_*, y, t), \tag{5}$$

arising from the integrations in the three paths indicated in the figure.

1. G_F is the part of the Green faction arising from the C_F. Since $|\omega|$ is large, this term contributes to the short time, thus, it does not affect the late time behavior of the system.
2. G_Q arises from the poles of the Green function. These poles are exactly at the zeroes of the Wroskian. In this case, the two solutions $u_\ell^{in}(r_*, \omega)$ and $u_\ell^{out}(r_*, \omega)$ used to construct the Green function are linear depended. So they correspond to the same unique solution that satisfies both boundary conditions, ingoing at the horizon and outgoing at infinity. These are just by definition the boundary conditions of QNMs. So G_Q arises for the QNM contribution.
3. G_B is the contribution from the branch cut. The branch cut is responsible for the tail.

We are interested in wave tails which appear at large times, that is at small ω, and therefore large r. The relevant part of the Green function is then G_B as we have already mentioned. At large r, Eq. (2) is written as

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{\ell(\ell+1)}{r_*^2} - \frac{4M\ell(\ell+1)}{r_*^3}\ln\left(\frac{r_*}{2M}\right)\right]\psi_\ell(t, r_*) \approx 0, \tag{6}$$

or,

$$\left(\frac{\partial^2}{\partial r^2} + \omega^2 + \frac{4M\omega}{r} - \frac{\ell(\ell+1)}{r^2}\right)\hat{\psi}_\ell(r, \omega) \approx 0, \tag{7}$$

whereas the corresponding Green function satisfies (Andersson, 1997)

$$\left(\frac{\partial^2}{\partial r^2} + \omega^2 + \frac{4M\omega}{r} - \frac{\ell(\ell+1)}{r^2}\right)G(r_*, y, t) = \delta(t)\delta(r_* - r'_*). \tag{8}$$

Note that although we have considered a scalar field above, at spatial infinity Eqs. (6) and (8) also describe the equation and Green's function of tensor perturbations (GW) at large r. In other words, scalar and tensor perturbations exhibit identical tails, simply because the latter are determined by scattering off the asymptotic potential.

The contribution of the branch cut has been calculated by various methods at various moments (Price, 1972; Leaver, 1986; Andersson, 1997). The result is

$$G_B(r_*; r'_*, t) = 4iMr_*r'_* \int_0^{-i\infty} \mathrm{d}\omega \, e^{-i\omega t} \omega^2 j_\ell(\omega r_*) j_\ell(\omega r'_*).$$

The form of the G_B given in (Andersson, 1997) is

$$G_B(t, r; r') = M \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^{\ell+1} 2^{2-m} (2\ell + 2m + 2)!}{n!(m-n)!(2\ell + 2n + 1)!!(2\ell + 2m - 2n + 1)!!} \frac{r_*^{\ell+2m-2n+1} y^{\ell+1+2n}}{t^{2\ell+3+2n}}. \quad (9)$$

At very late times, it turns out

$$G_B(r_*; r'_*, t) = (-1)^{\ell+1} \frac{(2\ell + 1)!}{[(2l + 1)!!]^2} \frac{4M(r_* r'_*)^{\ell+1}}{t^{2\ell+3}},$$

so that we recover Price scaling

$$G_B \sim t^{-2\ell-3}.$$

An important step we took was the evaluation the double sum (A.K. & A. Riotto, 2025). We found that

$$
\begin{aligned}
G_B(t, r; r') =& M \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^{\ell+1} 2^{2-m} (2\ell + 2m + 2)!}{n!(m-n)!(2\ell + 2n + 1)!!(2\ell + 2m - 2n + 1)!!} \frac{r_*^{\ell+2m-2n+1} y^{\ell+1+2n}}{t^{2\ell+3+2n}} \\
=& - \frac{M}{2^\ell \sqrt{\pi}} \frac{t}{r_* r'_*} \frac{\Gamma(\ell+1)}{\Gamma(l+\frac{3}{2})} \frac{\chi^{-\ell-1}}{\sqrt{\chi^2 - 1}} {}_2F_1\left(\frac{\ell}{2} + \frac{1}{2}, \frac{\ell}{2} + 1, \ell + \frac{3}{2}, \frac{1}{\chi^2}\right). \qquad (10)
\end{aligned}
$$

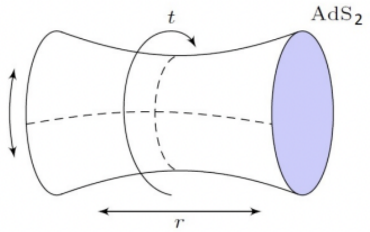
This can be written even in a more compact form in terms of Legendre functions as

$$
\boxed{G_B(t, r; r') = -\frac{2M}{\pi} \frac{t}{r_* r'_*} \frac{Q_\ell(\chi)}{\sqrt{\chi^2 - 1}},} \qquad (11)
$$

where

$$
\chi = \frac{-t^2 + r_*^2 + r'^2_*}{2 r_* r'_*}. \qquad (12)
$$

AdS$_2$



The AdS$_2$ spacetime (of unit radius) is the hyperboloid

$$X_0^2 - X_1^2 + X_2^2 = 1, \tag{13}$$

embedded into 3D spacetime with metric

$$ds^2 = \eta_{MN} dX^M dX^N = -dX_0^2 + dX_1^2 - dX_2^2,$$

so that $\eta_{MN} = (-1, 1, -1)$, $(M, N = 0, 1, 2)$. We can parametrize the AdS$_2$ hyperboloid (13) by the Poincaré coordinates (t, r) as

$$\begin{aligned} X_0 &= \frac{1}{2r}\left(r^2 - t^2 + 1\right), \\ X_1 &= \frac{1}{2r}\left(r^2 - t^2 - 1\right), \\ X_2 &= \frac{t}{r}. \end{aligned} \tag{14}$$

Then, the induced metric on the AdS_2 is

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}r^2}{r^2}. \tag{15}$$

The Sygne's world function on the AdS_2 hyperboloid between (X_0, X_1, X_2) and (X'_0, X'_1, X'_2) is

$$\sigma(X, X') = \frac{1}{2}\eta_{MN}(X - X')^M (X - X')^N,$$

which, by using the parametrization (14) is written as

$$\sigma = \frac{-(t - t')^2 + (r - r')^2}{2rr'}.$$

One quantity that appears frequently is the invariant distance

$$1 + \sigma = \frac{-(t - t')^2 + r^2 + r'^2}{2rr'}.$$

σ is invariant under the SL(2) isometry group of AdS_2 corresponding to the invariance of the metric (15) under dilations $x^i \to \lambda x^i$, time shift $t \to t + \text{constant}$, and conformal inversions.

We easily recognize that

$$1 + \sigma = \chi,$$

we encountered in the Green function G_B

$$G_B(t, r; r') = -\frac{2M}{\pi} \frac{t}{r_* r'_*} \frac{Q_\ell(\chi)}{\sqrt{\chi^2 - 1}},$$

So, one obvious question is: what AdS_2 has to do with the tails of BH ringdown.

To undertand this, we should go to the next, non-linear level.

The Nonlinear Tails

We assume that a source S_L is initiated at a specific moment t_0 and radial coordinate r_0 and then propagates outward at the speed of light. Its influence is confined in retarded time, denoted by $u' \equiv (t' - r') \simeq u_0 \pm \delta$, where δ indicate the small support of the propagating source. In contrast, the advanced time variable $v' \equiv (t' + r')$ spans the range from $(t - r)$ to $(t + r)$, see Fig. We also assume that the observer's position is such that the time coordinate is significantly larger than the radial coordinate, i.e., $t \gg r$. Under these circumstances, Eq. (6) will take the form

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{L(L+1)}{r_*^2}\right)\psi_L \approx S_L,$$

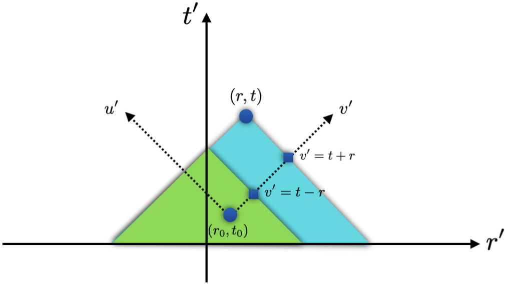


Figure: The causality diagram concerning an outgoing signal propagating from the time t_0, emitted at the point r_0, at the speed of light along the null direction $u' = (t_0 - r_0) = u_0$.

Imagine now that the source S_L is a second-order source, composed of the combination of two linear QNMs (e.g. an $L = 4$ mode made out of two linear $\ell = 2$ modes). As the second-order source originates from the product of two linear modes, each of frequency ω, it will necessarily take the form at spatial infinity (Nakano, 2007)

$$S_L \approx \frac{A_L}{r_*^2} e^{-2i\omega u}. \tag{16}$$

Therefore, the nonlinear quadratic L-mode at large times and distances satisfies an equation of the form

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{L(L+1)}{r_*^2}\right)\psi_L \approx \frac{A_L}{r_*^2} e^{-2i\omega u},$$

which, after multiplying with r_*^2 turns out to be

$$\boxed{\left[r_*^2\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2}\right) - L(L+1)\right]\psi_L \approx A_L e^{-2i\omega u}.} \tag{17}$$

The 4D $AdS_2 \times S^2$

Let us now consider an exact $AdS_2 \times S^2$ spacetime with metric in Poincaré coordinates for the AdS_2

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}r_*^2}{r_*^2} + \mathrm{d}\Omega_2^2. \tag{18}$$

A massless scalar field Ψ on $AdS_2 \times S^2$ obeys

$$\Box\Psi = \Box_{AdS_2}\Psi + \nabla^2_{S^2}\Psi = 0. \tag{19}$$

By using the metric (18) and expanding Ψ as $\Psi = \sum \psi_\ell(r_*, t) Y_{\ell m}(\theta, \phi)$ in spherical harmonics, we find that Eq. (19) turns out to be

$$\Box_{AdS_2}\psi_\ell - \ell(\ell+1)\psi_\ell = 0. \tag{20}$$

Therefore, the modes ψ_ℓ are massive scalars on AdS_2, with square mass $m^2 = \ell(\ell+1)$.

The scaling dimension of a field of mass m in AdS_2 is

$$\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}, \tag{21}$$

which for $m^2 = \ell(\ell+1)$, gives

$$\Delta = \ell + 1. \tag{22}$$

Then we have, in a presence of a source ρ_ℓ,

$$\Box_{\mathrm{AdS}_2}\psi_\ell - \ell(\ell+1)\psi_\ell = \boxed{r_*^2\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2}\right)\psi_\ell - \ell(\ell+1)\psi_\ell = \rho_\ell .} \tag{23}$$

which coincides with the equation for the 2nd-order perturbation with $\ell = L$ and $\rho_\ell = A_L e^{-2i\omega u}$.

In other words, the equation that dominates the dynamics at large r of second-order perturbations in the Schwarzschild BH background, is identical to the equation which governs a massive scalar with mass square $m^2 = \ell(\ell+1)$ on an AdS_2 spacetime.

AdS_2 Green functions

In other words, the problem of determining the second-order tails has been reduce in solving a sourced massive field of dimension $\ell + 1$ on AdS_2

$$\left(\Box_{\mathrm{AdS}_2} - m^2\right)\psi_\ell = \rho_\ell, \qquad \rho_\ell = A_L e^{-2i\omega u}. \tag{24}$$

As usual in order to find ψ_ℓ, we need the retarded Green function $G^{\mathrm{ret}}_{\mathrm{AdS}}$ on AdS_2

$$\left(\Box_{\mathrm{AdS}_2} - m^2\right) G^{\mathrm{ret}}_{\mathrm{AdS}_2} = \frac{1}{\sqrt{-g}}\delta(t-t')\delta(r_* - r'_*). \tag{25}$$

One can try build $G^{\mathrm{ret}}_{\mathrm{AdS}_2}$ by the usual method we did for the linear tail as a product of solutions of the sourceless problem. Although one can find the Green function this way for specific ℓ, we didn't succeeded to get a closed formula for any ℓ. For example, we found that

$$G^{\mathrm{ret}}_{\mathrm{AdS}_2}(t,r;t',r') = \sqrt{\frac{\pi}{2}}\frac{3(t-t')^4 + 3r^4 + 2r^2r'^2 + 3r'^4 - 6(t-t')^2(r^2+r'^2)}{16r^2r'^2}, \qquad \ell = 2,$$

$$
\begin{aligned}
G^{\rm ret}_{\rm AdS_2}(t,r;t',r') =& \sqrt{\frac{\pi}{2}}\frac{1}{256\, r^4 r'^4}\left[35r^8 + 20r^6(r'^2 - 7(t-t')^2)\right.\\
&+ +20r^2(r'^2 - 7(t-t')^2)(r'^2 - (t-t')^2) + 35(r'^2 - (t-t')^2)^4\\
&\left.+ 6r^4(3r'^4 - 30r'^2(t-t')^2 + 35(t-t')^4)\right]. \qquad \ell = 4.
\end{aligned}
$$

On the other hand, many Green functions has been calculated for massive scalars like two-point functions or Hadamard functions but the retarded Green function has not appeared in the literature. So in the following we will construct the retarder Green function from the AdS_2 two-point function. The latter has been calculated in various occasions (Burgess, 1985; Satoh, 2003) and reads

$$
\boxed{G_{\rm AdS_2}(\chi) = \frac{1}{2^\ell\sqrt{\pi}}\frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})}\chi^{-\ell-1}\;{}_2F_1\left(\frac{\ell}{2}+\frac{1}{2},\frac{\ell}{2}+1,\ell+\frac{3}{2},\frac{1}{\chi^2}\right).} \tag{26}
$$

This is actually the two-point function calculated in the $SL(2,R)$-invariant vacuum in Poincaré coordinates $|0_{\rm Poincare}\rangle$ (Spradlin, 1999).

By using the following relation between Legendre and hypergeometric functions

$$Q_\ell(z) = 2^{-\ell-1}\frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})}z^{-\ell-1}\,{}_2F_1\left(1+\frac{\ell}{2}+\frac{1}{2}+\frac{\ell}{2},\ell+\frac{3}{2},\frac{1}{z^2}\right), \qquad |z|>1, \tag{27}$$

we can express Eq. (26) as

$$\boxed{G_{\mathrm{AdS}_2}(\chi) = G^{\ell}_{\mathrm{AdS}_2}(\chi) = \frac{1}{2\pi}Q_\ell(\chi).} \tag{28}$$

We stress that the above two-point function is a function of the AdS_2 invariant distance χ and therefore is the $SL(2,R)$ invariant. We can also compare the branch-cut contribution

$$G_B = -\frac{2M}{\pi}\frac{t}{r_* r'_*}\frac{Q_\ell(\chi)}{\sqrt{\chi^2-1}} = -\frac{4M}{r_* r'_*}\frac{t}{\sqrt{\chi^2-1}}G_{\mathrm{AdS}_2} \approx -\frac{4M}{(r_* r'_*)^{1/2}}G_{\mathrm{AdS}_2} \ll G_{\mathrm{AdS}_2}, \tag{29}$$

which for $t \gg r_*, r'_*$ and $r_*, r'_* \gg M$ gives $G_B(t,r;r') \ll G_{\mathrm{AdS}_2}(\chi)$.

Algebraic Structure

There is an underlying symmetry which can become manifest by defining raising and lowering ladder operators $L_\pm$ as

$$\begin{aligned} L_+ &= (\chi^2 - 1)\frac{\mathrm{d}}{\mathrm{d}\chi} + (\ell + 1)\chi, \\ L_- &= (1 - \chi^2)\frac{\mathrm{d}}{\mathrm{d}\chi} + \ell\chi, \end{aligned} \tag{30}$$

These operators satisfy

$$\begin{aligned} L_+ G^{\ell}_{\mathrm{AdS}_2}(\chi) &= (\ell + 1) G^{\ell+1}_{\mathrm{AdS}_2}(\chi), \\ L_- G^{\ell}_{\mathrm{AdS}_2}(\chi) &= \ell\, G^{\ell-1}_{\mathrm{AdS}_2}(\chi). \end{aligned} \tag{31}$$

In addition, we further introduce the operator L_0 defined by

$$L_0\, G^{\ell}_{\mathrm{AdS}_2}(\chi) = \left(\ell + \frac{1}{2}\right) G^{\ell}_{\mathrm{AdS}_2}(\chi). \tag{32}$$

Then it is easy to verify that $L_\pm$ and L_0 satisfy the $SL(2,R)$ algebra

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = -2L_0. \tag{33}$$

Therefore, starting with the lowest $\ell = 0$ Green function $G^0_{\mathrm{AdS}_2}(\chi)$, we can construct all $G^\ell_{\mathrm{AdS}_2}(\chi)$'s by repeated applications of the raising operator L_+ as

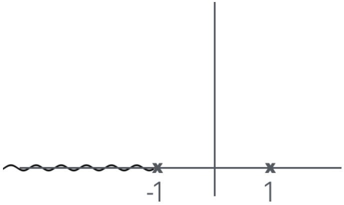
$$G^\ell_{\mathrm{AdS}_2}(\chi) = \frac{1}{\ell!} L_+^\ell G^{\ell=0}_{\mathrm{AdS}_2}(\chi). \tag{34}$$

Since, $G^{\ell=0}_{\mathrm{AdS}_2}(\chi) = \frac{1}{4\pi} \ln\left(\frac{1+\chi}{1-\chi}\right)$, we find for example

$$G^{\ell=1}_{\mathrm{AdS}_2}(\chi) = \frac{x}{4\pi} \ln\left(\frac{\chi+1}{\chi-1}\right) - 1, \tag{35}$$

$$G^{\ell=2}_{\mathrm{AdS}_2}(\chi) = \frac{1}{8\pi}\left(3\chi^2 - 1\right)\ln\left(\frac{\chi+1}{\chi-1}\right) - \frac{3}{4\pi}\chi. \tag{36}$$

Obviously, the $SL(2,R)$ "spectrum generating symmetry" mentioned above is related to the hidden AdS_2 structure underlying our problem.



The Legendre function $Q_\ell(z)$
is analytic in the whole complex z-plane with branch points at

$$z = \pm 1 \qquad \text{and} \qquad \infty.$$

In the complex z plane, $Q_\ell(z)$ is defined through its relation to the corresponding hypergeometric function. The latter has a branch cut from $z = 1$ to $z = \infty$ along the real axis, and defining $z^{-\ell-1} = e^{-(\ell+1)\log z}$, the branch cut of the $Q_\ell(z)$ is taken to be $(-\infty, -1]$. Then, the Feynman propagator is the limit of $G_{\mathrm{AdS}_2}(\chi + i0)$ above the cut, whereas the retarded Green function is

$$G^{\mathrm{ret}}_{\mathrm{AdS}_2}(\chi) \quad = \quad i\theta(t-t')\Big[G_{\mathrm{AdS}_2}(\chi + i0) - G_{\mathrm{AdS}_2}(\chi - i0)\Big]. \tag{37}$$

Then by using the relation (Abramowitz, 1964)

$$Q_\ell(z + i0) = Q_\ell(z - i0) - i\pi P_\ell(\chi), \tag{38}$$

we find that the retarded Green function is given by

$$\boxed{G^{\mathrm{ret}}_{\mathrm{AdS}_2}(\chi) = \frac{1}{2}\theta(t-t')P_L(\chi).} \tag{39}$$

The nonlinear quadratic tail is therefore fully dictated by the behavior of the AdS_2 retarded Green function

$$\psi_L = A_L \int_{u_0-\delta}^{u_0+\delta} \mathrm{d}u'\, e^{-2i\omega_\ell u'} \int_{t-r}^{t+r} \frac{1}{2} P_L(\chi) \left(1 - \frac{2M}{r'}\right)^{-1} \mathrm{d}\upsilon', \tag{40}$$

We write the second integral as

$$I_2(u') = \int_{t-r}^{t+r} \frac{1}{2} P_L(\chi) \left(1 - \frac{2M}{r'}\right)^{-1} \mathrm{d}\upsilon' = \frac{1}{2} \sum_{n=0}^{\infty} \int_{t-r}^{t+r} P_L(\chi) \left(\frac{2M}{r'}\right)^{n} \mathrm{d}\upsilon'. \tag{41}$$

In terms of u' and v', χ is written as

$$\chi = \frac{-(t-t') + r^2 + r'^2}{2rr'} \quad \Longrightarrow \quad \chi = \frac{r^2 - (t-u')(t-\upsilon')}{r(u'-\upsilon')} \tag{42}$$

and if we trade υ' for χ in the integral of Eq. (41), we get

$$I_2 = \sum_{n=0}^{\infty} r \left[r^2 - (t-u')^2 \right]^{-1-n} (2M)^n$$

$$\int_{-1}^{1} \mathrm{d}\chi P_L(\chi)(r\chi + t - u')^n. \tag{43}$$

One can prove the relations

$$\int_{-1}^{1} \mathrm{d}\chi P_L(\chi)(a\chi + b)^n = 0, \qquad \text{for} \qquad n < L, \tag{44}$$

and

$$\int_{-1}^{1} P_L(\chi)(a\chi + b)^L \mathrm{d}\chi = \frac{2^L L!(L-1)!}{(2L+1)(2L-1)!!} a^L, \tag{45}$$

Therefore, using the above, we find that the first non zero contribution appears for $n = L$ for which

$$\begin{aligned} I_2 &\approx r\Big[r^2 - (t-u')^2\Big]^{-1-L} (2M)^L \int_{-1}^{1} P_L(\chi)(r\chi + t - u')^L \mathrm{d}\chi \\ &= \frac{2^L L!(L-1)!}{(2L+1)(2L-1)!!} \frac{(2M)^L\, r^{L+1}}{\Big[r^2 - (t-u')^2\Big]^{1+L}}. \end{aligned} \tag{46}$$

Finally, the solution for the second order perturbation ψ_L is

$$\boxed{\psi_L \approx A_L \omega_L^{-1} e^{-2i\omega_L u_0} \sin\left(2\omega_L \delta\right) \frac{2^L L!(L-1)!}{(2L+1)(2L-1)!!} \frac{(2M)^L\, r^{L+1}}{\left[r^2 - (t-u_0)^2\right]^{1+L}}.} \tag{47}$$

For $t \gg r$ we find

$$\psi_L \sim (-)^{L+1} \frac{2^L L!(L-1)!(2M)^L r^{L+1}}{(2L+1)(2L-1)!!} \frac{1}{t^{2L+2}},$$

scaling that violates Price's law. It should be stressed that the generation mechanism for the nonlinear quadratic tail differs from that of the linear Price's tail. The latter is sourced by back-scattered at low frequencies from the potential at infinity. This potential peaks at small, non-vanishing frequencies, resulting in an amplitude proportional to the BH mass. Conversely, the nonlinear tail originates from the quadratic QNM itself. In the exact solution for the second perturbation, there exist a factor that do not depend on the details of the BH it self and the first order perturbations. We would like to understand the universal factor

$$\frac{2^L L!(L-1)!}{(2L+1)(2L-1)!!} \tag{48}$$

Aretakis Constant

In AdS_2 we have

$$\Box_{\mathrm{AdS}_2} = \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\right) = r^2\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2}\right), \tag{49}$$

so that the equation for a scalar of dimension $\ell + 1$ (or $mass^2 = \ell(\ell+1)$ is explicitly written as

$$r^2\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2}\right)\psi_\ell - \ell(\ell+1)\psi_\ell = 0, \tag{50}$$

In Poincaré (Fefferman-Graham) coordinates $R = 1/r$ and $T = t$, the AdS_2 metric can be written as

$$\mathrm{d}s^2 \quad = \quad -R^2\mathrm{d}T^2 + \frac{\mathrm{d}R^2}{R^2}, \tag{51}$$

Furthermore, by defining

$$u = T + \frac{1}{R} = t + r \qquad \text{and} \qquad v = T - \frac{1}{R} = t - r,$$

the AdS_2 metric is written in incoming Eddington–Finkelstein coordinates as

$$\begin{aligned} \mathrm{d}s^2 = &- R^2\mathrm{d}T^2 + \frac{\mathrm{d}R^2}{R^2} = -R^2\left(\mathrm{d}\upsilon^2 + \frac{\mathrm{d}R^2}{R^4} - 2\mathrm{d}\upsilon\frac{\mathrm{d}R}{R^2}\right) \\ &+ \frac{\mathrm{d}R^2}{R^2} = -R^2\mathrm{d}\upsilon^2 + 2\mathrm{d}\upsilon\mathrm{d}R. \end{aligned} \tag{52}$$

In these coordinates the equation for a massive scalar field is written as

$$2\partial_\upsilon\partial_R\psi_\ell + \partial_R(R^2\partial_R\psi_\ell) - \ell(\ell+1)\psi_\ell = 0. \tag{53}$$

Operating both sides with ∂_R^ℓ, we get

$$2\partial_\upsilon\partial_R^{\ell+1}\psi_\ell + \partial_R^{\ell+1}(R^2\partial_R\psi_\ell) - \ell(\ell+1)\partial_R^\ell\psi_\ell = 0. \tag{54}$$

In particular, we have

$$\partial_R^{\ell+1}(R^2\partial_R\psi_\ell) = \sum_{k=0}^{\ell+1}\binom{\ell+1}{k}\partial_R^{\ell+1-k}(R^2)\partial_R^{k+1}\psi_\ell. \tag{55}$$

For $R \simeq 0$, only the term with $(\ell + 1 - k) = 2$ in the sum is not vanishing. all other terms in the sum vanishes. We find in particular

$$\partial_R^{\ell+1}(R^2\partial_R\psi_\ell)\Big|_{R=0} \simeq 2\begin{pmatrix} \ell+1 \\ \ell-1 \end{pmatrix}\partial_R^\ell\psi_\ell\Big|_{R=0} = \ell(\ell+1)\partial_R^\ell\psi_\ell\Big|_{R=0}. \tag{56}$$

Therefore, Eq. (53) turns out to be

$$2\partial_v\partial_R^{\ell+1}\psi_\ell + \partial_R^{\ell+1}(R^2\partial_R\psi_\ell) - \ell(\ell+1)\partial_R^\ell\psi_\ell\Big|_{R=0} = 2\partial_v\left(\partial_R^{\ell+1}\psi_\ell\right)\Big|_{R=0} = 0. \tag{57}$$

This means that

$$\boxed{\mathcal{H}_\ell \equiv \partial_R^{\ell+1}\psi_\ell\Big|_{R=0} = \text{constant.}}$$

The constant $\mathcal{H}_\ell$ is called Aretakis number (Aretakis, 2015)

Nonlinear Tails and the Aretakis Constants

We have seen that the Aretakis number $\mathcal{H}_L$ (Aretakis, 2011; Aretakis, 2011) for a field in AdS_2 of squared mass $L(L+1)$ reads

$$\mathcal{H}_L = \left(\partial_R^{L+1}\psi_L\right)\Big|_{R=0}, \tag{58}$$

and is constant

$$\partial_v \mathcal{H}_L = 0. \tag{59}$$

In particular, for our second-order perturbation ψ_L at late times we have

$$\psi_L \approx (-)^{L+1}\frac{2^L L!(L-1)!(2M)^L r^{L+1}}{(2L+1)(2L-1)!!}\,\frac{1}{t^{2L+2}},$$

which, in Eddington-Finkelstein coordinates (R, v), is written as

$$\psi_L \approx \frac{(-)^{L+1}2^L L!(L-1)!}{(2L+1)(2L-1)!!}\,\frac{(2M)^L R^{L+1}}{(Rv-1)^{2L+2}}. \tag{60}$$

Then we find that

$$\left(\partial_R^{L+1}\psi_L\right)\Big|_{R=0} = (-)^{L+1}\frac{2^L\,(L!)^2\,(L-1)!(2M)^L}{(2L+1)(2L-1)!!}. \tag{61}$$

so that

$$\mathcal{H}_L = -(-)^L\frac{2^L\,(L!)^2\,(L-1)!(2M)^L}{(2L+1)(2L-1)!!}. \tag{62}$$

Finally, in the original (t, r) coordinates, we can express the late time solution as

$$\boxed{\psi_L \approx L!\,\mathcal{H}_L\,\frac{r^{L+1}}{t^{2L+2}},}$$

and the nonlinear source induces a tail proportional to the Aretakis constant.

Conclusions

Two messages :

1. The Price law t^{-2L-3} at linear order for Schwarzschild BHs is violated at the non-linear level by a t^{-2L-2} scaling.
2. The $\mathrm{AdS}_2 \times \mathrm{S}^2$ geometry is behind the t^{-2L-2} scaling.

And one question:

1. Why AdS_2 appears again and again in BH theory?