

# Acceleration of charged particles in the Kerr magnetosphere: An electrogeodesic approach

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IDP - Tours

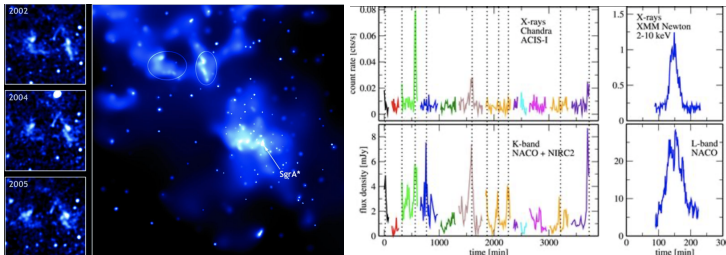
4th April 2026

Based on arXiv:2601.05048 [astro-ph.HE]

In collaboration with I. El Mellah and E. Gourgoulhon

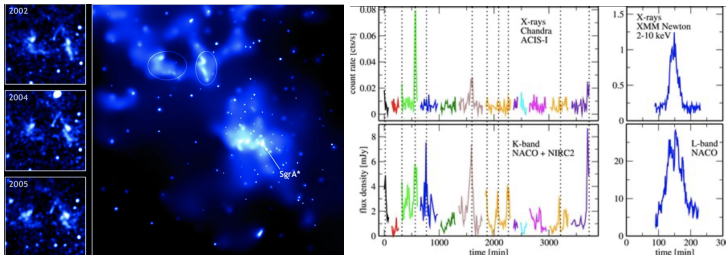
# The black hole accretion/ejection paradigm

- Black holes accrete and eject matter, giving rise to a complicated spectrum in all frequency bands not yet understood
- Variability of the radiation emitted by the center of the galaxy:
  - 1) Infrared and X emission:  $\Delta\tau \sim \text{few sec} - 1h$
  - 2) Radio emission:  $\Delta\tau \sim 1h - 1d$
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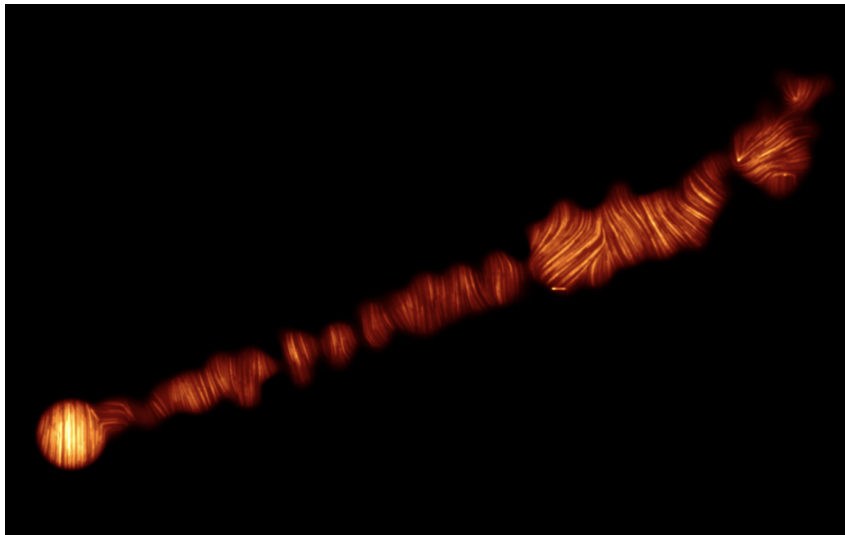
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## Black hole as astrophysical sources of extreme phenomena

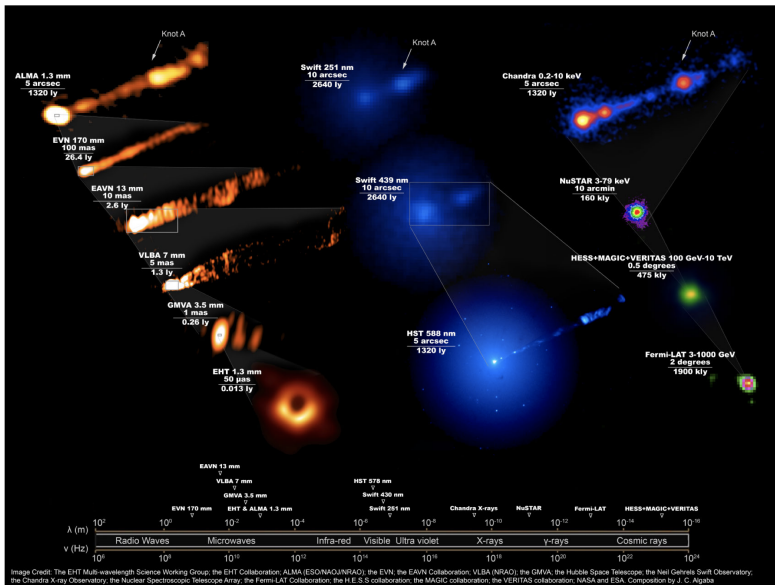
- Black holes are believed to play a key role in very high energy astrophysical phenomena
- Rotating black holes stand as the most important reservoir of extractable energy: natural accelerator of particles
- Compelling evidence that very high energy phenomena are triggered in the surrounding region
  - Relativistic jets of plasma from active galactic nuclei (AGN) and X-ray binaries systems
  - $\gamma$ -ray bursts



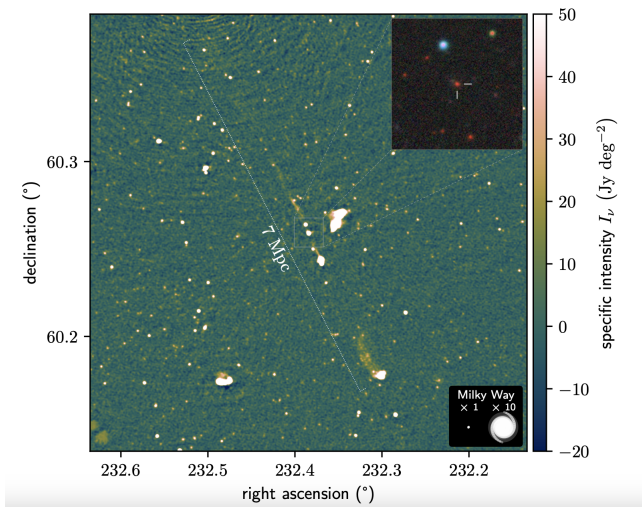
Credit: [Atacama Large Millimeter/submillimeter Array \(ALMA\)](#)

- Radio observation of the jet through polarized light
- Scale: 6000 LY

# The jet from Messier 87 at different wavelengths



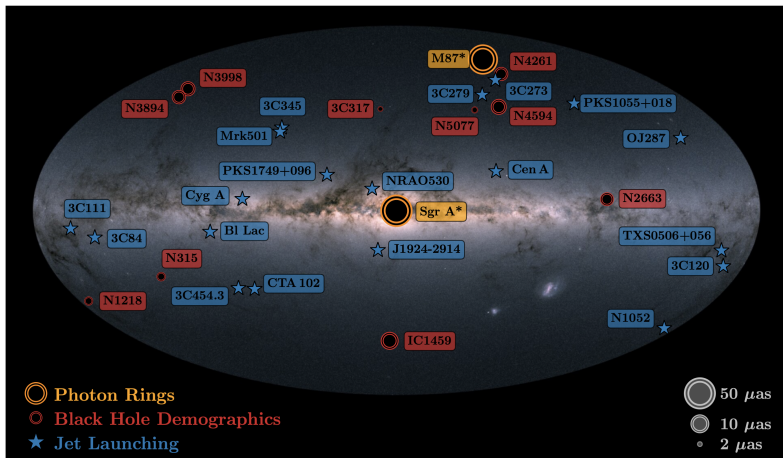
# The largest jet ever recorded



Credit Oei et al '24

- Demonstrate that the jet launching mechanism can efficiently redistribute matter on cosmological scales
- Can it seed the filaments in the early phase of the universe ?

# Population of bright sources interpreted as black holes



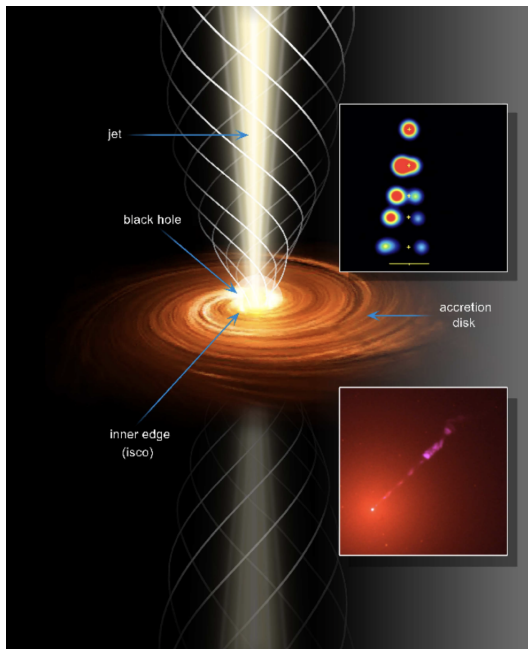
## Main challenges : resolve the supermassive black hole environment

- Event Horizon Telescope: EHT (on-going), nEHT (2030) ground based
- Black Hole Explorer (BHEX): 2032 ? mixed ground-space based
- SHARP ... space-based

## Current astrophysical model

- Active galactic nuclei powered by a spinning magnetized supermassive black hole
- Trigger collimated jets which redistribute the matter on large scales
- Many observations / catalogue
- Theoretically: confirmed by numerical simulations .... but still no well defined theory for this paradigm so far
  - 1) GR-magneto-hydrodynamics simulations: fluid based approaches
  - 2) GR-Particle-in-Cells simulations: kinetic approaches
- Require to understand the complicated physics between General Relativity + Electromagnetism + Fluid dynamics / microphysics

# Schematic picture of the black hole accretion/ejection paradigm



## Open questions

- Which mechanism allows for such jet launching ?
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## Plan

- Review the maximal extractible energy in a rotating black hole
- Review energy extraction mechanism: Blandford-Znajek model and its limits
- Analytic model for the motion of charged particles in a rotating magnetized black hole  
[J. BA, I. El Mellah, E.ourgoulhon '26]
- Few concrete predictions opening dialogue with numerical simulations

Accessible rotational energy from the Kerr black hole

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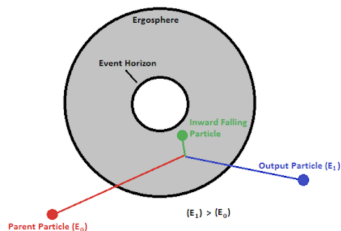
Apply this to supermassive black holes with  $M \sim 10^6 M_{\odot}$  ... Huge amount of energy

Review analytic models for black hole energy extraction

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## Many different mechanisms to extract the energy of black holes

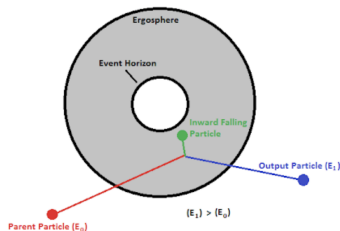
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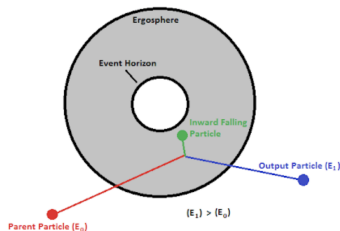


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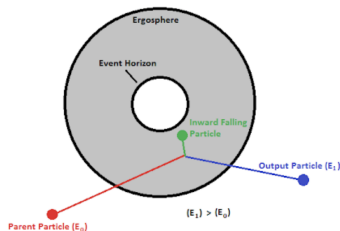


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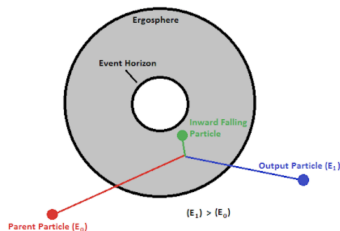


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- Magnetic field reconnections and negative energy plasmoid [Comisso, Asenjo '21]  
→ fluid GRMHD approach

Review Blandford-Znajek model and its limits

# Maxwell equations and Poynting flux in short

## Basic equations

- The Maxwell equations read

$$\nabla_{[\mu} F_{\alpha\beta]} = 0 \quad \nabla^{\mu} F_{\mu\nu} = j_{\nu} \quad \nabla_{\mu} j^{\mu} = 0 \quad (5)$$

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- Energy-momentum tensor

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- Can be decomposed into a non-perfect radiation fluid

$$T_{\mu\nu} = \frac{1}{2} (E^2 + B^2) u_{\mu} u_{\nu} + \frac{1}{6} (E^2 + B^2) h_{\mu\nu} + 2Q_{(\mu} u_{\nu)} + \pi_{\mu\nu} \quad (9)$$

with

$$P = \frac{\rho}{3} = \frac{1}{6} (E^2 + B^2) \quad Q_{\mu} = \epsilon_{\mu\nu\rho\sigma} E^{\nu} B^{\rho} u^{\sigma} \quad \pi_{\mu\nu} = 2Ph_{\mu\nu} - E_{\mu} E_{\nu} - B_{\mu} B_{\nu} \quad (10)$$

## Energy flux from toroidal electric and magnetic fields

- Consider a stationary and axisymmetric spacetime with timelike Killing vector  $\xi^\mu \partial_\mu = \partial_t$ :

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 \quad (11)$$

- Let us evaluate the flux of electromagnetic energy at infinity

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- For a stationary and axi-symmetric metric, the flux component  $T_{tr}$  reads

$$T_{tr} = -g^{\phi t} F_{t\phi} F_{tr} - g^{\theta\theta} F_{t\theta} F_{\theta r} \quad (14)$$

# Energy flux from toroidal electric and magnetic fields

- Consider a stationary and axisymmetric spacetime with timelike Killing vector  $\xi^\mu \partial_\mu = \partial_t$ :

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Let see some concrete examples

## Concrete examples

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- **Poynting flux at fourth order in spin  $a$ :**

$$P^{BZ} = \frac{\pi}{24} \frac{\mathcal{G}^2 a^2}{M^2} + \frac{(56 - 3\pi^2)\pi}{1080} a^4 + \mathcal{O}(a^5) \quad (26)$$

## Limitations

- BZ only valid for small spin: Open question so far  
Is there a fully exact BZ-like solution on Kerr not perturbative in the spin  $a$  ?  
Crucial as it could well change the prediction  $P_{\text{BZ}} = \frac{G^2 a^2}{M^2}$  for highly spinning black hole !
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- Under which conditions can we integrate the electrogeodesic motion in a Kerr magnetosphere model ?
- First review the case of geodesic and the Carter miracle [Carter '68]

Carter miracle for the geodesic motion on Kerr

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- Consider a test neutral particle of mass  $m$  with 4-velocity  $u^\mu \partial_\mu$ : timelike  $u^\mu u_\mu = -1$

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## General properties of geodesic motion on Kerr

- Consider a test neutral particle of mass  $m$  with 4-velocity  $u^\mu \partial_\mu$ : timelike  $u^\mu u_\mu = -1$
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- Two more conserved charges: the mass ... and **the Carter's constant**

$$m^2 = -\frac{1}{2} g^{\mu\nu} p_\mu p_\nu \quad \mathcal{K} = K^{\mu\nu} p_\mu p_\nu \quad (31)$$

with  $K^{\mu\nu}$  a generalization of a Killing vector called a hidden symmetry called a Killing tensor

$$\nabla_{(\rho} K_{\mu\nu)} = 0 \quad (32)$$

**Crucial to solve the geodesic motion !**

# Geodesic motion on Kerr

## Geodesic equations on Kerr

- Separable system in term of Mino time  $d\tau = d\lambda/\Sigma$  with  $\Sigma = r^2 + a^2 \cos^2 \theta$

$$\frac{dt}{d\tau} = \frac{1}{\Delta(r)} [(r^2 + a^2)^2 E - 2MarL] - E \sin^2 \theta \quad (33)$$

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with the radial and polar potentials given by

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## First step for electrogeodesics

- Under which condition a magnetosphere preserves the Carter constant for electrogeodesic ?

## Electrogeodesic phase space

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- In term of this momenta, the hamiltonian takes the usual form  $H = \frac{1}{2}g^{\mu\nu} p_\mu p_\nu$

## Condition to preserve the Carter-like constant

### Geometrical condition on the magnetosphere

- Consider the Killing-Yano tensor and its associated Killing tensor in the Kerr geometry:

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The Kerr magnetosphere : Building exact solutions from symmetries

# Maxwell solution from Killing vectors

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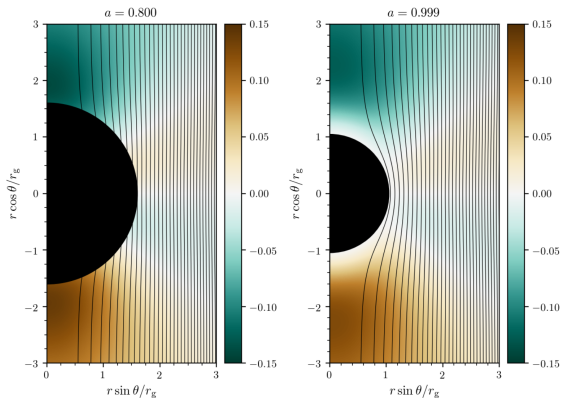
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- Most studied solution so far ! Symmetries of the metric naturally carry the Maxwell solution

# Magnetic field lines for Wald solution



[Credit: B. Crinquand thesis '21]

## Killing-Yano tensors

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Let us apply this to Kerr !

The Kerr monopole magnetosphere

# The Kerr monopole

- The rotating Kerr monopole solving the source-free Maxwell equation for any spin  $a$

$$F = -\frac{8a\mathcal{G}r \cos \theta}{\Sigma^2} dt \wedge dr + \frac{4a\mathcal{G}(r^2 - a^2 \cos^2 \theta) \sin \theta}{\Sigma^2} dt \wedge d\theta \\ - \frac{8a^2\mathcal{G}r \sin^2 \theta \cos \theta}{\Sigma^2} dr \wedge d\varphi - \frac{4\mathcal{G}(r^2 + a^2)(r^2 - a^2 \cos^2 \theta) \sin \theta}{\Sigma^2} d\theta \wedge d\varphi \quad (60)$$

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- What are its properties ?

## Electromagnetic configuration seen from the Carter observer

- Introduce the Carter's observer located at some fix  $r$  and rotating with the BH: Carter tetrad

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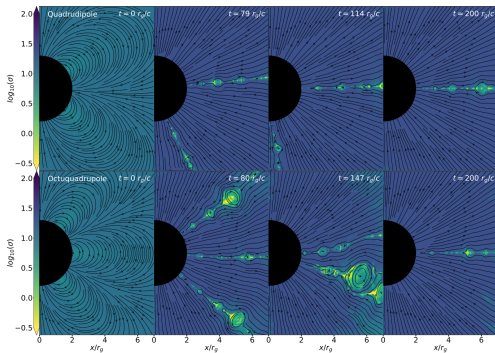
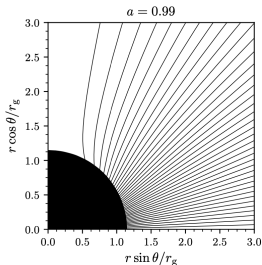
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- Exact Maxwell solution on the Kerr black hole for any spin  $a$ :  
→ unique and selected by the Kerr symmetry !

# Representation of the magnetosphere

- Magnetic field lines of the Kerr magnetic monopole : [Credit: B. Crinquand thesis '21]



- Directly applicable to the split monopole
- Universality of the split monopole configuration at late time: [Credit: Selvi et all '25]
- Universal starting point to build force-free magnetosphere:

$$F = F_{\text{monopole}} + F_{\text{flux}} \quad (71)$$

with  $F_{\text{monopole}}$  a vacuum solution /  $F_{\text{flux}}$  a force-free solution (Michel, Blandford-Znajek)

Exact solution for charged particle motion (and ejection)

# The separable electrogeodesic equations in the Kerr monopole

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## Separable electrogeodesic equations [J. BA, I. El Mellah, E.ourgoulhon '25]

- Introducing the Mino time  $d\tau = d\lambda/\Sigma(r(\lambda), \theta(\lambda))$ , the electrogeodesic separable equations are

$$\frac{dt}{d\tau} = \frac{1}{\Delta(r)} [(r^2 + a^2)^2 E_\kappa - 2MarL_\kappa] - E_\kappa \sin^2 \theta + 2\kappa a \mathcal{G} \cos \theta \quad (72)$$

$$\frac{dr}{d\tau} = \epsilon_r \sqrt{R(r)} \quad (73)$$

$$\frac{d\theta}{d\tau} = \epsilon_\theta \sqrt{\Theta(\theta)} \quad (74)$$

$$\frac{d\varphi}{d\tau} = \frac{L_\kappa}{\sin^2 \theta} + \frac{a}{\Delta(r)} (2MrE_\kappa - aL_\kappa) + \kappa \frac{4\mathcal{G} \cos \theta}{\sin^2 \theta} \quad (75)$$

with the radial and polar potentials given by

$$R(r) = [(r^2 + a^2)E_\kappa - aL_\kappa]^2 - \Delta(r)(r^2 m^2 + \mathcal{K}_\kappa)$$

$$\Theta(\theta) = \mathcal{K}_\kappa - m^2 a^2 \cos^2 \theta - \left( \frac{L_\kappa - 2\kappa \mathcal{G} \cos \theta}{\sin \theta} - aE_\kappa \sin \theta \right)^2$$

- Labelled by the constants  $(E_\kappa, L_\kappa, \mathcal{K}_\kappa, m, \kappa \mathcal{G})$
- Only particles with  $\ell = -\kappa \mathcal{P}$  can cross the axis of rotation  
Can be ejected along the rotation axis

## Recall the open questions

- How charged particles are accelerated in the black hole magnetosphere ?
- Can we characterize the zone of acceleration ?
- Can we deduce the angular velocity of the jet precession ?

Can be address for the split Kerr monopole magnetosphere

Acceleration of charged particles in the Kerr magnetosphere

# Properties of the acceleration of charged particles

- Exact solution for the electrogeodesic 4-velocity for the charged particle everywhere outside Kerr

$$u = \frac{\varepsilon(r^2 + a^2) - a\ell}{\sqrt{\Sigma}\sqrt{\Delta}} e_0 + \epsilon_r \frac{\sqrt{\mathcal{R}(r)}}{\sqrt{\Sigma}\sqrt{\Delta}} e_1 + \epsilon_\theta \frac{\sqrt{\tilde{\Theta}(\theta)}}{\sqrt{\Sigma}} e_2 + \frac{\ell + \kappa\mathcal{P} \cos \theta - a\varepsilon \sin^2 \theta}{\sqrt{\Sigma} \sin \theta} e_3. \quad (76)$$

- The acceleration field for any charge in the black hole magnetosphere is given by

$$a^\mu = \kappa F^\mu{}_\nu u^\nu. \quad (77)$$

- Component by component, it reads

$$a^0 = -\epsilon_r \frac{2a\kappa\mathcal{P}r \cos \theta \sqrt{\mathcal{R}(r)}}{\Sigma^{5/2} \sqrt{\Delta}} \quad (78)$$

$$a^1 = -\frac{2a\kappa\mathcal{P} [(r^2 + a^2)\varepsilon - a\ell] r \cos \theta}{\Sigma^{5/2} \sqrt{\Delta}} \quad (79)$$

$$a^2 = \frac{\kappa\mathcal{P}(\ell + \kappa\mathcal{P} \cos \theta - a\varepsilon \sin^2 \theta)(r^2 - a^2 \cos^2 \theta)}{\Sigma^{5/2} \sin \theta} \quad (80)$$

$$a^3 = -\epsilon_\theta \frac{\kappa\mathcal{P} \sqrt{\tilde{\Theta}(\theta)}(r^2 - a^2 \cos^2 \theta)}{\Sigma^{5/2}}. \quad (81)$$

- Notice that the radial acceleration is proportional to :  $a^1 \propto a\kappa\mathcal{P} \cos \theta$
- Radial acceleration triggered by spin, magnetization and maximal at the poles !
- Similar to the Poynting flux in Blandford-Znajek magnetosphere

Characterizing the zone of acceleration around the black hole

# Geometry for the zone of acceleration

- What a Carter observer would measure when a charged particle crosses its trajectory ?
- Consider the redshift  $z$  by

$$1 + z = \frac{E_{\text{em}}}{E_{\text{obs}}}, \quad (82)$$

→  $E_{\text{em}}$  is the energy of the particle when emitted

→  $E_{\text{obs}}$  the energy of the particle when observed

- The particle is blueshifted when  $E_{\text{obs}} > E_{\text{em}}$  which corresponds to  $z < 0$ .
- The redshift/blueshift factor  $z$  reads

$$1 + z = \frac{[u^\mu(e_0)_\mu]_{\text{em}}}{[u^\mu(e_0)_\mu]_{\infty}}. \quad (83)$$

- For the case  $\ell = -\kappa\mathcal{P} = a\beta$  allowing particle to cross the axis of rotation, one has

$$z = \frac{1}{\sqrt{\Sigma\Delta}} \left[ r_{\text{em}}^2 + a^2 \left( 1 - \frac{\beta}{\varepsilon} \right) - \sqrt{(r_{\text{em}}^2 + a^2 \cos^2 \theta)(r_{\text{em}}^2 - 2Mr_{\text{em}} + a^2)} \right]. \quad (84)$$

[J. BA, I. El Mellah, E. Gourgoulhon '25]

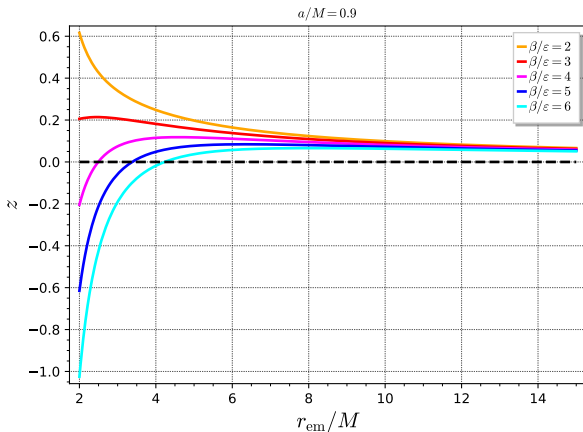
- This happens for sufficiently large values of

$$\frac{\beta}{\varepsilon} = -\frac{\kappa\mathcal{P}}{a\varepsilon} \quad (85)$$

and  $r_{\text{em}}$  sufficiently small,

Let us plot the observed asymptotic redshift/blueshift w.r.t to the position  $r_{\text{em}}$  of emission

# Critical magnetization threshold and asymptotic blueshift



- At a given spin, i.e.  $a/M = 0.9$ , we find a critical threshold allowing  $z < 0$
- The charged particles seen with  $z < 0$ , hence blueshifted asymptotically, are emitted in a region with maximal extent  $r_{em}$  which depends on the spin  $a$  and magnetization  $\beta$
- Provide a first analytic characterization of the zone of ejection of accelerated particles in the Kerr monopole magnetosphere [J. BA, I. El Mellah, E. Gourgoulhon '25]

What about the precession of the ejected particles ?

# Precession of charged particles: A new magnetic frame dragging effect

- The frame dragging is obtained by computing the angular velocity  $\Omega = d\varphi/dt$  seen for an asymptotic observer
- Consider the two key electrogeodesic equations:

$$\frac{dt}{d\tau} = \frac{1}{\Delta} [(r^2 + a^2)^2 \varepsilon + 2\kappa a M \mathcal{P} r] - a^2 \varepsilon \sin^2 \theta + a \kappa \mathcal{P} \cos \theta \quad (86)$$

$$\frac{d\varphi}{d\tau} = \frac{a}{\Delta} (2Mr\varepsilon + a\kappa\mathcal{P}) + \frac{\kappa\mathcal{P}(\cos\theta - 1)}{\sin^2\theta}. \quad (87)$$

- The angular velocity is then given by

$$\Omega(r, \theta) = \frac{d\varphi}{dt} = \frac{(2Mr\varepsilon + a\kappa\mathcal{P}) a \sin^2 \theta + \kappa\mathcal{P}(\cos\theta - 1) \Delta}{[(r^2 + a^2)^2 \varepsilon + 2\kappa a M \mathcal{P} r + a\Delta(\kappa\mathcal{P} \cos\theta - a\varepsilon \sin^2 \theta)] \sin^2 \theta}. \quad (88)$$

- At large distance, the standard frame-dragging effect is recovered for  $\mathcal{P} = 0$ , i.e.

$$\Omega_\infty(r, \theta) \sim \frac{2aM}{r^3}. \quad (89)$$

- When the magnetosphere is turned on, the large distance behavior is dominated by the electromagnetic interaction

$$\Omega_\infty^{\text{MK}}(r, \theta) \sim -\frac{\kappa\mathcal{P}}{\varepsilon r^2} \frac{(1 - \cos\theta)}{\sin^2\theta}. \quad (90)$$

Conclusion

# Conclusions and take away messages

## Results

- Identify the condition on the magnetosphere to integrate the motion of charged particles
- Give one concrete example with the Kerr monopole magnetosphere
- Fully analytical trajectories of charged particles + acceleration
- Reveal that charged particles which are observed blueshifted by asymptotic observers can only come from a bounded region around the horizon whose radius is known analytically
- Reveal a rich structure in ISCO for charged particles / polar equilibrium positions / precession of charged particles / new zero-acceleration layer
- Open the road for several generalizations and discussion with PIC simulations

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## Questions for PIC simulations

- Can we resolve this bounded region and follow its dynamics ?
- Can we see zero-acceleration locus for charged particles ?

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## Generalizations

- Identify the generalisation of the Blandford-Znajek solution non-linear in the spin  $a$
- Identify which force-free magnetosphere allows for analytical integration of electrogeodesics
- Generalization to neutron star model : Hartle-Thorne solution
- Application of the symmetry method to disc models and gas dynamics on Kerr
- Understand the map between charged particles motion and light curve

Thank you

- The acceleration vanishes at the following locus of points

$$r = r_*(\theta) = a |\cos \theta| \sqrt{\frac{a^2 \cos^2 \theta - k}{3a^2 \cos^2 \theta + k}}. \quad (91)$$

Recall that  $k \geq 0$ ; hence we must have a not too large Carter constant:  $0 \leq k \leq a^2 \cos^2 \theta$ .

- At  $r = r_*(\theta)$ , the electrogeodesics reduces to a free-fall motion.
- What is the physical interpretation of this hypersurface ?
  - Charges can evolve as if no Lorenz force was present
  - Is it stable ?
  - Only at a fixed radius  $r_*$  if the particles stay at a stable polar equilibrium position  $\theta_*$
- Need more investigation but reveal a new structure of the magnetosphere !

## Idealized jet and stable polar positions

- Consider the charged particles which can cross the rotation axis so that  $\ell = -\kappa\mathcal{P}$  and assuming  $a \neq 0$ , the latitudinal equation of motion can be rewritten as

$$\left(\frac{d\theta}{d\tau}\right)^2 + V(\theta) = k \quad (92)$$

with the effective potential<sup>1</sup>

$$V(\theta) = a^2 \left[ \cos^2 \theta + \left( \varepsilon \sin \theta - \beta \tan \frac{\theta}{2} \right)^2 \right] \quad (93)$$

where  $\beta$  is the dimensionless quantity defined by

$$\beta = -\frac{\kappa\mathcal{P}}{a} = \frac{\ell}{a}. \quad (94)$$

- General condition on the energy of the particle

$$\varepsilon > a\Omega_C\beta. \quad (95)$$

By definition, an equilibrium latitudinal position for the charged particles satisfies

$$V(\theta) = k, \quad V'(\theta) = 0 \quad (96)$$

and it is stable if  $V'''(\theta) > 0$  and unstable if  $V'''(\theta) < 0$ .

- Consider the North pole  $\theta = 0$ . One has  $d\theta/d\tau = 0$  and  $V'''(0) > 0$ , if and only if, the electrogeodesic  $(\varepsilon, \ell, k)$  satisfies

$$k = a^2 \quad \text{and} \quad \varepsilon < \frac{\beta}{2} - 1 \quad \text{or} \quad \varepsilon > 1 + \frac{\beta}{2}. \quad (97)$$

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<sup>1</sup>Use has been made of the identity  $(1 - \cos\theta)/\sin\theta = \tan(\theta/2)$ .

# Hartle-Thorne solution for neutron star

- The Hartle-Thorne metric reads

$$ds^2 = -f \left[ 1 + 2k_1 P_2(\cos \Theta) - \frac{2J^2}{R^4 f} (2 \cos^2 \Theta - 1) \right] dt^2 + f^{-1} \left[ 1 - 2k_2 P_2(\cos \Theta) - \frac{2J^2}{R^4 f} \right] dR^2 + R^2 [1 - 2k_3 P_2(\cos \Theta)] d\Omega^2 - \frac{4J}{R} \sin^2 \Theta dt d\phi \quad (98)$$

where

$$k_1 = \frac{J^2}{MR^3} \left( 1 + \frac{M}{R} \right) - \frac{5}{8} \frac{Q - J^2/M}{M^3} Q^2 {}_2 \left( \frac{R}{M} - 1 \right) \quad (99)$$

$$k_2 = k_1 - \frac{6J^2}{r^4} \quad (100)$$

$$k_3 = k_1 + \frac{J^2}{R^4} - \frac{5}{4} \frac{Q - J^2/M}{M^2 R \sqrt{f}} Q^1 {}_2 \left( \frac{R}{M} - 1 \right) \quad (101)$$

- Reduce to the Kerr geometry up to second order in the spin in Boyer-Lindquist coordinates  $(t, r, \theta, \phi)$  when

$$a = -\frac{J}{M} \quad Q = \frac{J^2}{M} \quad (102)$$

and performing the change of coordinates

$$r = R + \frac{a^2}{2R} \left[ \left( 1 + \frac{2M}{R} \right) \left( 1 - \frac{M}{R} \right) - \cos^2 \Theta \left( 1 - \frac{2M}{R} \right) \left( 1 + \frac{3M}{R} \right) \right] \quad (103)$$

$$\theta = \Theta + \frac{a^2}{2R^2} \left( 1 + \frac{2M}{R} \right) \sin \Theta \cos \Theta \quad (104)$$