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Beyond circularity

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based on arxiv:2505.08880
+ work in progress

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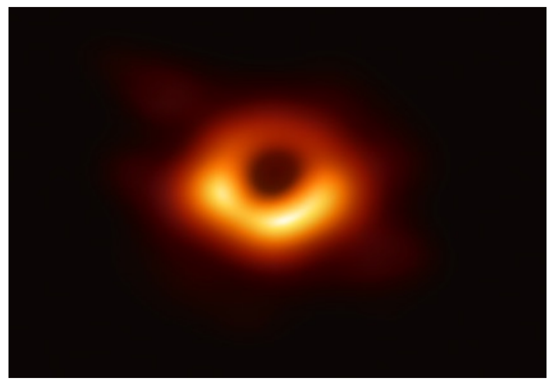
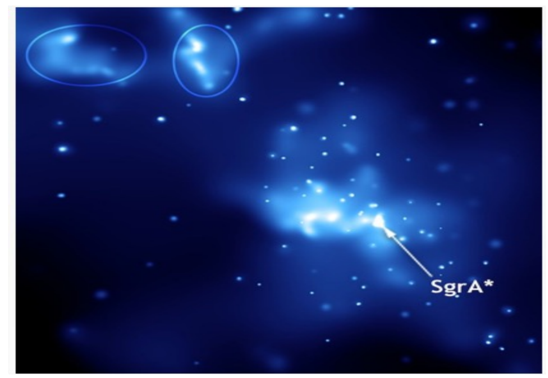
Motivation: Test gravity with BHs

Modern observations enable unprecedented tests of GR:

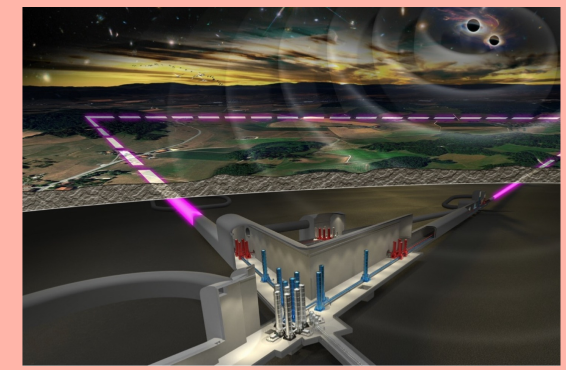
Gravitational Waves
LIGO/Virgo: binary mergers



BH imaging
EHT: M87*, Sgr A*



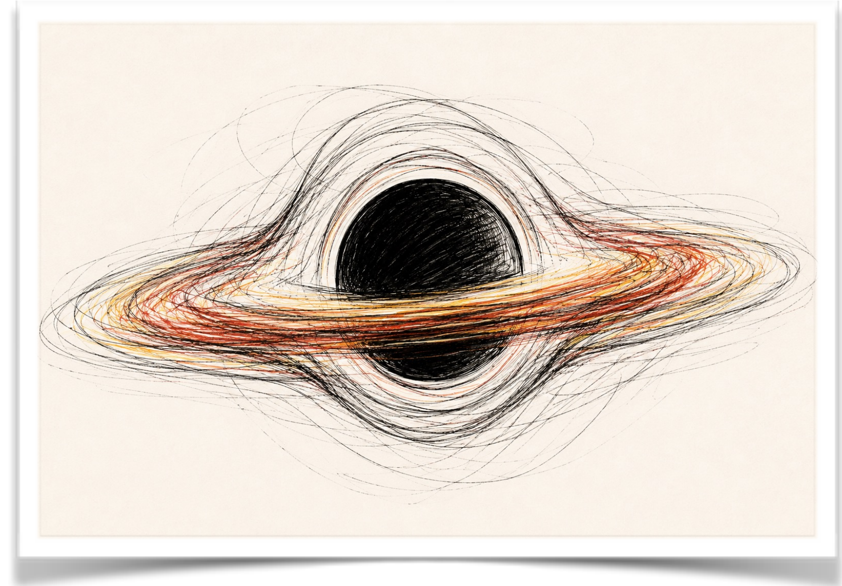
Stellar orbits
GRAVITY: S2 star



Future detectors
LISA, Einstein Telescope

- GR has passed all tests so far, but is unlikely to be the ultimate theory of gravity
- Dark energy, quantum gravity, and renormalisability remain unresolved issues
- We need metrics that deviate from Kerr to design non-null tests of GR
- The key question: what is the most general rotating black hole metric beyond GR?

Motivation: Going beyond Kerr



- In General Relativity the Kerr solution describes vacuum rotating black hole
- It is important to construct deformations of the Kerr spacetime to have a benchmark for testing General Relativity and look for signatures of modified gravity.

Direct approach

- Start from a specific modified gravity theory
- Search for exact rotating black hole solutions
- Pro: all physical properties accessible
- Pro: gravitational degrees of freedom included
- Con: very few known exact solutions
- Main example: Disformal Kerr (scalar–tensor gravity)

Agnostic Approach

- Construct an ad hoc metric deviating from Kerr
- Various parametrisations proposed in literature (but circular cases only)
- Pro: more general, broader phenomenology
- Con: dynamical problems require extra assumptions
- Con: many examples found to be pathological (naked singularities)
- Root cause: using singular coordinates at the horizon

Motivation: Circularity

- Circularity -- accidental symmetry of Kerr solution. A symmetry that is in addition to stationarity and axisymmetry
- In GR, for vacuum solutions circularity is automatic
- Beyond GR : do we keep circularity or not? Usually yes, but this is not always justified.

Outline

- Circularity vs non-circularity
- General non-circular metric and gauge
- Solving circularity condition for the general metric
- Examples
- Outlook

Circularity vs non-circularity

Circularity vs. non-circularity

Most general stationary axisymmetric spacetime: two Killing vectors associated to the symmetries

- ξ^μ (stationarity: timelike at infinity)
- ψ^μ (axial symmetry: spacelike, closed orbits)

Geometric notion

- The 2D surfaces generated by the Killing isometries are everywhere orthogonal to a family of codimension-2 surfaces.
- Frobenius' theorem:

$$\xi_{[\mu}\psi_\nu\partial_\rho\xi_{\sigma]} = 0$$
$$\xi_{[\mu}\psi_\nu\partial_\rho\psi_{\sigma]} = 0$$

Matter notion

- A flow vector u^μ (the velocity of a particle or of a cloud of gas)
- u^μ is circular if it is a linear combination of ξ^μ and ψ^μ
- In GR: circular sources $\leftrightarrow$ circular spacetime.
- All vacuum GR solutions (with or without Λ) are automatically circular.
- The Kerr metric is the prime example.

Carter, Black holes equilibrium states, Les Houces lecture notes'73

Why circularity?

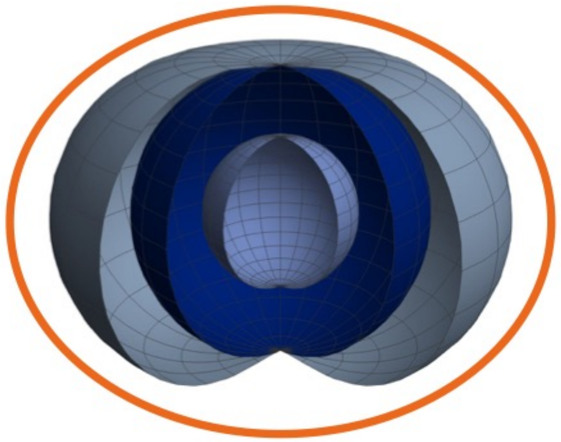
- Vacuum GR solutions are circular
- Some solutions to alternative theories of gravity are circular:
  – stealth solutions, whereby the metric is that of Kerr;
  – numerical solutions in dilatonic EGB theory;
  – perturbative rotating solutions in sEGB and dynamical Chern-Simons gravity;
- beyond-GR solutions that are continuously connected to GR are circular [Y. Xie et al'21]

The metric is
block diagonal

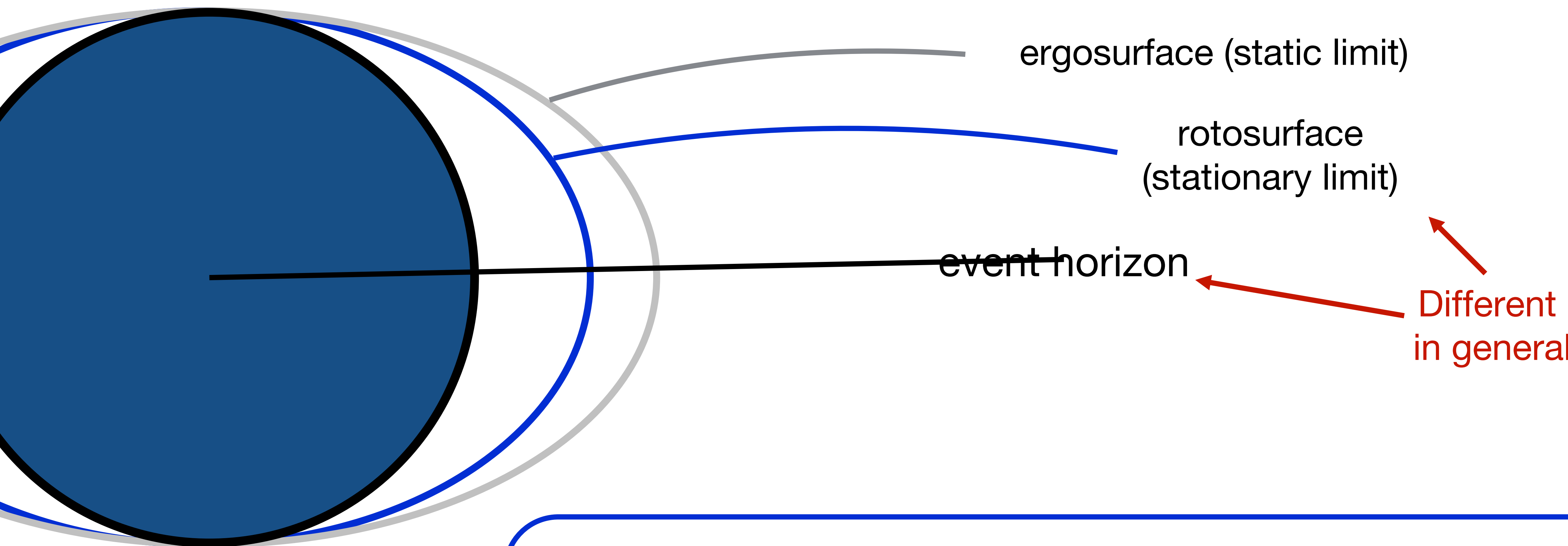
$$g_{\mu\nu} = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}$$

$t-\phi$ symmetry ($t \to -t$, $\phi \to -\phi$)
5 'free' functions of two variables (r, θ)

For Kerr metric
Boyer-Lindquist coordinates



Circularity means simplicity



If circularity, then:

- rotosurface $\equiv$ horizon
- frame dragging $\Omega =$ const. on the horizon (Rigidity theorem)
- event horizon is Killing
- different notions of surface gravity agree
- surface gravity is constant (0th law of BH mechanics)

NB Circularity is necessary (although not sufficient) condition for separability
(Metric can be circular but still not separable)

Why circularity may fail?

Even in GR non-vacuum solutions may have a non-circular metric Gourgoulhon, Bonazzola'93

In modified gravity:
– Disformed Kerr black hole Anson et al'21 (also Ben Achour at al'21);
– Numerical rotating BHs in Einstein-Aether Adam et al'21

Circularity is an extra assumption

In modified gravity this assumption translates to the symmetry assumption on extra fields.
– E.g. in scalar-tensor theory one needs $\xi^\mu \partial_\mu \varphi = \psi^\mu \partial_\mu \varphi = 0$ to fall into class of theories with circular solutions.

Conclusion: Assuming circularity *a priori* is often not justified

General metric and gauge choice

Gauge choice for general axisymmetric stationary metric

- Coordinates adapted to symmetries, i.e. v and ϕ along the Killing orbits,

$$\xi^\mu \partial_\mu = \partial_v \quad \text{and} \quad \psi^\mu \partial_\mu = \partial_\phi$$

Other coordinates r and θ, so that

$$g_{\mu\nu}\left(x^\alpha\right) = g_{\mu\nu}(r,\theta)\,.$$

- These coordinates are defined up to

$$\tilde{v} = v + V\left(r,\theta\right)$$
$$\tilde{r} = R\left(r,\theta\right)$$
$$\tilde{\theta} = \Theta\left(r,\theta\right)$$
$$\tilde{\phi} = \phi + \Phi\left(r,\theta\right)$$

$$g_{\mu\nu} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

where $* = *(r,\theta)$

Most general choice of coords adapted to the symmetries (10 components of metric)

Gauge choice for general axisymmetric stationary metric

Look for horizon penetrating coord

$$g_{\mu\nu} = \begin{pmatrix} g_{vv} & g_{vr} & 0 & g_{v\phi} \\ * & g_{rr} & 0 & g_{r\phi} \\ 0 & 0 & g_{\theta\theta} & 0 \\ * & * & 0 & g_{\phi\phi} \end{pmatrix}$$

- Orthogonal gauge (3 components $= 0,\ g_{\mu\theta} \propto \delta^{\theta}_{\mu}$)
- Kerr-like gauge (4 components $= 0,\ g_{\mu\theta} \propto \delta^{\theta}_{\mu},\ g_{rr} = 0$)

$$g_{\tilde{\mu}\tilde{\nu}}\left(\tilde{x}^{\alpha}\right) \xrightarrow{\text{change of coordinates}} g_{\mu\nu}\left(x^{\alpha}\right)$$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\tilde{v} = v + V\left(r,\theta\right)$$
$$\tilde{r} = R\left(r,\theta\right)$$
$$\tilde{\theta} = \Theta\left(r,\theta\right)$$
$$\tilde{\phi} = \phi + \Phi\left(r,\theta\right)$$

$$\begin{pmatrix} * & * & 0 & * \\ * & 0 & 0 & * \\ 0 & 0 & * & 0 \\ * & * & 0 & * \end{pmatrix}$$

such coord transformation exist
if a solution of 4 PDE in terms of
4 unknown functions exist

$$\begin{cases} g_{v\theta} = 0 \\ g_{r\theta} = 0 \\ g_{\theta\phi} = 0 \\ g_{rr} = 0 \end{cases}$$

Proof of existence

Naively: 4 equations in 4 functions, solution should exist?

For Kerr-like (or orthogonal) gauge the existence of diffeo can be proven by the use of *Cauchy-Kovalevskaya theorem*

yes, but no:

$$g_{\mu\nu} = \begin{pmatrix} g_{vv} & 0 & 0 & g_{v\phi} \\ 0 & g_{rr} & g_{r\theta} & 0 \\ 0 & * & g_{\theta\theta} & 0 \\ * & 0 & 0 & g_{\phi\phi} \end{pmatrix}$$

Also 4 equations, but metric is circular

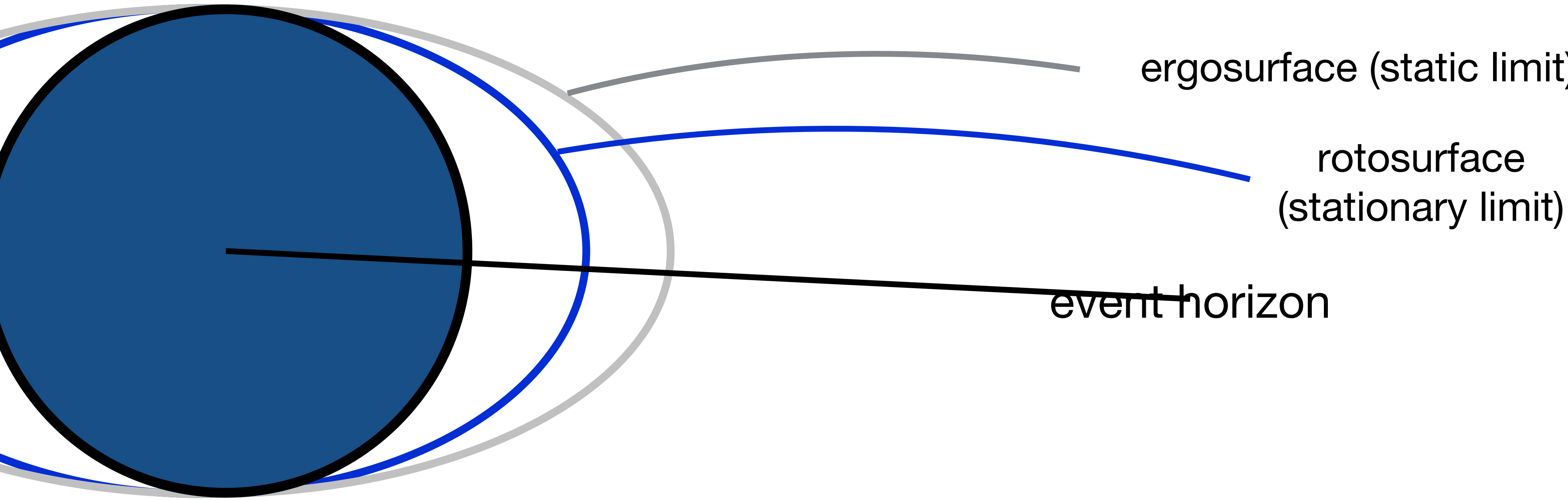
This gauge cannot be obtained by coordinate change for non-circular metric

Local result, since

- Cauchy-Kovalevskaya theorem ensures local existence
- Initial coordinate chart might no be global

! There is a residual gauge that can be related to `initial conditions' of 4 PDEs

Notable Surfaces



Ergosurface

$g_{vv}\big|_{r=r_{\rm erg}} = 0.$

Locus where the Killing vector ξ_μ becomes null. Inside: no static observers.

Rotosurface

$(\xi_\mu + \Omega\psi_\mu)(\xi^\mu + \Omega\psi^\mu) = 0 \quad \Leftrightarrow \quad g^{rr} = 0$ (in orthogonal gauge)

Locus where locally non-rotating observers $\propto \xi^\mu + \Omega\psi^\mu$ cease to exist.

Horizon $\quad g^{rr}\big|_{r=H} + g^{\theta\theta}\big|_{r=H}\left(\frac{\mathrm{d}H}{\mathrm{d}\theta}\right)^2 = 0$

A null hypersurface. In circular spacetimes: horizon = rotosurface = Killing horizon with constant Ω. In non-circular spacetimes these need not coincide.

Solving circularity conditions

Circularity conditions

The circularity condition are PDEs

$$\xi_{[\mu}\psi_\nu\partial_\rho\xi_{\sigma]} = 0$$
$$\xi_{[\mu}\psi_\nu\partial_\rho\psi_{\sigma]} = 0$$

Given a metric it is easy to check circularity

It is difficult to construct simple non-circular metrics

! Can we "solve" conditions to get algebraic relations?

Depends on choice of "good" gauge

Solving circularity conditions

Kerr-like gauge is great

$$\xi_{[\mu}\psi_\nu\partial_\rho\xi_{\sigma]} = 0$$

$$\xi_{[\mu}\psi_\nu\partial_\rho\psi_{\sigma]} = 0$$

Plug into Kerr (orthogonal) gauge

Long calculations

$$g^{vr} = f(r)g^{rr}$$

$$g^{r\phi} = h(r)g^{rr}$$

Solved circularity conditions

upper indices

$$\frac{g^{vr}}{g^{rr}} = f(r)$$

$$\frac{g^{r\phi}}{g^{rr}} = h(r)$$

lower indices

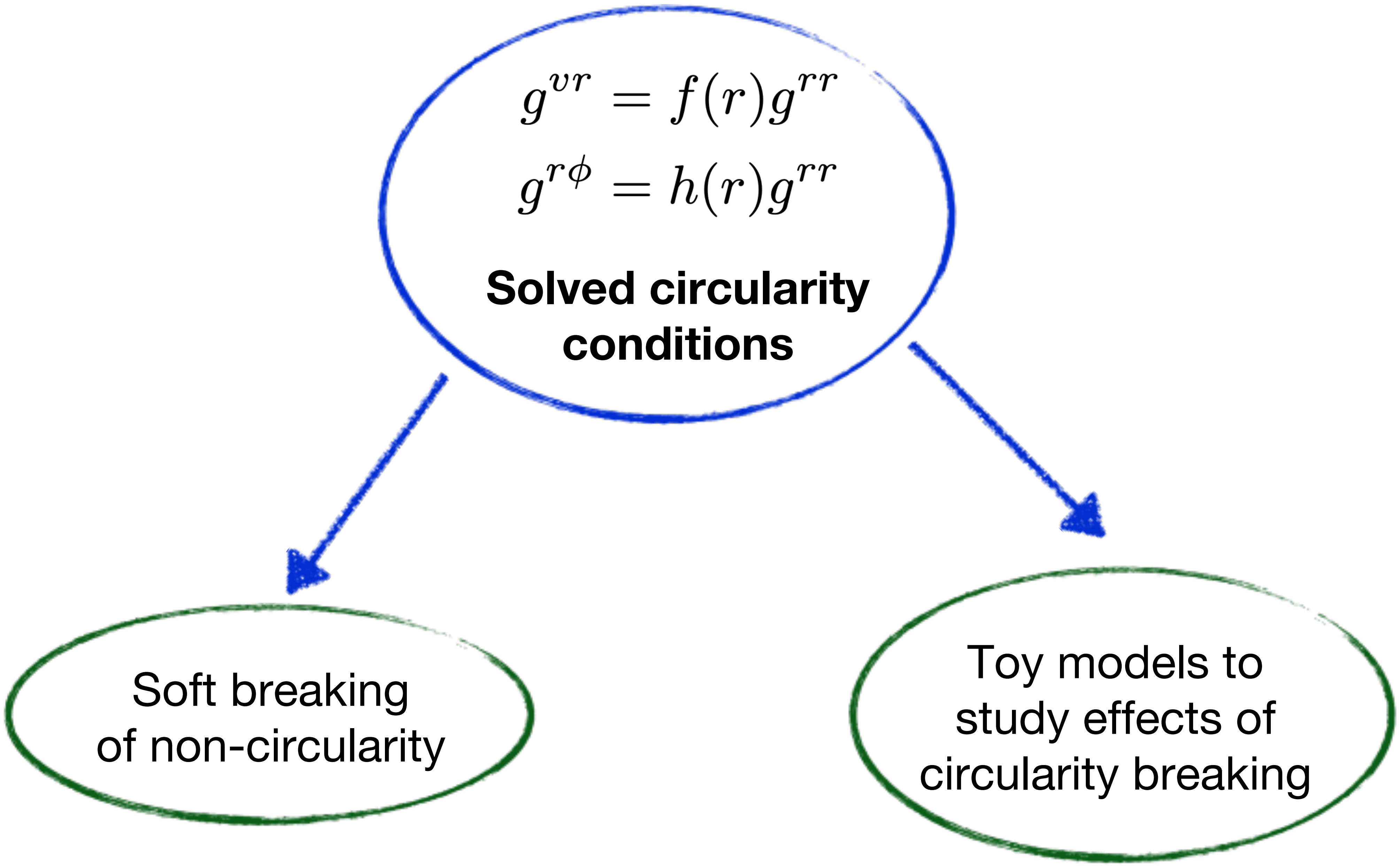
$$\frac{g_{vr}g_{\phi\phi} - g_{r\phi}g_{v\phi}}{g_{v\phi}^2 - g_{vv}g_{\phi\phi}} = f(r)$$

$$\frac{g_{r\phi}g_{vv} - g_{vr}g_{v\phi}}{g_{v\phi}^2 - g_{vv}g_{\phi\phi}} = h(r)$$

These are exactly conditions to go from Kerr-like (orthogonal) to `Boyer-Lindquist' form

$$\begin{pmatrix} * & * & 0 & * \\ * & \square & 0 & * \\ 0 & 0 & * & 0 \\ * & * & 0 & * \end{pmatrix} \longrightarrow \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}$$

Why is this interesting?



Constructing non-circular deformations

How do we deform Kerr metric

The algebraic circularity condition gives us a precise handle on circularity breaking

Circularity is broken when $g^{vr}/g^{rr} = f$ or $g^{r\phi}/g^{rr} = h$ acquire θ-dependence.

For Kerr metric in ingoing coordinates:

$f_{\text{Kerr}} = \frac{r^2+a^2}{\Delta}$ and $h_{\text{Kerr}} = \frac{a}{\Delta}$, where $\Delta := r^2 - 2Mr + a^2$

Two complementary examples of circularity breaking:

1 **"Minimal" deformation**

Modify g^{vr} while keeping all surface locations (almost) as in Kerr. Horizon exists but is no longer Killing.

2 **"Not-so-minimal" deformation**

Replace mass M with angle-dependent mass function $m(r,\theta)$. Horizon and rotosurface are deformed (but all analytic). Horizon profile can be chosen freely.

Example 1: Minimal deformation

Deformation $g^{vr} = f_{\rm Kerr} g^{rr}_{\rm Kerr} + f(r,\theta)_{\rm deform}$ with $f(r,\theta)_{\rm deform} = \frac{\delta(r,\theta)}{\Sigma}$

Properties:

- Static limit unchanged: $r_{\rm erg} = M + \sqrt{M^2 - a^2\cos^2\theta}$ (same as Kerr)
- Rotosurface unchanged: $R = M + \sqrt{M^2 - a^2}$ (constant-r sphere, same as Kerr)
- Event horizon $=$ rotosurface (as in Kerr), but it is NOT a Killing horizon
- Angular velocity Ω on the horizon inherits θ-dependence from $\delta \Rightarrow$ not constant
- Surface gravity $\kappa_H = \left.\frac{r-M}{2Mr+\delta}\right|_H$ depends on $\theta \Rightarrow$ zeroth law fails
- Equatorial geodesics completely indistinguishable from Kerr
- Extremal limit $a = M$ still exists, but circularity is not restored unless $\delta \to 0$

Example 2: Not-so-minimal deformation

Deformation: in Kerr metric replace $M \to m(r,\theta)$

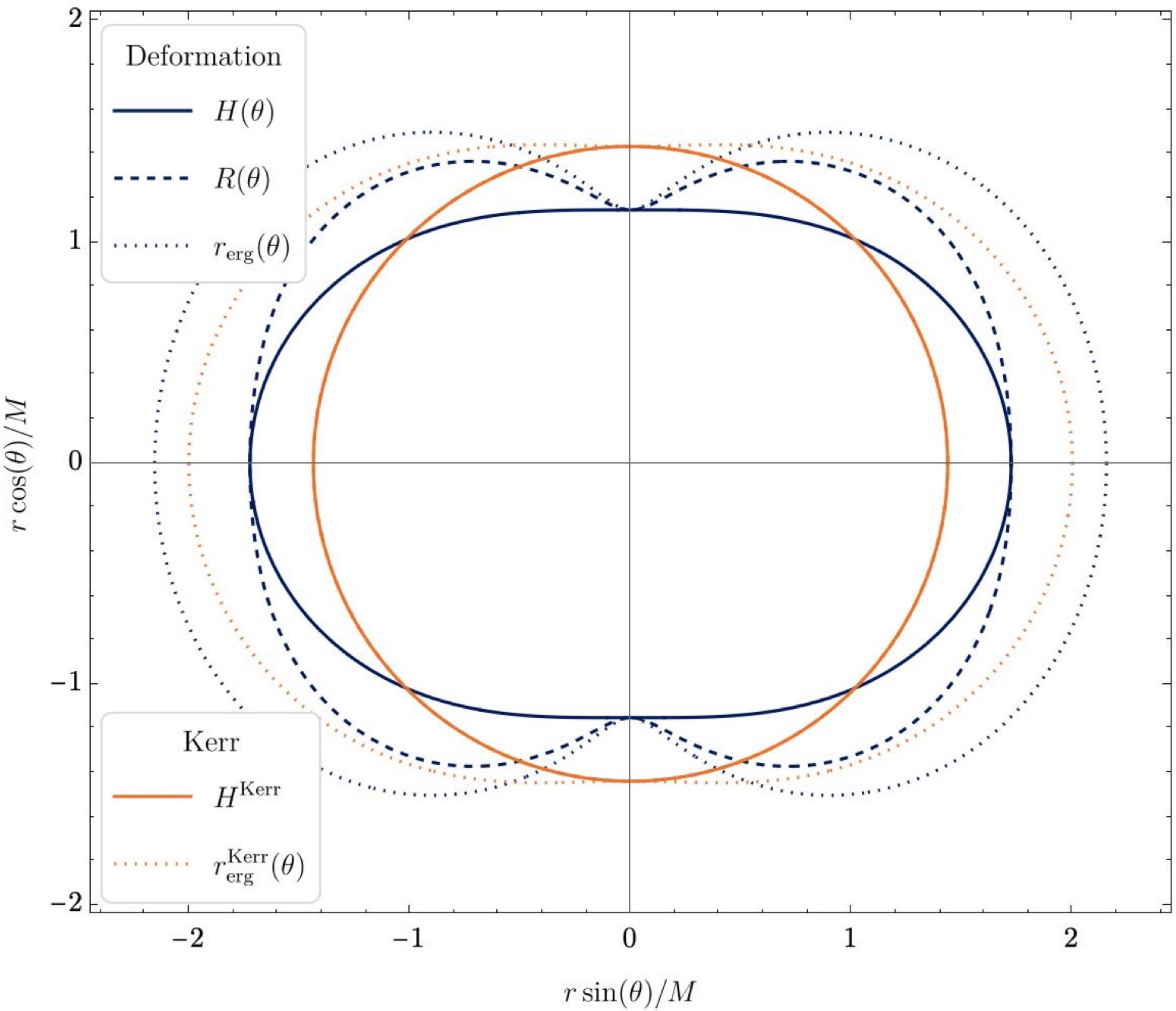
- Regular black hole
- The circularity conditioon is broken because m depends non-trivially on θ
- Reverse-engineering approach: given $H(\theta)$, solve for $m(r,\theta)$ analytically

Worked example: $H(\theta) = H_{\textsf{Kerr}}\left[1 - \epsilon\cos(2\theta)\right]$

- Rotosurface: $R = M + \sqrt{(H-M)^2 + (H')^2}$ — displaced from Kerr's constant-r sphere
- Static limit: $r_{\text{erg}} = M + \sqrt{(H-M)^2 + (H')^2 + a^2\sin^2\theta}$
- Horizon and rotosurface no longer coincide — a new feature absent in circular spacetimes

NB Choosing $m(r,\theta) = \mathcal{O}\left(r^3\right)$ as $r \to 0$ regularises the ring singularity $\Rightarrow$ regular BH

Not-so-minimal deformation



Black hole thermodynamics?

- In a circular spacetime: the horizon generator is Killing, with $\Omega =$ const. over the horizon
- In non-circular spacetimes (including both deformations): the horizon is not a Killing horizon
- Angular velocity Ω varies over the horizon $\rightarrow$ no well-defined 'horizon angular velocity'
- Surface gravity is not constant (moreover, different notions of surface gravity need not agree) $\rightarrow$ the zeroth law of black hole mechanics fails
- Possible physical picture: non-constant $\kappa \Rightarrow$ angle-dependent Hawking temperature
- Non-circular BHs may be 'out-of-equilibrium thermodynamic systems'

Hawking radiation calculated via tunnelling method F. Del Porro, J. Mazza [2511.02911]

$$T_H = \frac{\kappa_{\rm p}}{2\pi}$$

$\kappa_{\rm p}$ is peeling surface gravity, depends on θ

Some phenomenology?

Observational signatures of non-circularity

This framework opens the door to systematic phenomenological studies:

Black hole shadows

Non-circular metrics break geodesic separability → more complex photon orbits.

Accretion disk spectra

The ISCO location from equatorial geodesics is unchanged by some deformations, but off-equatorial structure may differ.

Pulsar timing around Sgr A*

Orbital precession (Lense–Thirring effect) and pulse arrival times can constrain non-circular deviations.

Gravitational waves

Waveforms encode non-circularity. Resonant orbits and inspiral dynamics are sensitive probes of the spacetime symmetry.

Summary of main results

- Proved local existence of Kerr-like gauge for any stationary axisymmetric metric (6 free functions of 2 variables)
- Solved the differential circularity conditions analytically -> algebraic conditions
- Showed that circularity <-> existence of Boyer-Lindquist-like coordinates
- Constructed “minimal” and “non-minimal” non-circular Kerr deformations

Current limitations of our work

- Proof of gauge existence is only local (Cauchy-Kovalevskaya theorem)
- Explicit form of the coordinate change is generally unavailable analytically
- No dynamical (theory-based) derivation of the examples

Our current study

What about slow rotation?

Usually, slow rotation means Hartle-Thorne [Lense-Thirring] metric

$$ds^2 = -f(r)\mathrm{d}t^2 + \tilde{f}(r)\mathrm{d}r^2 + r^2\left[\,\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right]$$
$$-2r^2\sin^2(\theta)\Omega(r,\theta)\mathrm{d}t\;\mathrm{d}\phi$$
$$[\Omega(r,\vartheta) = \mathcal{O}(a)]$$

circular

$$ds^2 = -f(r)\mathrm{d}v^2 + 2h(r)\mathrm{d}v\;\mathrm{d}r + r^2\left[\,\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right]$$
$$-2r^2\sin^2(\theta)[\Omega(r,\theta)\mathrm{d}t\;\mathrm{d}\phi + \Lambda(r,\theta)\mathrm{d}r\;\mathrm{d}\phi]$$
$$[\Omega(r,\vartheta), \Lambda(r,\vartheta) = \mathcal{O}(a)]$$

non-circular

Is there something ‘protecting’ circularity at $\mathcal{O}(a)$?

or are we missing some slowly rotating solutions?