Existence and Uniqueness of solutions to the conformal constraint equations.

Lichnerowicz Conference : Journées relativistes de Tours 2026.
Collaboration with Romain Gicquaud. *Arxiv:2603.21214*

Armand Coudray

Université de Tours, Institut Denis Poisson.

Vendredi 5 Juin 2026

Motivations: mathematical and physical reasons

Why study the solutions to the constraint equations?

The goal:

Find the initial data for general relativity.

- Initial data sets = the initial states of physical systems.
- Understand the space of all **globally hyperbolic solutions** to Einstein's equations.
  Because space of initial data is *simpler* than the space of whole solutions.

Plan of the presentation

1. Conformal constraint equations
   - Constraint equations from geometry
   - The conformal method.
   - Classification of solutions and state of the art.
2. Existence and uniqueness
   - Assumptions
   - Theorem
3. Sketch of the Proof
   - Existence of a contractive map
   - Lower and upper bounds

Conformal constraint equations

Einstein's equations

$\mathcal{M}$: a $n+1$ manifold. $\tilde{g}$: a Lorentzian metric.

Einstein's vacuum equations

$$R_{ab} - \frac{1}{2}\mathrm{Scal}_{\tilde{g}}\,\tilde{g}_{ab} = 0$$

- R_{ab} : Ricci curvature tensor.
- $\mathrm{Scal}_{\tilde{g}}$: scalar curvature. $\mathrm{Scal}_{\tilde{g}} = \mathrm{tr}_{\tilde{g}} R_{ab}$.

The Cauchy Problem

Question: Is the Cauchy problem of the vacuum Einstein equations well-posed?

Initial data

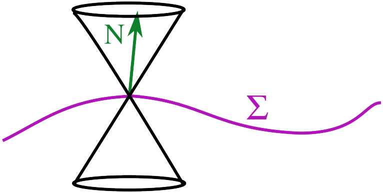
- Σ a spacelike hypersurface. $\dim(\Sigma) = n$.
- $\tilde{\gamma}$: the induced metric on Σ.
- K: the second fundamental form on Σ.

Problem: The initial data cannot be chosen freely.

Constraint equations from geometric conditions

Spacelike hypersurface

- Σ is spacelike : $\tilde{\gamma} := \tilde{g}|_{\Sigma}$ is Riemannian.
- **N** is the future unit normal vector.



Geometric approach to constraint equations: Can the curvature components of the manifold $\mathcal{M}$ be expressed in terms of data intrinsic to the surface Σ?

Gauss and Codazzi equations

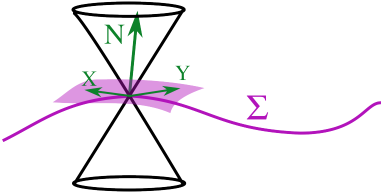
Gauss-Codazzi equations = coupling of **intrinsic** and **extrinsic** quantities:

- the *first* and *second* fundamental forms on Σ : $\tilde{\gamma}$ and K.
- the geometry of the *ambient manifold* $\mathcal{M}$: the Riemmann curvature tensor on $\mathcal{M}$.

Second fundamental form

Definition

- Σ a spacelike hypersurface.
- X, Y tangent vector fields on Σ.
- D: affine connection on $\mathcal{M}$.
- $\tilde{\nabla}$: affine connection on Σ.



Decomposition of connections

$$D_X Y = \tilde{\nabla}_X Y + K(X, Y)\mathbf{N}, \quad K(X, Y) := g(D_X \mathbf{N}, Y)$$

Gauss and Codazzi equations

Gauss equation

$X, Y, Z, W \in T\Sigma$,

$$\begin{aligned}&\mathrm{Riem}_{\mathcal{M}}(X, Y, Z, W) = \\ &\mathrm{Riem}_{\Sigma}(X, Y, Z, W) + K(X, W)K(Y, Z) - K(X, Z)K(Y, W).\end{aligned}$$

Codazzi equation

$$\mathrm{Riem}_{\mathcal{M}}(X, Y, N, Z) = \tilde{\nabla}_X K(Y, Z) - \tilde{\nabla}_Y K(X, Z).$$

Constraint equations

Gauss + Codazzi + Einstein :

Constraint equations

$\Sigma, \tilde{\gamma}, K$: initial data

$$\begin{aligned} \mathrm{Scal}_{\Sigma,\tilde{\gamma}} + (\mathrm{tr}_{\tilde{\gamma}} K)^2 - |K|^2_{\tilde{\gamma}} &= 0, \\ \mathrm{div}_{\tilde{\gamma}} K - d(\mathrm{tr}_{\tilde{\gamma}} K) &= 0. \end{aligned}$$

- **The hamiltonian constraint** : scalar equation.
- **The momentum equation** : vector equation.

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We obtain constraints on the initial data. What are the **evolution equations** ?

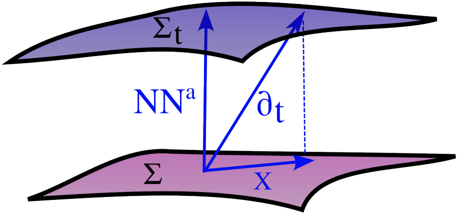
$1 + N$ formulation ?

We introduce : t, a time function *adapted to* Σ. Σ is a level hypersurface of t.

Shift and Lapse

$$\tilde{g}_{ab} = \tilde{\gamma}_{ab} + \mathbf{N}_a \mathbf{N}_b,$$
$$\partial_t = N\mathbf{N} + X.$$

- N: lapse function.
- X: shift vector.



Equivalent definition

In abstract index notation

$$K_{ab} = \frac{1}{2}\mathcal{L}_{\mathbf{N}}\tilde{\gamma}_{ab}.$$

Evolution equations

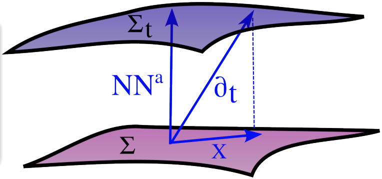
ADM equations

t-evolution of $(\tilde{\gamma}, K)$ are:

$$\partial_t \tilde{\gamma}_{ab} = 2NK_{ab} + \mathcal{L}_X \tilde{\gamma}_{ab}.$$

$$\partial_t K_{ab} = \nabla_a \nabla_b N + \mathcal{L}_X K_{ab} + N \left(-\mathrm{Ric}^{\Sigma}_{ab} + K_a^c K_{bc} - \mathrm{tr}_{\tilde{\gamma}} K K_{ab} \right).$$

- We have shown that the Einstein constraint equations are **necessary** for the data.
- Are these conditions **sufficient** ?



Well-posedness of the Cauchy problem

The well-posedness of the Cauchy problem for general relativity is well known:

Y. Choquet-Bruhat, 1952

The Cauchy formulation of Einstein's vacuum equations, for smooth initial data, is well-posed locally.
Existence and uniqueness of the solution in a small neighbourhood of Σ_0.

Y. Choquet-Bruhat and R. Geroch, 1969

Given any set of initial data for the vacuum Einstein's equations satisfying the constraint equations, then there exists a development of that data that is maximal.

Solving the constraint equations

Constraint equations

$\Sigma, \tilde{\gamma}, K$: initial data

$$\begin{aligned} \mathrm{Scal}_{\Sigma,\tilde{\gamma}} + (\mathrm{tr}_{\tilde{\gamma}} K)^2 - |K|^2_{\tilde{\gamma}} &= 0, \\ \mathrm{div}_{\tilde{\gamma}} K - d(\mathrm{tr}_{\tilde{\gamma}} K) &= 0. \end{aligned}$$

Various approaches :

- **The conformal approach**.
- gluing methods.
- thin sandwich methods.

Solving the constraint equations

Constraint equations

$\Sigma, \tilde{\gamma}, K$: initial data

$$\begin{aligned}\mathrm{Scal}_{\Sigma,\tilde{\gamma}} + (\mathrm{tr}_{\tilde{\gamma}} K)^2 - |K|^2_{\tilde{\gamma}} &= 0,\\ \mathrm{div}_{\tilde{\gamma}} K - d(\mathrm{tr}_{\tilde{\gamma}} K) &= 0.\end{aligned}$$

Various approaches :

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Conformal approach: Decompose initial data into parameters and unknows.

The conformal method

Conformal method = change of variables. **Transformation of the induced metric $\tilde{\gamma}$.**

The conformal class

$$\tilde{\gamma} = \varphi^{\frac{4}{n-2}} \gamma.$$

- We are not working with the entire conformal class.
- Parameter : a given metric γ. Riemannian (on Σ).
- Unknown : φ a positive function. Conformal factor.

The York's decomposition

Transformation of the second fundamental form K.

1. Extraction of the trace $\tau = \mathrm{tr}_{\tilde{\gamma}} K$:

$$K = \frac{\tau}{n}\tilde{\gamma} + L$$

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2. L is a **traceless symmetric** tensor (w. respect to $\tilde{\gamma}$ or γ) : $L \in \mathrm{Sym}^2(\Sigma)$.

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2. L is a **traceless symmetric** tensor (w. respect to $\tilde{\gamma}$ or γ) : $L \in \mathrm{Sym}^2(\Sigma)$.

3. $\mathbb{L}$ Conformal Killing operator (differential op. deg. 1).

Definition

In abstract index notation:

$$(\mathbb{L}W)_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{n}\nabla^c W_c g_{ab}.$$

where ∇ : Levi-Civita for γ.

The York's decomposition

Transformation of the second fundamental form K.

4. York's decomposition, L^2-orthogonal decomposition of set $\mathrm{Sym}^2(\Sigma)$

$$\mathrm{Sym}^2(\Sigma) = \mathcal{T}_\gamma(\Sigma) \perp_{L^2} \mathrm{Im}\,\mathbb{L}.$$

The York's decomposition

Transformation of the second fundamental form K.

4. York's decomposition, L^2-orthogonal decomposition of set $\mathrm{Sym}^2(\Sigma)$
$$\mathrm{Sym}^2(\Sigma) = \mathcal{T}_\gamma(\Sigma) \perp_{L^2} \mathrm{Im}\,\mathbb{L}.$$

5. $\mathcal{T}_\gamma(\Sigma)$ denotes the set of **transverse** and **traceless** tensors: TT-tensors. If $\sigma \in \mathcal{T}_\gamma(\Sigma)$, then :

$$\mathrm{tr}_\gamma\sigma = 0, \quad \mathrm{div}_\gamma\sigma = 0.$$

The York's decomposition

Transformation of the second fundamental form K.

Decomposition

$$K = \frac{\tau}{n}\tilde{\gamma} + \varphi^{-2}(\sigma + \mathbb{L}W). \qquad \text{(dec. York)}$$

- $\mathbb{L}$ Conformal Killing operator (differential op. deg 1).
- γ : Riemannian metric, $\tau = tr_\gamma K$, σ : TT-tensor (Traceless, Transverse) ($tr_\gamma \sigma = \mathrm{div}_\gamma \sigma = 0$). (γ, τ, σ) : Parameters.
- φ : conformal factor, W : vector field. (φ, W) : Variables.

In our work

γ does not have a **conformal Killing vector field**, i.e.

$$\mathbb{L}X \equiv 0 \Rightarrow X \equiv 0.$$

The conformal constraint equations

By setting (γ, σ, τ), we look to the solution (φ, W) :

Elliptic PDE

$$-\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} = \frac{|\sigma + \mathbb{L}W|^2}{\varphi^{\frac{3n-2}{n-2}}}, \quad \text{(Lichnerowicz)}$$

$$\Delta_{\mathbb{L}} W = \frac{n-1}{n}\varphi^{\frac{2n}{n-2}} d\tau. \quad \text{(Vector)}$$

- Δ : Laplacian operator.
- $\Delta_{\mathbb{L}} := -\frac{1}{2}\mathbb{L}^*\mathbb{L}$.
- All of these quantities are defined in terms of γ.

Constant mean curvature

Conformal method adapted to CMC solutions : $\tau = Cte$.

$$-\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} = \frac{|\sigma + \mathbb{L}W|^2}{\varphi^{\frac{3n-2}{n-2}}}, \quad \text{(Lichnerowicz)}$$

$$\Delta_{\mathbb{L}} W = 0. \quad \text{(Vector)}$$

Since γ is assumed to have no Killing vector fields :

$$W \equiv 0.$$

Then, it remains **one scalar equation**:

$$-\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\varphi + \frac{n-1}{n}\tau^2\varphi^{N-1} = \frac{|\sigma|^2}{\varphi^{\frac{3n-2}{n-2}}},$$

Classification of solutions

For CMC ($\tau = cte$) some results have been obtained *e.g.*:

- Closed manifold. [Y. Choquet-Bruhat, J.W. York, **1979**], [J. Isenberg, **1995**].
- Asymptotically Euclidean. [M. Cantor, **1977**].
- Asymptotically hyperbolic. [L. Andersson, P.T. Chruściel, H. Friedrich, **1992**], [L. Andersson, P.T. Chruściel, **1996**].

In our work

Σ : a closed manifold + non CMC.

Near-CMC solutions

Different results for the near-CMC case. Assuming that:

$$\frac{\max|\nabla\tau|}{\min|\tau|} \quad \text{is small.}$$

Results for :

- Closed manifold : [J. Isenberg, **2003**], [J. Isenberg and V. Moncrief, **1996**].
- Asymptotically Euclidean. [Y. Choquet-Bruhat, J. Isenberg, and J. York. **2000**].
- Asymptotically hyperbolic. [J. Isenberg and J. Park, **1997**].

Yamabe invariant

Yamabe Theorem

Let (Σ, γ) be a closed Riemannian manifold of dimension $n \geq 3$. Then there exists a metric $\hat{\gamma}$ conformal to γ that has constant scalar curvature.

Moreover, we define the **Yamabe invariant**:

$$\mathcal{Y}_\gamma(\Sigma) = \inf_{u \not\equiv 0, u \in C^\infty(\Sigma)} \frac{\int_\Sigma \left[\frac{4(n-1)}{n-2}|du|^2 + \mathrm{Scal} u^2\right] d\mu^\gamma}{\left(\int_\Sigma u^N d\mu^\gamma\right)^{2/N}},$$

that is a **conformal invariant.** And

$$N := \frac{2n}{n-2}, \quad \frac{1}{N} = \frac{1}{2} - \frac{1}{n}.$$

The minimizer u of $\mathcal{Y}_\gamma(\Sigma)$ gives:

$$\hat{\gamma} = u^{N-2}\gamma.$$

In our work

For the rest of this presentation : $\mathcal{Y}_\gamma(\Sigma) > 0$.

Non CMC solutions.

When $d\tau \neq 0$: the PDEs are coupled.

Maxwell's result (2009)

It is possible to renormalise τ and σ such that

- Arbitrarly large τ.
- Small $\|\sigma\|_{L^\infty}$.

In our work

We will work with $d\tau \neq 0$. Small TT-tensor σ.

State of the art

- M. Holst, G. Nagy and G. Tsogtgerel : First results for non-CMC manifolds (2008, 2009).
- D. Maxwell : a simplified version of the proof (2009).

Existence thms based on **Schauder fixed point** $\Rightarrow$ failed to proof uniqueness.

Attempts for uniqueness:

- R. Gicquaud and Q.A. Ngô : Unified version of the two previous results (2014) + proof of the local uniqueness.
- R. Gicquaud (2025): uniqueness results but additional assumptions $\mathrm{essinf}_{\Sigma}|\sigma| > c > 0$.

Existence and uniqueness

Assumptions

Recall :

- Σ is a closed manifold of dimension n.
- γ has no Killing vector field.
- **Positive Yamabe invariant:** $\mathcal{Y}_\gamma(\Sigma) > 0$.
- **Smallness of the TT-tensor:** $\|\sigma\|_{L^{2p}} \leq c_1$.
- $\|\sigma\|_{L^2} \leq c_2 \|\sigma\|_{L^{2p}}$.
- **Control of the volume of the solution:** $V_{\tilde{\gamma}}(\Sigma) = \int_\Sigma \varphi^N d\mu^\gamma$ such that:

$$V_{\tilde{\gamma}}(\Sigma) \leq V_{max}. \quad V_{max} \text{ small enough.}$$

Main Theorem

Theorem [A.C. R. Gicquaud, 2026]

There exists a unique solution $(\varphi, W) \in W^{2,p}(\Sigma, \mathbb{R}) \times W^{2,p}(\Sigma, T\Sigma)$ to the conformal constraint equations.

Main Theorem

Theorem [A.C. R. Gicquaud, 2026]

There exists a unique solution $(\varphi, W) \in W^{2,p}(\Sigma, \mathbb{R}) \times W^{2,p}(\Sigma, T\Sigma)$ to the conformal constraint equations.

- Our existence theorem relies on a **Banach fixed point** argument.
- The key ingredients : lower and upper bound on $\|\varphi\|$ + estimates on the sub-solution .

Sketch of the Proof

Banach fixed point theorem

Consider (X, d) a metric space, with metric $d(x, y)$. Let $T : X \to X$ be a contracting map, i.e. it exists $q < 1$ such that :

$$d(T(x), T(y)) \leq q\, d(x, y), \quad \text{for all } x, y \in X$$

Then T admits a **unique fixed point** $x \in X$ such that:

$$T(x) = x.$$

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Goal: Find a contractive application such that x is a solution to the CCE. Uniqueness follows from the Banach fixed point theorem.

The map Φ

The map Vect

Given $\varphi \in L^r(\Sigma, \mathbb{R}_+)$. Let $W = \mathrm{Vect}(\varphi)$ be the unique solution to:

$$\Delta_{\mathbb{L}} W = \frac{n-1}{n} \varphi^N d\tau$$

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The map Lich

Now let W such that $\mathbb{L}W \in L^{2p}(\Sigma, T\Sigma)$. Let $\varphi' = \mathrm{Lich}(W)$ be the unique solution to :

$$\frac{4(n-1)}{n-2} \Delta \varphi' + \mathrm{Scal}\varphi' + \frac{n-1}{n} \tau^2 (\varphi')^{N-1} = \frac{|\sigma + \mathbb{L}W|^2}{(\varphi')^{N+1}}.$$

The map Φ

The map Φ

Let $r = 2pN$. Φ is the map

$$\Phi : L^r(\Sigma, \mathbb{R}_+) \to L^r(\Sigma, \mathbb{R}_+).$$

defined by

$$\Phi(\varphi) = (\mathrm{Lich} \circ \mathbb{L} \circ \mathrm{Vect})\,(\varphi).$$

Objective : Is Φ a contractive map ?

Let φ_1, φ_2 two solutions to the CCE, we want to prove that for $\lambda < 1$:

$$\|\Phi(\varphi_1) - \Phi(\varphi_1)\|_{L^r} \leq \lambda \|\varphi_1' - \varphi_2'\|_{L^r}.$$

Uniqueness

Let (φ'_1, W_1) and (φ'_2, W_2) two solutions to CCE, s.t.

$$\varphi'_1 = \Phi(\varphi_1), \quad \varphi'_2 = \Phi(\varphi_2).$$

We can prove that for $\alpha \geq 1$:

$$\|\varphi'_1 - \varphi'_2\|^{2\alpha}_{L^{N\alpha}} \lesssim \int_\Sigma (\varphi'_1 - \varphi'_2)^{2\alpha-1} \, \frac{\mathbb{L}W_1^2 - \mathbb{L}W_2^2}{(\varphi'_1)^{N+1}} d\mu^\gamma.$$

Uniqueness

Let (φ_1', W_1) and (φ_2', W_2) two solutions to CCE, s.t.

$$\varphi_1' = \Phi(\varphi_1), \quad \varphi_2' = \Phi(\varphi_2).$$

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$$\|\varphi_1' - \varphi_2'\|_{L^{N\alpha}}^{2\alpha} \lesssim \int_\Sigma (\varphi_1' - \varphi_2')^{2\alpha-1} \, \frac{\mathbb{L}W_1^2 - \mathbb{L}W_2^2}{(\varphi_1')^{N+1}} d\mu^\gamma.$$

By applying Hölder estimate, we need to control:

- $\|\mathbb{L}W\|$.
- $\|\varphi^{-(N+1)}\|$. Need to obtain lower bounds for $\|\varphi\|$.

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Remark: Bounding from above $\|LW\|$ needs to obtain upper bound on $\|\varphi\|$.

Useful estimates

In order to establish the previous inequality : **lower and upper estimates**.

In particular,

- The gap phenomenon : from $V_\gamma(\Sigma) \leq V_{max}$.
- The bootstrap argument.
- Lower bound : the importance of sub-solutions and Green functions.

The gap phenomenon

Once we state :

$$V_{max} \leq C \left(\frac{n-1}{n} \right)^{n} \|d\tau\|_{L^n}^{n}$$

C: mix of different Sobolev constants.

Then, estimates follow :

The size of (φ, W) is controlled by the size of TT-tensor σ:

$$\|\varphi^N\|_{L^{\frac{N}{2}+1}}^{2\frac{n-1}{n}} \leq C_1(n, \tau, V_{max})\|\sigma\|_{L^2}^2.$$

$$\|\mathbb{L}W\|_{L^2}^2 \leq C_2(n, \tau, V_{max})\|\varphi^N\|_{L^{\frac{N}{2}+1}}^{2\frac{n-1}{n}} \lesssim \|\sigma\|_{L^2}^2$$

The bootstrap argument

For any σ such that $\|\sigma\|_{L^{2p}} \leq 1$, if (φ, W) is a solution satisfying $V_\gamma(\Sigma) \leq V_{max}$, then

Promotion of previous estimates

$$\|\mathbb{L}W\|_{L^\infty} \leq C\|\sigma\|_{L^{2p}}^{\frac{n}{n-1}}.$$

And for any $r \in]1, +\infty[$:

$$\|\varphi^N\|_{L^r} \leq C'(r)\|\sigma\|_{L^{2p}}^{\frac{n}{n-1}}.$$

Proof's ideas : Induction reasonement. Successive Hölder and Sobolev inequalities.

Sub-solution

We recall that a sub-solution $\varphi_- \leq \varphi$ satisfies :

$$-\frac{4(n-1)}{n-2}\Delta\varphi_- + \mathrm{Scal}\varphi_- + \frac{n-1}{n}\tau^2\varphi_-^{N-1} \leq \frac{|\sigma + \mathbb{L}W|^2}{\varphi_-^{N+1}}.$$

We prove $\varphi_- = \lambda v$, with v the solution to :

$$-\frac{4(n-1)}{n-2}\Delta v + \mathrm{Scal} v + \frac{n-1}{n}\tau^2 v = A^2, \quad A^2 := |\sigma + \mathbb{L}W|^2$$

Maxwell's results

$$v(x) = \int_\Sigma G_\tau(x,y)A^2(y)d\mu^\gamma(y), \quad \textit{for a.e. } x \in \Sigma.$$

Green's function is **bounded from below**:

$$G_\tau(x,y) \geq m_{\gamma,\tau} \quad \forall x,y \in \Sigma, x \neq y.$$

$$\varphi \geq \varphi_- \geq C(\gamma,\tau,\sigma)\|A^2\|_{L^2}^{\frac{n-2}{2(n-1)}}.$$

Conclusion of the proof.

We define the contraction constant to be :

$$\lambda := \frac{1}{(\inf_{\Sigma} \varphi_1')^{N+1}} \|\mathbb{L}W_1 + \mathbb{L}W^2\|_{L^{2p}} \|\sigma\|_{L^2p}^{\frac{n+2}{2(n-1)}}.$$

From the previous estimates :

$$\inf_{\Sigma} \varphi_1' \gtrsim \|\sigma\|_{L^{2p}}^{\frac{n-2}{2(n-1)}}.$$

Taking $\|\sigma\|_{L^{2p}}$ sufficiently small ensures

$$\lambda < 1.$$

Hence,

$$\|\Phi(\varphi_1) - \Phi(\varphi_2)\|_{L^r} \leq \lambda \|\varphi_1' - \varphi_2'\|_{L^r}. \qquad \square$$

Conclusion

- Proof of existence and uniqueness of solutions to conformal constraint equations.
- Banach fixed point theorem.
- Importance of the smallness of σ.
- Construction of solutions.
- Analytical construction ? Numerical implementation ?