# The quantum vacuum:
# from the laboratory to the cosmos

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The quantum vacuum is intimately linked to Heisenberg's uncertainty principle and ultimately to the fundamental relation between conjugate variables:

$$[A, \pi_A] = i\hbar$$

$$\Pi_A = \frac{\partial \mathcal{L}}{\partial \dot{A}}$$

This implies the uncertainty relation:

$$\Delta A \Delta \Pi_A \geq \frac{\hbar}{2}$$

This is true for any quantum state in which the mean root squared are calculated:

$$(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle, \quad \langle A\rangle = \langle\psi|A|\psi\rangle$$

In non-relativistic quantum mechanics, this is in particular the famous:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

And for non-relativistic wave packets, the energy is also constrained:

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

This is obviously at the origin of the quantum fluctuations in vacuum, so let us see step by step how this comes about
For the harmonic oscillator, the building block of particle quantisation, the vacuum is particular and is such that the inequality is saturated:

$$\Delta x \Delta p = \frac{\hbar}{2} \qquad \Delta t \Delta E = \frac{\hbar}{2}$$

*As we are amongst friends, from now on I will use natural units… Hoping you will not blame me ☺*

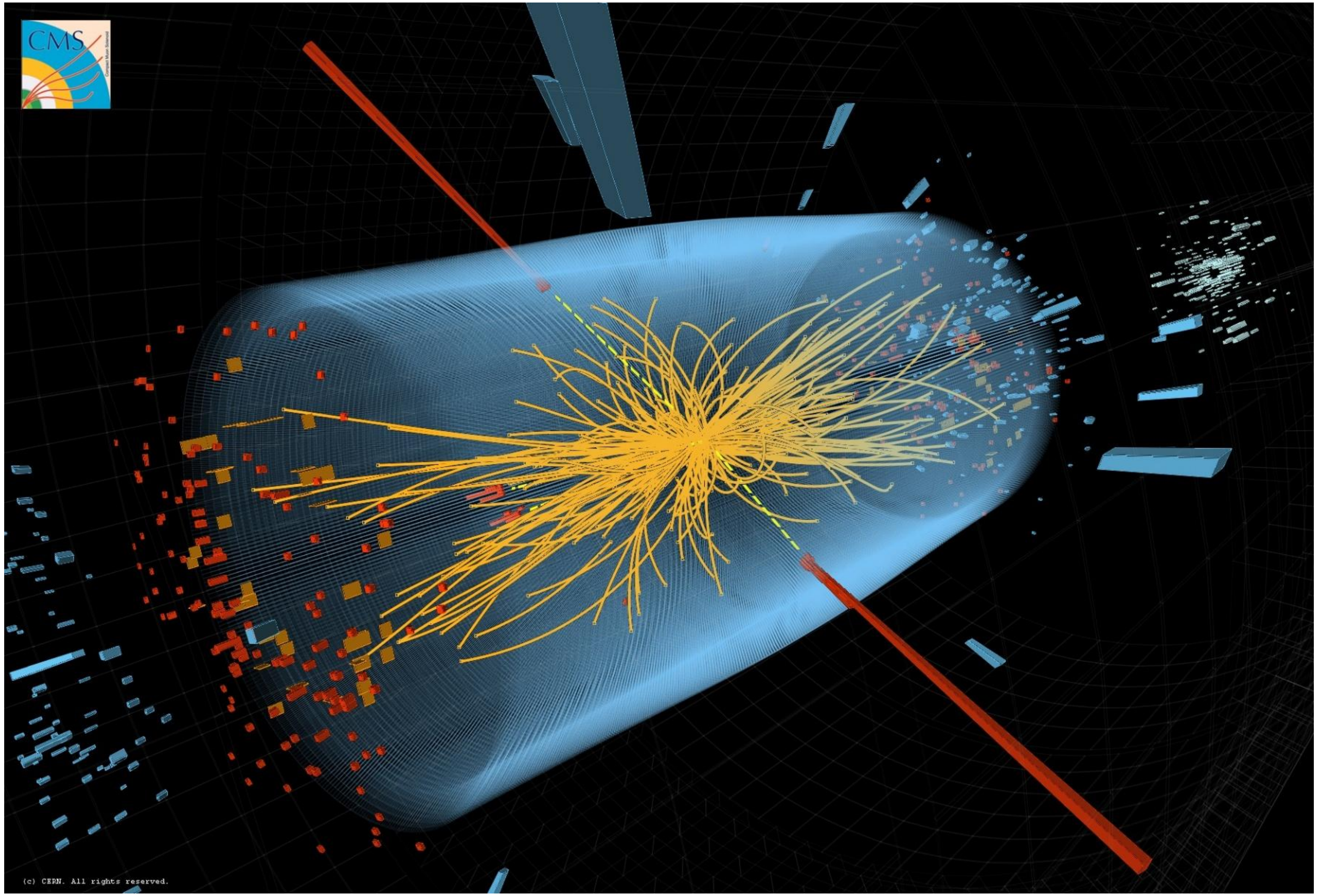
Probing the vacuum on short time scales will potentially release enough energy to create particles:

$$\Delta t \leq \frac{1}{2m} \Rightarrow \Delta E \geq m$$

This is compatible with the dispersion relation for a particle in special relativity:

$$E = \sqrt{\vec{p}^2 + m^2} \geq m$$

$$\Delta t \sim \frac{1}{E}$$




The vacuum reacting violently to a very energetic probe

“ Materia non proprie dici quod est, cum non sit nisi in potentia”

Aquinas “*De spiritualibus Creaturis*”

$$|0\rangle \to a^{\dagger}|0\rangle \to \ldots (a\dagger)^n|0\rangle \ldots$$

*The vacuum as the root of Fock space*

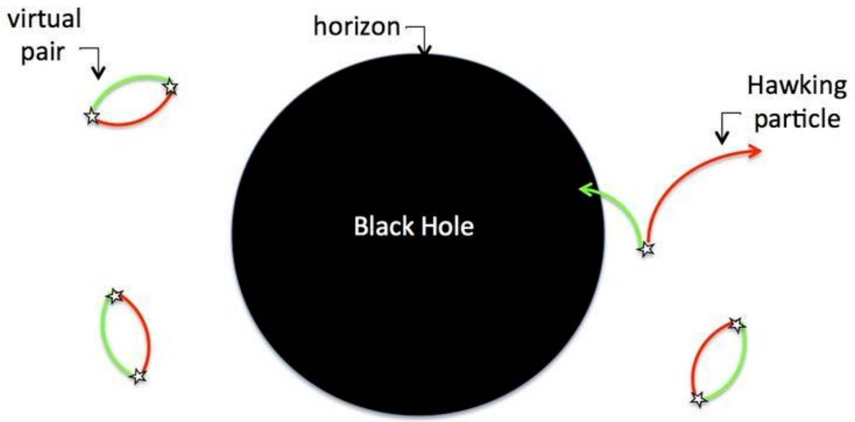
The quantum vacuum “knows” about the existence of particles which exist “in potentiality”.

The virtual particles are not on-shell and their energy and momenta do not satisfy the dispersion relation from special relativity….

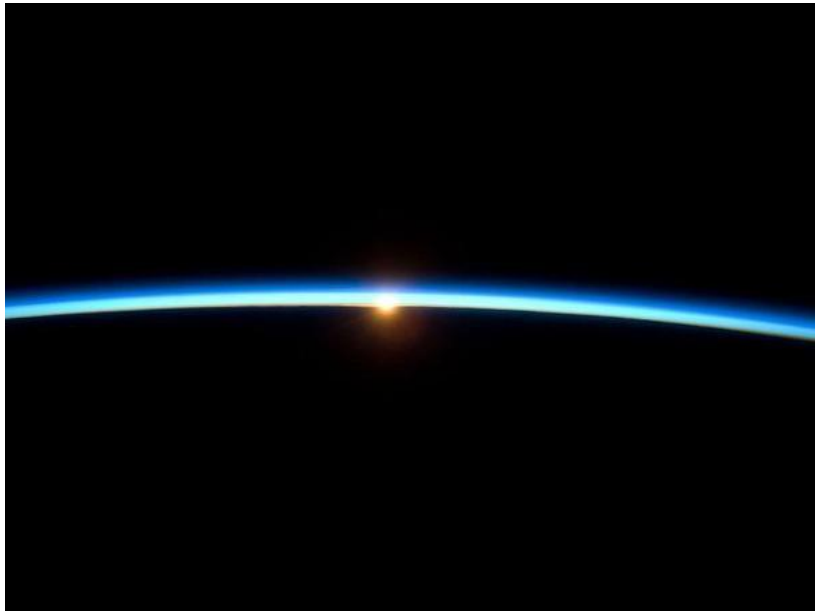
BUT this reasoning  applies when special phenomena render the virtual particles real!

See *Patrick Peter's* talk

# Horizons

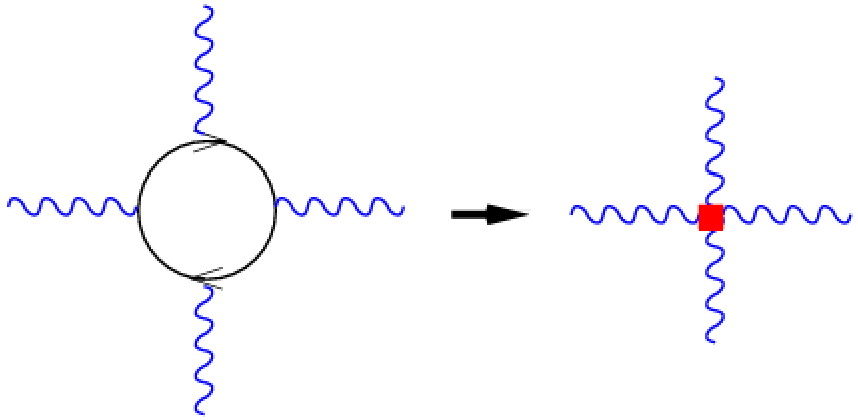



Hawking radiation at even horizon



Radiation from cosmological horizon

Virtual particles also modify particle interactions. For instance, the vacuum of QED acquires a non-trivial permittivity



Virtual particles lead to effective interactions

$$\mathcal{L}_{EH} = \frac{2\alpha^2}{45m_e^4}((\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}.\vec{B})^2) = \frac{\alpha^2}{720m_e^4}(8(F_{\mu\nu}F^{\mu\nu})^2 + 7(F_{\mu\nu}\tilde{F}^{\mu\nu})^2)$$

Birefringence of the vacuum…

See Xavier Sarazin's talk

# Effective field theories:

$$\mathcal{L}_{\text{eff}}(\phi_1) \qquad\qquad \mathcal{L}(\phi_1, \phi_2)$$

Mass m

Below the mass threshold when particles are “integrated out”, i.e. when they cannot be produced, the theory for the remaining light particles is modified:

$$\mathcal{L}_{\text{eff}} = \sum_{n=0}^{4} c_n(\mu)\mathcal{O}_n + \sum_{n>0} \frac{\mathcal{O}_{n+4}}{m^n}$$

*Renormalisable*
*Relevant*

Scale dependence

*Non renormalizable*
*Irrelevant*

$$\mathcal{O}_0 = \mathcal{I}$$

The rest of this talk will try to make sense of the coefficient multiplying the identity operator.

# The vacuum energy

The vacuum energy appears innocuously in the choice of normal ordering (or not) the Hamiltonian:

$$H = \frac{\omega}{2}(aa^\dagger + a^\dagger a)$$

Now we have: $[x,p] = i \Rightarrow [a, a^\dagger] = 1$

$$H = (a^\dagger a + \frac{1}{2})\omega$$

Normal ordered Hamiltonian:

$$\langle 0|H|0\rangle = \frac{\omega}{2}$$

**Cluster decomposition, causality, Wick's theorem**

In quantum field theory, the first calculation of the vacuum energy was performed by **Lenz** and then by **Jordan and Pauli:**

$$\rho_{\rm vac} = \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m_e^2}$$

Immediately arises the question: what is the upper bound of this integral? Jordan and Pauli used the electron mass… Not a bad choice when decoupling the electron!

$$\rho_{\rm vac} \sim \frac{m_e^4}{32\pi^2}$$

They applied it to Einstein's static Universe (it's 1928 and Lemaitre's paper is in 1927…)

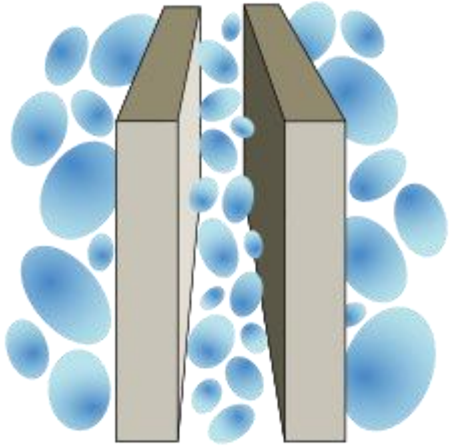
$$R = \frac{1}{\sqrt{8\pi G_N \rho_{\rm vac}}}$$

About 36 km…. makes no sense

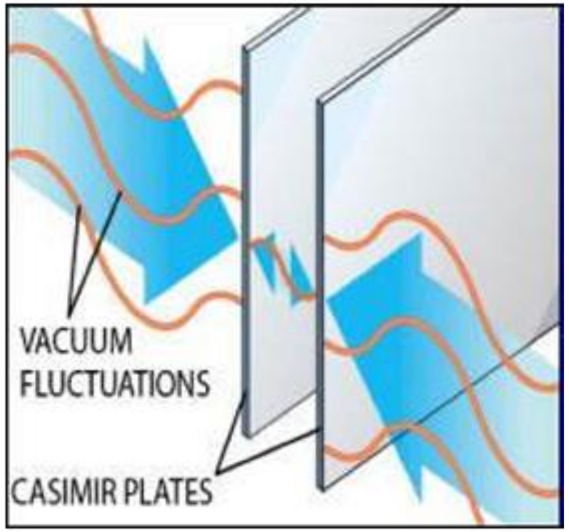
Contrary to the eigen-oscillations in a crystal lattice (where theoretical as well as empirical reasons speak to the presence of a zero-point energy), for the eigen-oscillations of the radiation no physical reality is associated to this "zero-point energy" of $\frac{1}{2}h\nu$ per degree of freedom. We are here doing with strictly harmonic oscillators, and since this "zero-point energy" can neither be absorbed nor reflected – and that includes its energy or mass – it seems to escape any possibility for detection. For this reason it is probably simpler and more satisfying to assume that for electromagnetic fields this zero-point radiation does not exist at all.[98]

Jordan and Pauli (1928)

# Subtle is the Lord…



This opinion was soon contradicted by the discovery of the Casimir effect, first theoretically in 1948 and then gradually confirmed experimentally (eventually Decca in 1997).




The original calculation involves ideal plates in which the electric field does not penetrate.

$$\omega_n(\vec{k}_\parallel) = \sqrt{\frac{n^2\pi^2}{d^2} + \vec{k}_\parallel^2}$$

$$E = \frac{2A}{2}\sum_n \frac{1}{(2\pi)^2}\int d^2k_\parallel \omega_n(\vec{k}_\parallel)$$

In the ideal case with no dissipation, the force on the plate is:

$$F = -\frac{\partial E}{\partial d} = \frac{A\pi^2}{2d^4}\sum_n n^3$$

$$\int d^2k_{\|}\omega_n(\vec{k}_{\|}) = 2\pi\int_{\frac{n\pi}{d}}^{\infty} d\omega\omega^2$$

At this point one could invoke Riemann's zeta function but this hides the physical origin of the result

- One must remove the pressure from the "other" side of the plate.
- Physically the material becomes transparent at high frequency

**Euler-Maclaurin:**

$$F = -\frac{\pi^2 A}{2d^4}\frac{g^{(3)}(0)}{6!}$$

$$F = \frac{\pi^2 A}{2d^4}\Big(\sum n^3\chi(n) - \int_0^\infty dx x^3\chi(x)\Big), \quad \lim_{x\to\infty}\chi(x) = 0$$

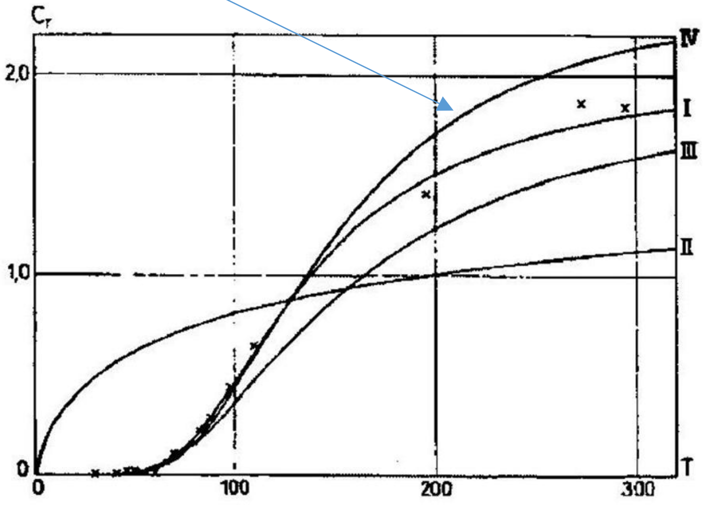
$g(n)$

$$F = -\frac{\pi^2 A}{240 d^4}$$

# INTERLUDE

The zero-point energy of the harmonic oscillator and the Euler-Maclaurin formula are embroiled in an astonishing piece of bluff, guess-work and eventually successful extravaganza of early quantum days

*The fit with the vacuum energy is better!*




$$Z = \sum_n e^{-J\frac{(2n+1)^2}{2T}}$$

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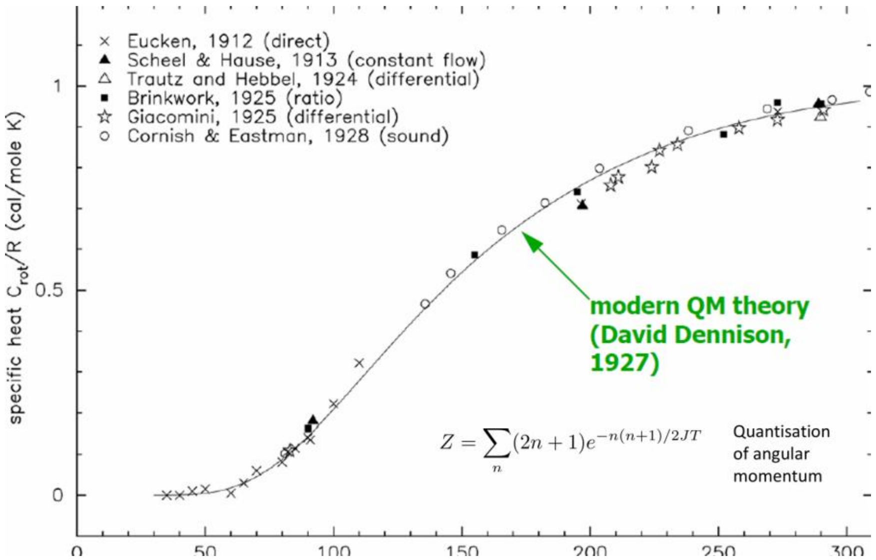


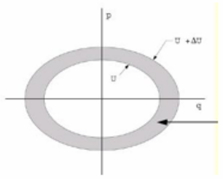

Around 1913, the heat capacity of the di-hydrogen molecule was measured.

More guess work!

$$E = n\omega + \frac{\omega}{2}$$

$$\frac{1}{2}kT = \frac{\hbar\omega}{2}$$

Planck: vacuum must receive an energy, thermal analogy



$$\# = (n+1)^2 - n^2 = 2n+1$$ Planck

Einstein and Stern (1913) quantised the angular motion:

$$\frac{1}{2}J\Omega^2 = (n+\frac{1}{2})\Omega \Rightarrow \Omega = \frac{(2n+1)}{J}$$

$$Z = \sum_n (2n+1)e^{-\frac{n(n+1)}{2JT}}$$

$$(n+\frac{1}{2})^2 - \frac{1}{4} = n(n+1)$$

END of the INTERLUDE

Understanding Casimir requires a hint of QFT:

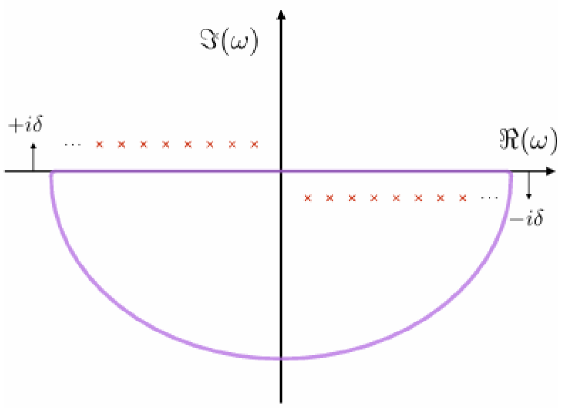
Regularised by point-splitting

$$E = \frac{i}{(2\pi)^3} \int dz d\omega d^2 p_{\|} \omega^2 (G_F(z;z,\omega,p_{\|}) - G_F^{(0)}(z;z,\omega,p_{\|}))$$

Feynman propagator for the scalars representing one of the two photons polarisations:

$$a^i = e^i_\alpha a^\alpha, \quad \alpha = 1,2$$

Polarisation vector



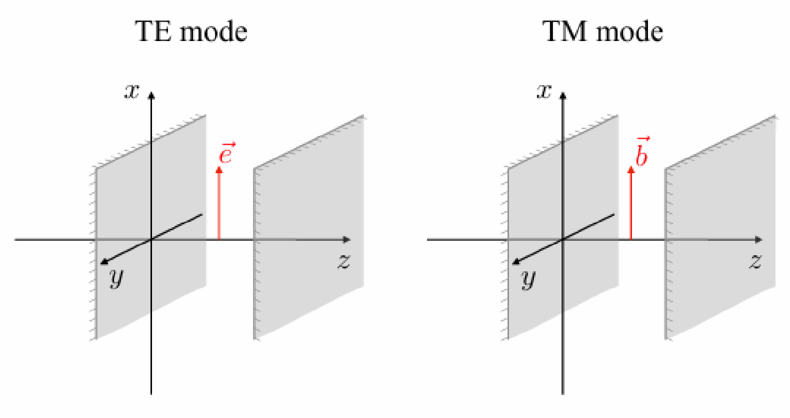

$$\frac{E}{A} = \int d^2 p_{\|} \omega(\vec{p}_{\|})$$

$$\frac{E}{A} = i \frac{d}{(2\pi)^3} \int d\omega d^2 p_{\|} \frac{\omega^2 \cos \Delta d}{\Delta \sin \Delta d}$$

$$\Delta = (\omega^2 - p_{\|}^2)^{1/2}$$

Poles

Let us come back to the question: Casimir and transparency at high frequency

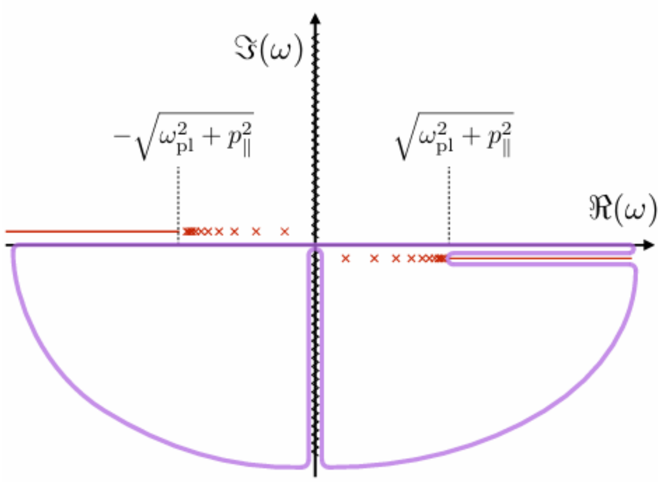



The two modes in the cavity

$$(\epsilon(\omega)w^2 - \vec{p}_{||} + \partial_z^2)a_\alpha = 0$$

**Drude model:** $\epsilon(\omega) = 1 - \frac{\omega_{\text{pl}}^2}{\omega(\omega + i\gamma)}$ $\omega_{\text{pl}}^2 = \frac{4\pi\alpha n}{m_e}$

Dissipation



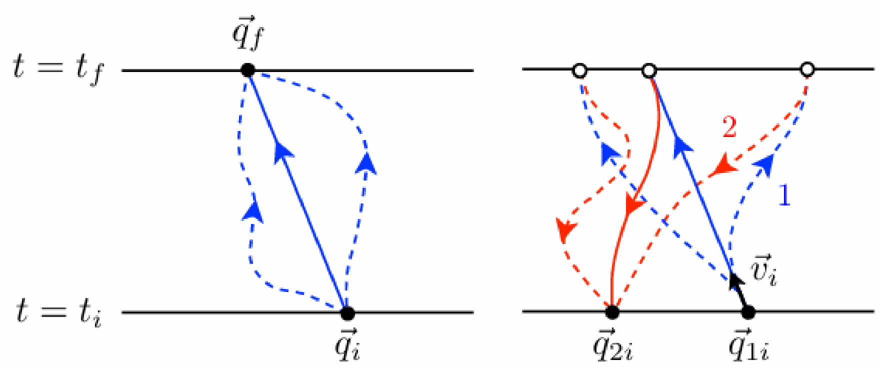

The branch cut below the real axis corresponds to transparency.

The Feynman propagator has a finite number of poles just below the real axis. The shift is given by the dissipation.

Dissipation makes QFT tricky. Indeed**, no lagrangian formulation** and the in and out **vacua are not unitarily related**. This can be cured in two steps:

A lagrangian formulation can be obtained by doubling the number of degrees of freedom:

$$a_\alpha \text{ and } \tilde{a}_\alpha$$




The new fields **bring energy back from infinity**

$$\tilde{a}_\alpha(t) \equiv a_\alpha(-t) \Rightarrow \mathcal{T}_{\mu\nu} = T_{\mu\nu}(t) + T_{\mu\nu}(-t)$$

Energy momentum of electrodynamics in matter (Minkowski) up to a gauge transformation

$$T_{zz} = -e_z d_z - b_z^2 + \frac{1}{2}(\vec{e}.\vec{d} + \vec{b}^2)$$

Quantum mechanically the evolution operator is not unitary. The “in” and “out” vacua are not unitarily related implying that Gellman-Low’s theorem fails.  Correlation functions cannot be calculated in the “in-out” fashion.

$$|\mathrm{out}\rangle = \mathrm{U_I}(+\infty, -\infty)|0\rangle \neq \mathrm{e}^{\mathrm{i}\alpha}|0\rangle$$

**S- matrix not unitary**

When dissipation is present, must distinguish pressure from energy:

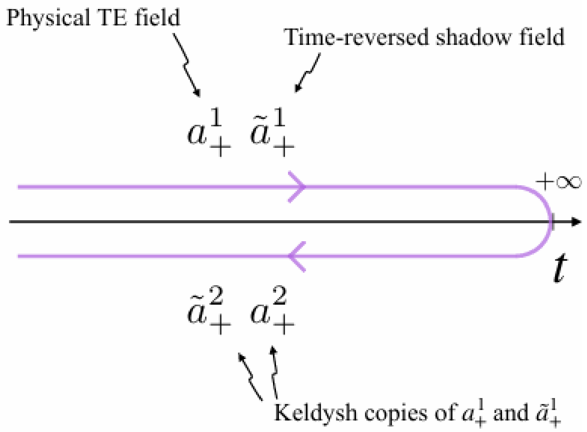
$$dE = -PdV + \delta Q$$

The pressure is given :

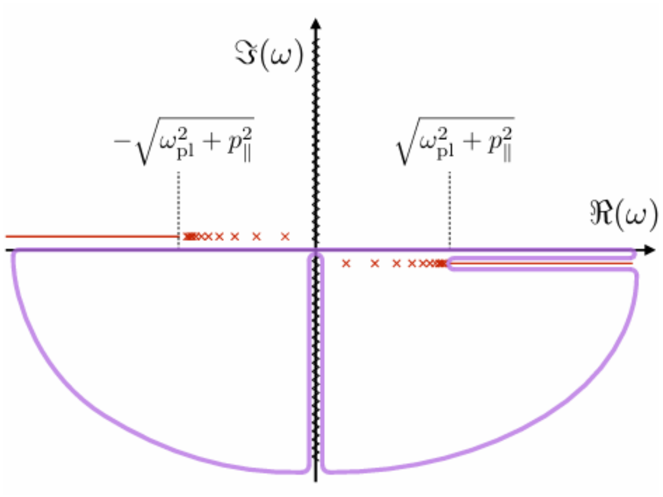
$$P = \langle 0|T_{zz}(+\infty)|0\rangle \longrightarrow P = \langle 0|U_I(-\infty,+\infty)T^I_{zz}(+\infty)U_I(+\infty,-\infty)|0\rangle$$

QED vacuum

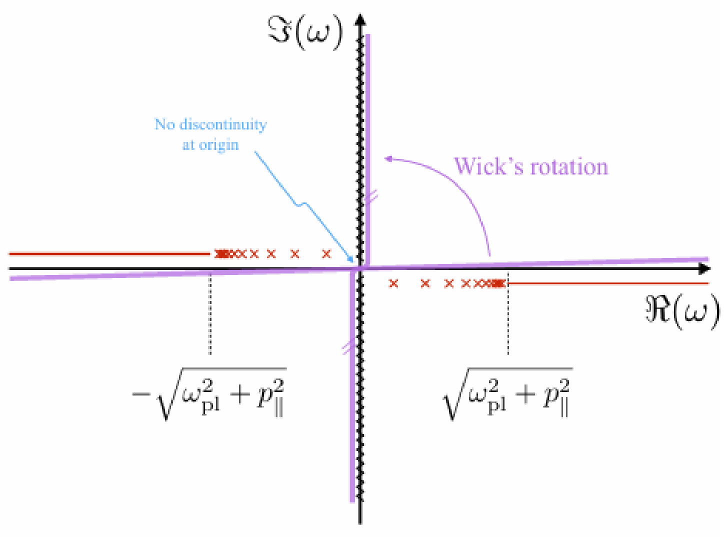
Expectation values are calculated in the **Schwinger-Keldysh** formalism



$$P = \frac{i}{2(2\pi)^3}\int d\omega d^2 p_{\|}(\partial^2 z - \vec{p}_{\|}^2 + \omega^2)(G_E + G_M)(z; z, \omega, p_{\|})$$




Pressure= poles + discontinuities




The Feynman propagator allows for a rotation of the contour in the complex plane: integral along the imaginary axis.

Doing all this properly, one ends up with **Lischitz's theory (1959)**

$$P_{E,M} = -\int \frac{d\omega d^2 p_{\|}}{(2\pi)^3} \frac{\Delta}{r_{E,M}^{-2} e^{2\Delta d} - 1}$$

$$\Delta = (\omega^2 + \vec{p}_{\|}^2)^{1/2}$$

This depends on the reflection coefficients r in the TE and TM cases.

In the ideal case r=1 and we retrieve:

$$P_{\text{ideal}} = -2\int \frac{d\omega d^2 p_{\|}}{(2\pi)^3} \frac{\Delta}{e^{2\Delta d} - 1} = -\frac{8\pi}{16(2\pi)^3 d^4}\Gamma(4)\zeta(4) = -\frac{\pi^2}{240 d^4}$$

This is all fine but what is it really that we have calculated?

# Vacuum fluctuations or matter-matter interactions?

Let us come back to the Drude model with no dissipation, the Maxwell equation in matter becomes

$$\epsilon(\omega) = 1 - \frac{\omega_{\rm pl}^2}{\omega^2} \qquad (\omega^2 - \vec{p}^2 - \omega_{\rm pl}^2)a_\alpha = 0$$

The photons become massive in matter. Hence Dyson's equation for the propagators:

$$\frac{1}{p^2+\omega_{\rm pl}^2} - \frac{1}{p^2} = -\frac{1}{p^2}\omega_{pl}^2\frac{1}{p^2} + \frac{1}{p^2}\omega_{\rm pl}^2\frac{1}{p^2}\omega_{\rm pl}^2\frac{1}{p^2} + \dots$$

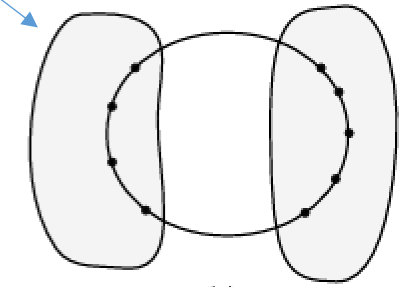
Recall that

$$\omega_{\rm pl}^2 = \frac{4\pi\alpha n}{m_e}$$

Mass insertion corresponding to an interaction between photons and the matter density of the material

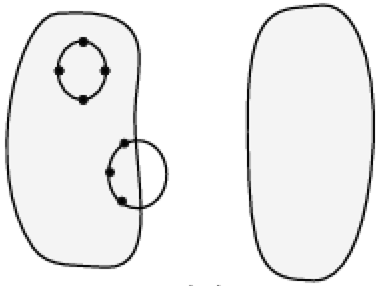
Free propagation

One of the infinite sum of one-loop diagram with matter interactions leading to Lifschitz



(i)

Diagrams giving no force



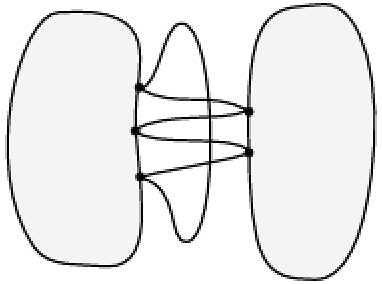
(ii)

When the density or the fine structure constant goes to infinity, the effective mass is infinite and the penetration length vanishes. This is equivalent to the Meissner effect for superconductors (London equation) :

$$\Delta\vec{B} = \omega_{\mathrm{pl}}^2 \vec{B}$$

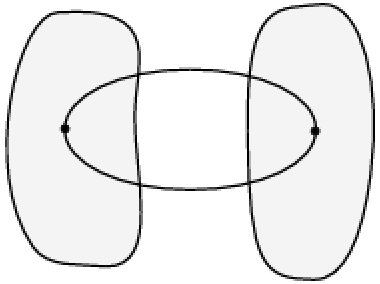
The penetration (London) length is:

$$\lambda_L = \frac{1}{\omega_{\mathrm{pl}}} \to 0$$



(iii)

The ideal case



(iv)

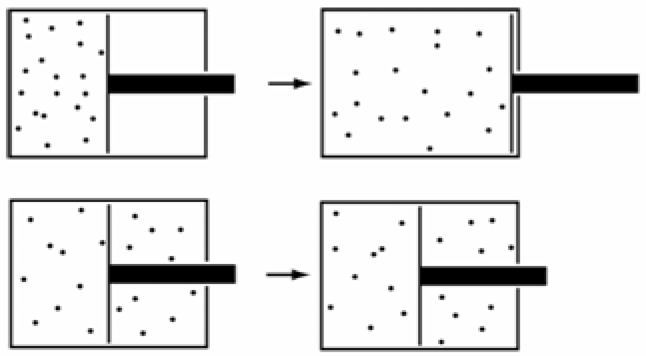
Casimir-Polder interaction. If not integrated over the bulk of the two objects, this is the Van der Waals interaction.

It is high time to come back to the real Mc Coy: the vacuum energy in empty space. And before doing this, we will revisit Jordan-Pauli's calculation. In absence of matter, Lorentz invariance is respected implying that the energy-momentum tensor of the vacuum fluctuations must be:

$$T_{\mu\nu} = -V g_{\mu\nu}$$

Locally gravity can be effaced and

$$T_{ab} = -V \eta_{ab}$$



This is the energy-momentum of a fluid with

$$p_{\text{vacuum}} = -\rho_{\text{vacuum}}$$

And this is pretty much all we know in general.

# BEWARE OF CUTOFFS !

$$\rho_{\text{vacuum}} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p \sim \frac{\Lambda^4}{16\pi^2}$$

$$p_{\text{vacuum}} = \frac{1}{6}\int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}^2}{\omega_p} \sim \frac{\Lambda^4}{48\pi^2}$$

$$p_{\text{vacuum}} = \frac{1}{3}\rho_{\text{vacuum}}$$

# COMPLETELY WRONG!

*Why? Because divergent integrals are nasty beasts and one should not break Lorentz invariance.* Need to use Lorentz-invariance preserving methods like dimensional regularisation.

Ref.TH.2857-CERN

# THE USE OF DIMENSIONAL RENORMALIZATION SCHEMES IN UNIFIED THEORIES

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## A B S T R A C T

We present details and clarify some points in a recently proposed modification of the minimal subtraction adapted to unified theories. As an application we give a precise value of the superheavy gauge boson masses in SU(5).

*) Partially supported by the Swiss National Science Foundation.

Ref.TH.2857-CERN
5 May 1980

$$\rho_{\text{vacuum}} = \int \frac{dEd^3p}{(2\pi)^4} \frac{\vec{p}^2 + m^2}{\vec{p}^2 + E^2 + m^2} = \int \frac{d^4p_E}{(2\pi)^4} \frac{\frac{3}{4}p_E^2 + m^2}{p_E^2 + m^2}$$

I walked in a bar and immediately I sensed danger.
The Specials "*Blanck Expression*"

Cauchy and 4d Euclidean integral.

I could use dimensional regularisation. I will use extensively:

$$\int \frac{d^4p_E}{p_E^n} = 0, \ \ n \neq 4$$

See 7.30 in **Zinn-Justin**

With this piece of wizardry, I can calculate:

$$\rho_{\text{vacuum}} = \int \frac{d^4p_E}{(2\pi)^4} \frac{\frac{3}{4}p_E^2 + m^2}{p_E^2 + m^2} = \frac{3}{4}\int \frac{d^4p_E}{(2\pi)^4} + \frac{m^2}{4}\int \frac{d^4p_E}{(2\pi)^4} \frac{1}{p_E^2 + m^2}$$

Leading to:

$$\rho_{\text{vacuum}} = \frac{m^2}{4}\int \frac{d^4p_E}{(2\pi)^4} \frac{1}{p_E^2 + m^2}$$

Let us crack on:

$$\rho_{\text{vacuum}} = \frac{m^2}{4}\int \frac{d^4p_E}{(2\pi)^4}\frac{1}{p_E^2+m^2} = \frac{m^2\pi^2 m^2}{4(2\pi)^4}\int dp_E \frac{p_E^3}{p_E^2+m^2} = \frac{m^2}{32\pi^2}(\int dp_E p_E - \int dp_E \frac{m^2 p_E}{p_E^2+m^2})$$

And again:

$$\rho_{\text{vacuum}} = -\frac{m^4}{32\pi^2}\int dp_E \frac{p_E}{p_E^2+m^2}$$

The same method shows that the equation of state is -1

This is non-vanishing… and can be calculated by using a cut-off in Euclidean 4-space preserving SO(4) invariance which is the Wick rotation of Lorentz invariance SO(3,1):

$$\rho_{\text{vacuum}} = -\frac{m^4}{64\pi^2}\ln(1+\frac{\mu^2}{m^2})$$

$$\mu \gg m,\ \rho_{\text{vacuum}} \sim -\frac{m^4}{64\pi^2}\ln\frac{\mu^2}{m^2}$$

Running under changes of scale (renormalisation group)

$$\mu \ll m,\ \rho_{\text{vacuum}} \sim 0$$

This is the result in the decoupling minimal subtraction scheme:

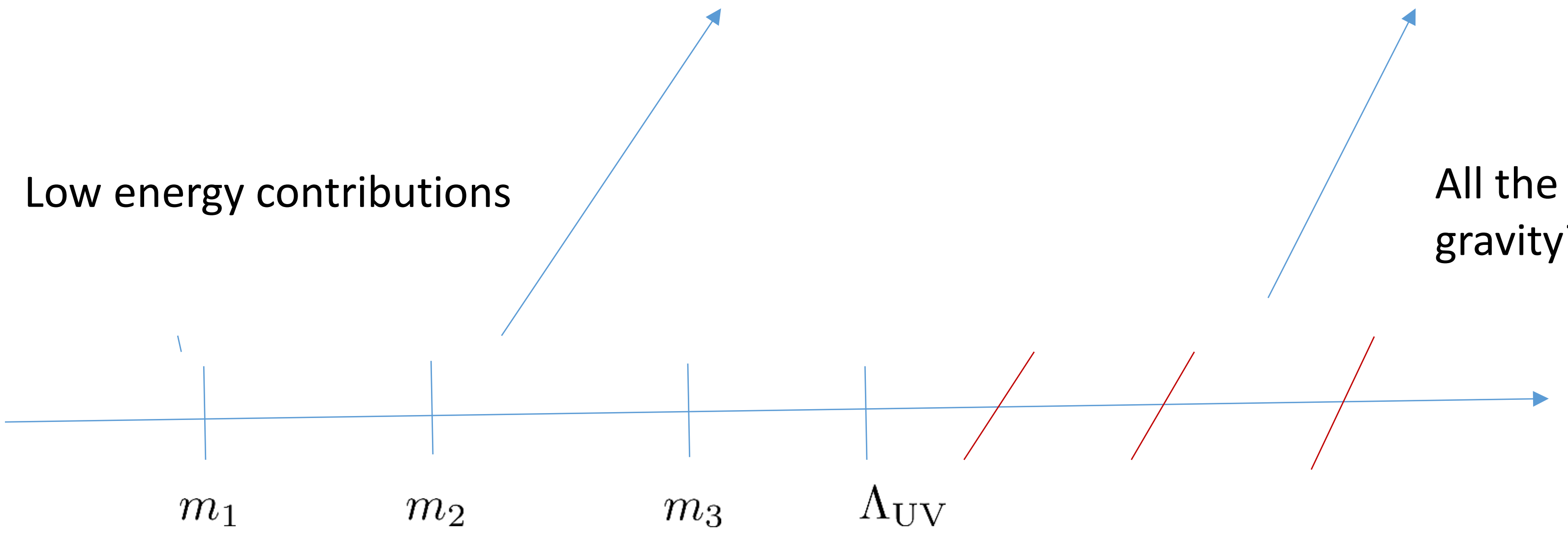
**Particles contribute to quantum corrections above their mass threshold**

We can express the vacuum energy in terms of the masses of the particles:

$$\rho_{\text{vacuum}} = \frac{1}{64\pi^2}\text{Str}(M^4 \ln(1 + \frac{\Lambda_{\text{UV}}^2}{M^2})) + \rho_{\text{vacuum}}(\Lambda_{\text{UV}})$$

Low energy contributions

All the UV physics. Quantum gravity?? String theory??

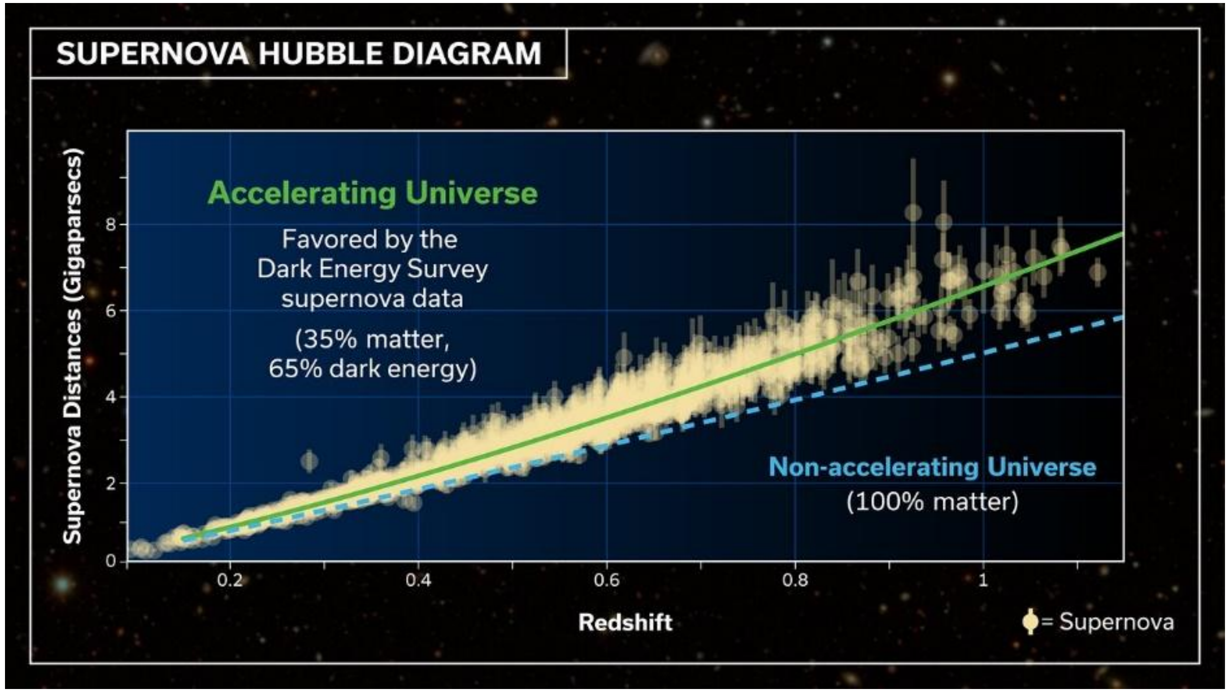


The sole contribution from the electron far exceed the cosmologically measured value!

$$\rho_{\text{vacuum}}(\text{electron}) \sim m_e^4 \sim 10^6 \text{g}/\text{cm}^3$$

$$\rho_c \sim 10^{-29} \text{g}/\text{cm}^3$$

Is trying to calculate the cosmological vacuum energy in vain?




Oh mama can this really be the end? To be stuck inside of Mobile with the Memphis blues again.

Bob Dylan "*Blonde on Blonde*"

Against anthropic arguments, S. Weinberg (1988) showed that quantum corrections could actually be used to understand the nature of the cosmological vacuum, with a twist.

To live outside the law, you must be honest

Bob Dylan, "*Absolutely sweet Marie*"

Consider a low energy theory with **NO scale:**

$$S = \int d^4x \sqrt{-g}\left(\frac{h(\phi^i)}{2}R - \frac{1}{2}(\partial\phi^i)^2 - V(\phi^i)\right)$$

Scaling dimension 2

Scaling dimension 4

Scale invariance will be broken by the vacuum leaving one Goldstone mode: the **dilaton.** Let us see this:

$$\phi^i = \rho y^i, \quad g^J_{\mu\nu} = \rho^{-2} g_{\mu\nu}$$

Vacua are obtained by redefining:

$$y^i = z^i y^0,\ V(y^0, y^0 z^i) = (y^0)^4 \tilde{V}(z^i) \ \Rightarrow \partial_{z^i}\tilde{V}(z^i) = 0,\ V(z^i) = 0$$

Vacua in this model are obtained by solving N equations for (N-1) fields: **TUNING !**
This requires a tuning between the coupling constants of the potential… Even if this is arranged at a given scale, this would be upset at a different scale because of the running of the couplings….

V=0

Only if underlying **symmetry** (like supersymmetry)….

This is **Weinberg's theorem**.

Once the vev's of the scalars are fixed, the effective theory for the dilaton is simply:

$$\begin{cases} h(y^0, y^0\langle z^i\rangle) = m_{\rm Pl}^2 \\ \rho = e^{-\gamma \frac{\phi}{m_{\rm Pl}}} \\ \gamma = (6 + \frac{(y^0)^2}{m_{\rm Pl}^2}(1 + z^i z^i))^{1/2} \end{cases}$$

$$S = \int d^4x \sqrt{-g_J}\left(\frac{m_{\rm Pl}^2}{2} e^{2\gamma \frac{\phi}{m_{\rm Pl}}} R_J - \frac{1}{2}(\partial\phi)^2\right)$$

Massless!

Let us now introduce fermions coupled to the scalars (Higgs mechanism):

$$S = -\int d^4x\sqrt{-g}(i\bar{\phi}^a\gamma^\mu D_\mu\psi^a - \lambda_{iab}\phi^i\bar{\psi}^a\psi^b)$$

Yukawa coupling

The fermions are massive when the scalars condense.

The next step is to normalise the Einstein-Hilbert term:

Still massless

$$g^J_{\mu\nu} = e^{-2\beta\frac{\varphi}{m_{\rm Pl}}}g^E_{\mu\nu}$$

$$\varphi = (1+6\gamma^2)^{1/2}\phi$$

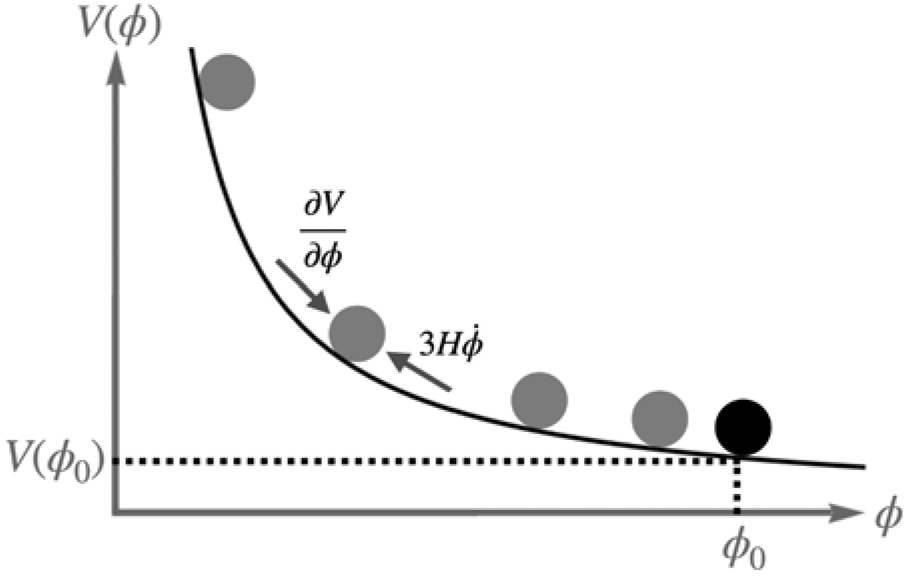
$$\beta = \frac{\gamma}{(1+6\gamma^2)^{1/2}}$$

$$S = \int\sqrt{-g_E}(\frac{m^2_{\rm Pl}}{2}R_E - (\partial\varphi)^2)$$

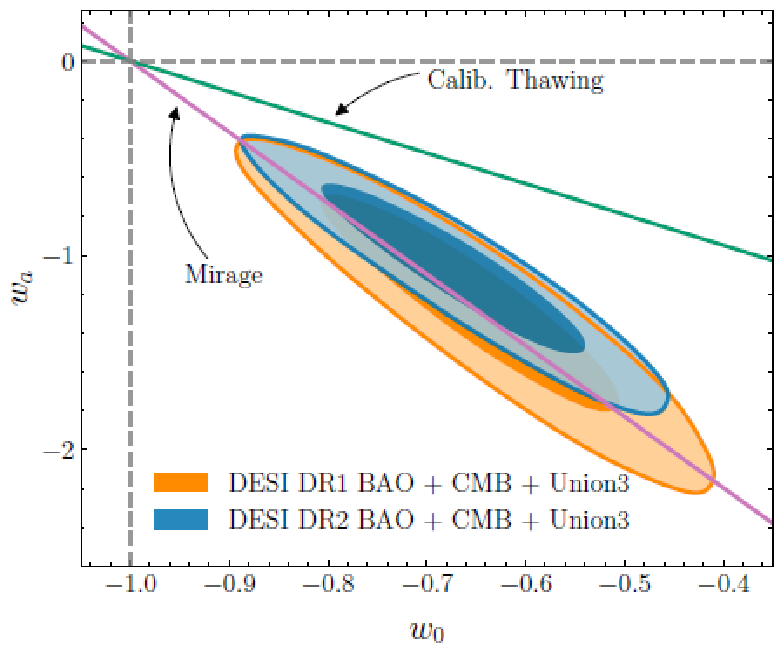
$$m_{ab} = e^{-\beta\frac{\varphi}{m_{\rm Pl}}}y^0(\lambda_{0ab} + \lambda_{iab}z^i)$$

**The quantum corrections give a potential to the dilaton**

$$V(\varphi) = V_0 e^{-4\beta\frac{\varphi}{m_{\rm Pl}}}$$




The dilaton could be at the origin of dark energy



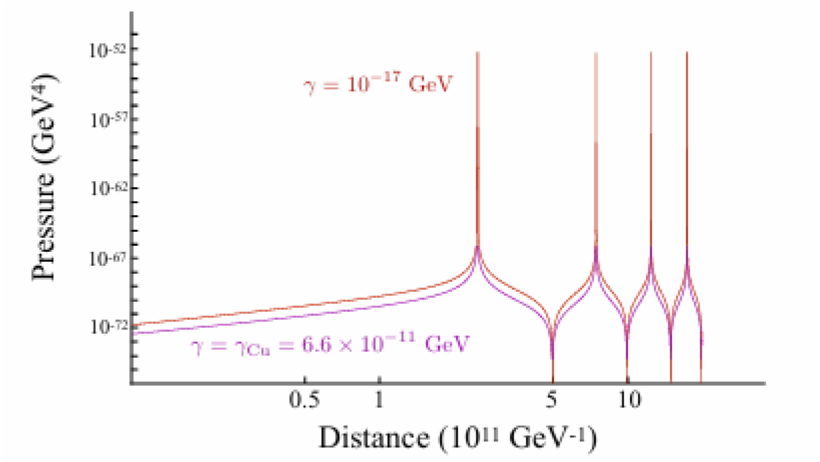

The dilaton is coupled to matter and this could be at the origin of the phantom crossing.

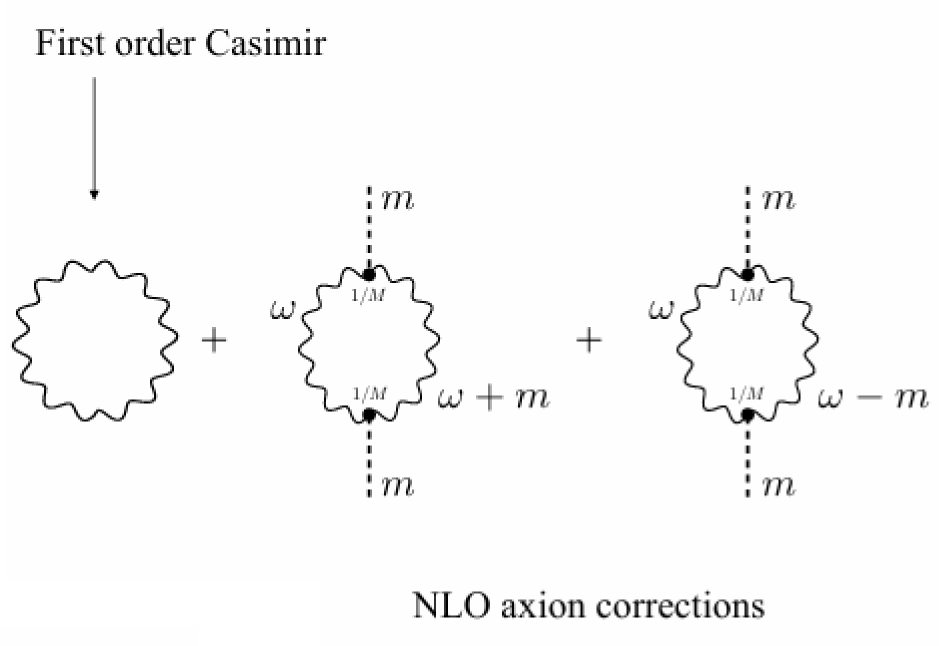
# Prospects

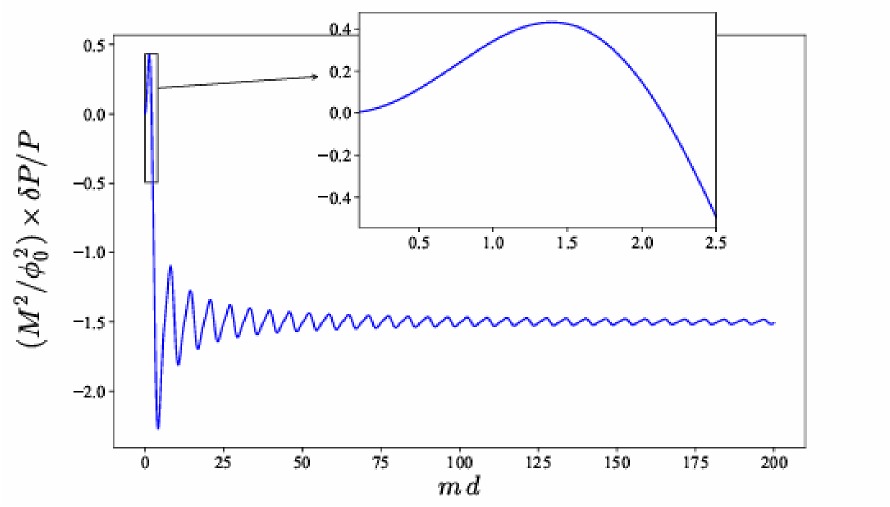
I have not mentioned that dark matter fills the Universe. How to use quantum effects to detect it?

One example: *axions and Casimir*

If dark energy is dynamical, would it lead to detectable quantum effects from its coupling to matter?


Pressure (GeV⁴)
10-52
10-57
10-62
10-67
10-72
$\gamma = 10^{-17}$ GeV
$\gamma = \gamma_{Cu} = 6.6 \times 10^{-11}$ GeV
0.5
1
5
10
Distance (10¹¹ GeV⁻¹)


First order Casimir
$m$
$\omega$
$1/M$
$\omega + m$
$\omega - m$
NLO axion corrections


$(M^2/\phi_0^2) \times \delta P/P$
$m\,d$
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
25
50
75
100
125
150
175
200
0.4
0.2
0.0
−0.2
−0.4
0.5
1.0
1.5
2.0
2.5