# Dark Matter Particle Scattering Processes and the Role of Sommerfeld Enhancement

UNIVERSITY OF TURIN
Department of Physics
Master's Degree in Theoretical Physics

July 21, 2026

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# Dark Matter

**Dark Matter**: a hypothetical new component of matter. Having suppressed (or null) electromagnetic interactions.

**WIMP Hypothesis** (*Weakly Interacting Massive Particle*): non-baryonic DM particles interacting only via weak interactions and having a sufficiently high mass ($m_x \geq$ a few GeV).

**Dark Matter Detection**: Production in Colliders, Direct detection, *Indirect detection*: searching for the products of WIMP particle annihilation processes.

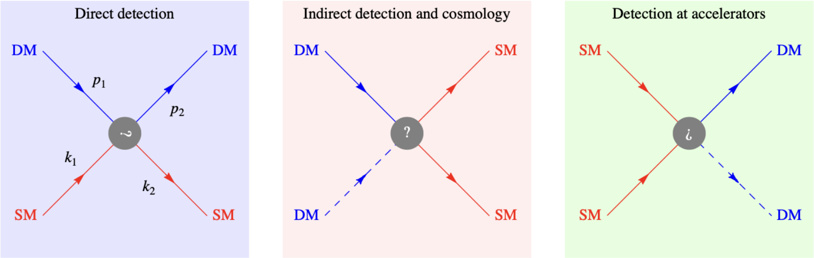



*Figure 1:* See Cirelli, Strumia, Zupan [3].

# The Hulthén Potential: the cross section and $S$

An attractive Hulthén potential can be considered together with an approximation of the centrifugal term:

$$V_H(r) = -\frac{\alpha_x \delta e^{-\delta r}}{1-e^{-\delta r}} \qquad , \qquad \tilde{V}_l(r) = \frac{l(l+1)}{m_x} \frac{\delta^2 e^{-\delta r}}{(1-e^{-\delta r})^2}$$

with $\delta = k m_\phi$, and solving the Schrödinger equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{m_x^2 v^2}{4} + \frac{m_x \alpha_x \delta e^{-\delta r}}{1-e^{-\delta r}} - \frac{l(l+1)\delta^2 e^{-\delta r}}{(1-e^{-\delta r})^2}\right] \chi_l(r) = 0.$$

This leads to:

$$\chi_l(t) = \frac{1}{2}\left|\frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}\right| t^{l+1}(1-t)^{iac}\, {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

This yields the phase shifts:

$$\delta_l = \arg\left(\frac{i\,\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)$$

# The Hulthén Potential: the cross section and $S$

and the cross-section:

$$\sigma_l = \frac{16\pi}{m_x^2 v^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \left\{ \arg \left( \frac{i\,\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\}.$$

**Sommerfeld Enhancement**: a typical non-relativistic QM phenomenon involving an increase in the annihilation cross-section, compared to the perturbative calculation, between particles in the presence of an interaction potential. The Sommerfeld factor is:

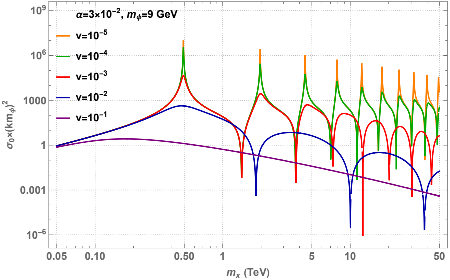
$$\begin{aligned} S_l &= \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(1+l)\Gamma(1+l+2iac)} \right|^2 \\ &= \left| \frac{(1+iac+\sqrt{c-a^2c^2})_l \ (1+iac-\sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times \\ &\times \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi\sqrt{a^2c^2-c})} \end{aligned}$$

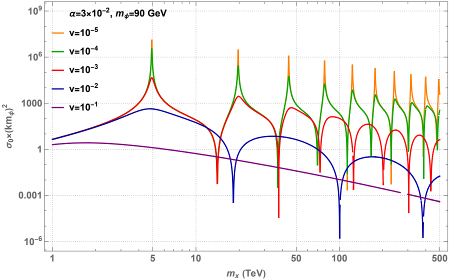
# The Hulthén Potential: the cross section and $S$

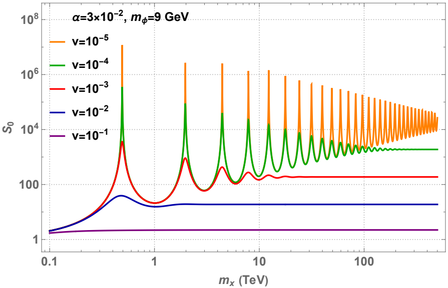
The resonant values for $S_l$ can be determined considering the low-velocity limit $v \to 0$ $(a \to 0)$. Resonance occurs for:

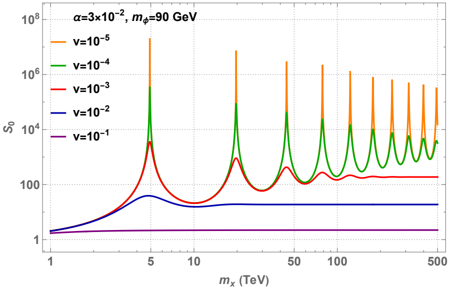
$$m_x = \frac{\pi^2 m_\phi}{6\alpha_x} n^2 \qquad , \qquad n \in \mathbb{N}.$$

# The Hulthén Potential: plots of $S$ factor and $\sigma$ for $l = 0$



# Conclusions

- The fundamental nature of Dark Matter remains one of the most significant open questions in contemporary physics.
- WIMP particles are among the leading candidates for DM.
- Sommerfeld enhancement could play a crucial role in the detection of indirect signals from WIMPs.

Thank You

# Appendix: table of contents

# Indications of Dark Matter

**Dark Matter**: a hypothetical new component of matter. Having suppressed (or null) electromagnetic interactions, its observational detection is highly complex.

Experimental evidence:

- Bullet Cluster
- Galaxy rotation curves
- Gravitational lensing of galaxy clusters
- Cosmic microwave background

# Indications of Dark Matter: Bullet Cluster

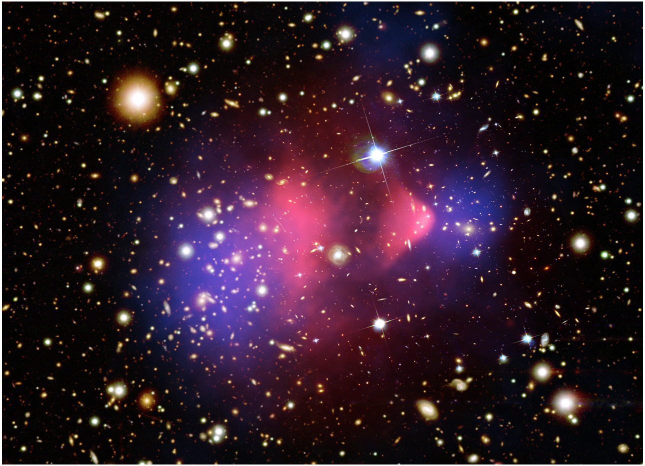


*Figure 2: The Bullet Cluster. The image shows the hot gas (in pink) traced via X-rays, galaxies and stars (the yellow and white areas), and the mass distribution (in blue). Credit: X-ray: NASA/CXC/CfA/M.Markevitch et al.; Optical: NASA/STScI; Magellan/U.Arizona/D.Clowe et al.; Lensing Map: NASA/STScI; ESO WFI; Magellan/U.Arizona/D.Clowe et al.*

# Indications of Dark Matter: Rotation Curves

Experimentally we observe:

$$M(r) \sim M \qquad r \sim \mathcal{O}(10) \text{ kpc.}$$

From Newton's equation:

$$G\frac{mM(r)}{r^2} = m\frac{v_c^2}{r}$$
$$v_c = \sqrt{\frac{GM(r)}{r}}.$$

Therefore, one must assume the existence of a DM component such that $\rho(r) \propto 1/r^2$ for $r \sim \mathcal{O}(10)$ kpc.

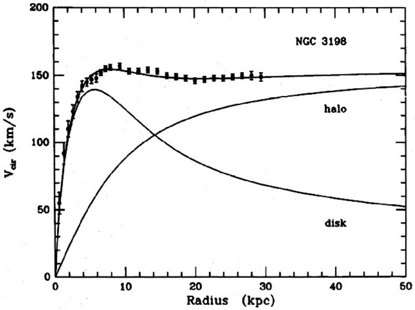



*Figure 3: Reference: link. Rotation curve of the spiral galaxy **NGC 3198** (located approximately $47$ million light-years from Earth).*

# WIMP Hypothesis and Scattering Processes of DM Particles

**WIMP Hypothesis** (Weakly Interacting Massive Particle): non-baryonic DM particles interacting only via weak interactions and having a sufficiently high mass ($m_x \geq$ a few GeV).

Dark Matter Detection:

- Production in Colliders
- Direct detection
- Indirect detection: searching for the products of WIMP particle annihilation processes.

**Sommerfeld Enhancement**: a typical non-relativistic QM phenomenon involving an increase in the annihilation cross-section, compared to the perturbative calculation, between particles in the presence of an interaction potential.

# The Sommerfeld Enhancement: $e^+ + e^- \to 2\gamma$ Annihilation

Annihilation Hamiltonian:

$$H_{ann} = U_0\,\delta^3(\boldsymbol{r}).$$

Annihilation cross-section:

$$\sigma_0 = |\psi_0(0)|^2.$$

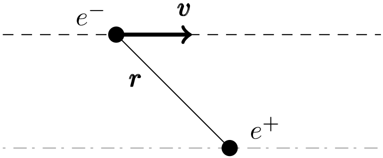


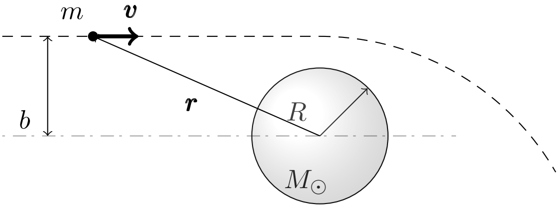

*Figure 4: Graphical representation of the $e^+ + e^- \to 2\gamma$ annihilation experiment.*

Introducing the interaction potential $V(r)$, we have:

$$\sigma = |\psi(0)|^2 \qquad \Longrightarrow \qquad S = \frac{\sigma}{\sigma_0} = \frac{|\psi(0)|^2}{|\psi_0(0)|^2}.$$

# The Sommerfeld Enhancement: Classical Analogy

A planet of mass $m$ interacts gravitationally with the Sun $M_\odot$.




Ignoring gravity:

$$\sigma_0 = \pi R^2.$$

Considering the gravitational interaction $V(r) = -G\frac{mM_\odot}{r}$:

$$\sigma = \pi R^2 \left(1 + \frac{v_{esc}^2}{v^2}\right) \qquad \Longrightarrow \qquad S = \frac{\sigma}{\sigma_0} = 1 + \frac{v_{esc}^2}{v^2}.$$

# The Hulthén Potential

An attractive Hulthén potential can be considered together with an approximation of the centrifugal term:

$$V_H(r) = -\frac{\alpha_x \delta e^{-\delta r}}{1-e^{-\delta r}} \qquad , \qquad \tilde{V}_l(r) = \frac{l(l+1)}{m_x}\frac{\delta^2 e^{-\delta r}}{(1-e^{-\delta r})^2}$$

and solving the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V(\mathbf{r})\right)\Psi(\mathbf{r}) = E\Psi(\mathbf{r}) \quad \Longrightarrow \quad \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \kappa^2 - 2\mu V_{eff.}(r)\right)\chi_l(r) = 0$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{m_x^2 v^2}{4} + \frac{m_x \alpha_x \delta e^{-\delta r}}{1-e^{-\delta r}} - \frac{l(l+1)\delta^2 e^{-\delta r}}{(1-e^{-\delta r})^2}\right]\chi_l(r) = 0.$$

where it is convenient to define:

$$a = \frac{v}{2\alpha_x} \qquad , \qquad c = \frac{\alpha_x m_x}{\delta} \qquad \delta = \frac{\pi^2}{6}m_\phi .$$

# Hulthén potential: Schrödinger Equation

Indeed, with an attractive Hulthén potential can be considered:

$$V_H(r) = -\frac{\alpha_x \delta e^{-\delta r}}{1 - e^{-\delta r}}$$

and the approximation of the centrifugal term:

$$\tilde{V}_l(r) = \frac{l(l+1)}{m_x} \frac{\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2}.$$

To solve the Schrödinger equation, one can define:

$$x = \alpha_x m_x r \qquad , \qquad a = \frac{v}{2\alpha_x} \qquad , \qquad c = \frac{\alpha_x m_x}{\delta}$$

so that:

$$\left(-\frac{\hbar^2}{2\mu}\Delta + V(\mathbf{r})\right)\Psi(\mathbf{r}) = E\Psi(\mathbf{r}) \implies \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \kappa^2 - 2\mu V_{eff.}(r)\right)\chi_l(r) = 0$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}+a^2+\frac{1}{c}\frac{e^{-x/c}}{1-e^{-x/c}}-\frac{l(l+1)}{c^2}\frac{e^{-x/c}}{(1-e^{-x/c})^2}\right]\chi_l(r)=0.$$

By changing the variable to:

$$t=1-e^{-x/c}$$

and reparameterizing the wavefunction:

$$\chi_l(x)=t^{l+1}(1-t)^{iac}f(t)$$

we obtain:

$$\left\{t(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\left[2+2l-(2+2l+2iac+1)t\right]\frac{\mathrm{d}}{\mathrm{d}t}-\right.$$
$$\left.-\left[l(l+1)+(l+1)(2iac+1)-c\right]\right\}f(t)=0$$

# Hulthén potential: Schrödinger Equation

where we can define:

$$\begin{cases} \lambda_{\pm}^{l} = 1 + l + iac \pm \sqrt{c - a^2c^2} \\ \omega = 2 + 2l \end{cases}$$

such that:

$$\left\{ t(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left[\omega - (\lambda_+^l + \lambda_-^l + 1)t\right]\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_+^l \lambda_-^l \right\} f(t) = 0.$$

Thus, the solution is:

$$\chi_l(t) = A\, t^{l+1}(1-t)^{iac}\, {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

The constant $A$ can be fixed by requiring that the part of $\chi_l(x)$ behaving as an outgoing plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ for $r \to \infty$ has a modulus of $1$.

# Hulthén potential: Wavefunction Normalization

The wavefunction is:

$$\chi_l(t) = A t^{l+1}(1-t)^{iac}\,{}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

The constant $A$ can be fixed by requiring that the part of $\chi_l(x)$ behaving as an outgoing plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ for $r \to \infty$ has a modulus of $1$.
Using one of the 5 Bolza formulas for the Hypergeometric function, along with ${}_2F_1(a,b,c,z) \to 1$ as $z \to 0$, we obtain the behavior for $t \to 1$ ($r \to \infty$):

$$\begin{aligned}\chi_l(t) = A t^{l+1}(1-t)^{iac}\Bigg\{ & \frac{\Gamma(\omega)\Gamma(\omega-\lambda_+^l-\lambda_-^l)}{\Gamma(\omega-\lambda_+^l)\Gamma(\omega-\lambda_-^l)} \times \\ & \times\ {}_2F_1(\lambda_+^l, \lambda_-^l, \lambda_+^l+\lambda_-^l-\omega+1, 1-t) + \\ & + \frac{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}(1-t)^{\omega-\lambda_+^l-\lambda_-^l} \times \\ & \times\ {}_2F_1(\omega-\lambda_+^l, \omega-\lambda_-^l, 1+\omega-\lambda_+^l-\lambda_-^l, 1-t)\Bigg\}.\end{aligned}$$

# Hulthén potential: Wavefunction Normalization

Note that:

$$\Gamma(\omega-\lambda_+^l-\lambda_-^l)=\Big(\Gamma(\lambda_+^l+\lambda_-^l-\omega)\Big)^*$$
$$\Gamma(\omega-\lambda_+^l)=\Big(\Gamma(\lambda_-^l)\Big)^*$$
$$\Gamma(\omega-\lambda_-^l)=\Big(\Gamma(\lambda_+^l)\Big)^*$$

to rewrite the wavefunction as:

$$\chi_l(t)=A\left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)^* t^{l+1}(1-t)^{iac}\times$$
$$\times\ {}_2F_1(\lambda_+^l,\lambda_-^l,\lambda_+^l+\lambda_-^l-\omega+1,1-t)+A\left(\frac{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)\times$$
$$\times\ t^{l+1}(1-t)^{-iac}\,{}_2F_1(\omega-\lambda_+^l,\omega-\lambda_-^l,1+\omega-\lambda_+^l-\lambda_-^l,1-t).$$

# Hulthén potential: Wavefunction Normalization

Taking the limit $t \to 1$ gives:

$$t^{l+1} \to 1 \qquad , \qquad (1-t)^{i\alpha} \to (e^{-x/c})^{i\alpha}$$

and the wavefunction becomes:

$$\chi_l(x) \sim A \left( \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right)^* e^{-iax} + \\ + A \left( \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) e^{iax}$$

# Hulthén potential: Wavefunction Normalization

$$\chi_l(x) \sim A \left| \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right| \times$$
$$\times \left[ e^{iax} \exp\left\{ i \arg\left( \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\} + \right.$$
$$\left. + e^{-iax} \exp\left\{ -i \arg\left( \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\} \right].$$

The constant $A$ is determined as:

$$A = \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|.$$

The asymptotic behavior of $\chi_l(x)$ can be rewritten as:

$$\chi_l(x) \sim \cos\left\{ ax + \arg\left( \frac{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)} \right) \right\}$$

# Hulthén potential: Wavefunction Normalization

using $\cos\alpha = \sin\left(\alpha + \frac{\pi}{2}\right)$ and $\arg(zw) = \arg z + \arg w$, it becomes:

$$\chi_l(x) \underset{x\to\infty}{\sim} \sin\left\{ax + \arg\left(\frac{i\,\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)\right\}.$$

Where we can notice $ax = kr$.

# Hulthén potential: The cross section

This leads to:

$$\chi_l(t) = \frac{1}{2}\left|\frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}\right| t^{l+1}(1-t)^{iac}\,{}_2F_1(\lambda_+^l,\lambda_-^l,\omega,t)$$

$$\chi_l(x) \underset{x\to\infty}{\sim} \sin\left\{kr + \arg\left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)\right\}.$$

Knowing the expected behavior from the theory:

$$\chi_l(r) \underset{r\to\infty}{\sim} \sin\left(kr - \frac{\pi}{2}l + \delta_l\right)$$

one can determine $\delta_l$ by computing:

$$\phi_l = \arg\left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right) - \arg\left(\frac{i\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)\Bigg|_{\alpha_x\to 0}.$$

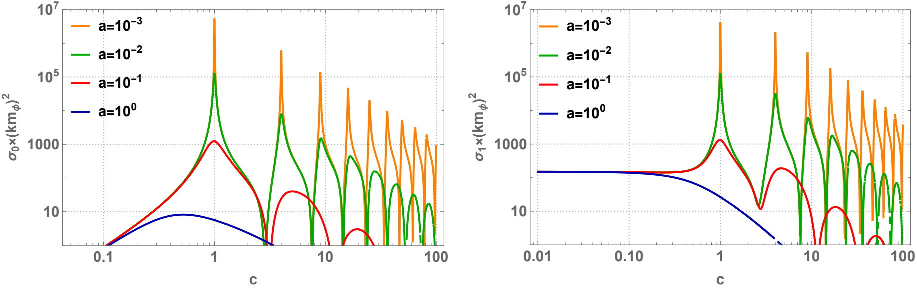
# Hulthén potential: The cross section

This yields the phase shifts:

$$\delta_l = \arg\left(\frac{i\,\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)$$

and the cross-section:

$$\sigma_l = \frac{16\pi}{m_x^2 v^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\left\{\arg\left(\frac{i\,\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)}{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}\right)\right\}.$$



# Hulthén potential: Wavefunction Behavior as $x \to 0$

Starting from:

$$\chi_l(t) = \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| t^{l+1}(1-t)^{iac} \, {}_2F_1(\lambda_+^l, \lambda_-^l, \omega, t)$$

we can expand it as $x \to 0$, i.e., as $t = 1 - e^{-x/c} \to 0$, to find:

$$\chi_l(r) \to B_l \frac{(\kappa r)^{l+1}}{(2l+1)!!}$$

and specifically the coefficient $B_l$, which represents the distortion introduced by the potential $V_H(r)$.

# Hulthén potential: Wavefunction Behavior as $x \to 0$

We have:

$$\chi_l(x) \underset{x\to 0}{\sim} \frac{1}{2}\left|\frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}\right|(1-e^{-x/c})^{l+1}\cdot 1$$

$$= \frac{1}{2}\left|\frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}\right|\left(1-\left(1-\frac{x}{c}+\dots\right)\right)^{l+1}$$

$$= \frac{1}{2}\left|\frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}\right|\left(\frac{x}{c}\right)^{l+1}$$

with the defined parameters, we have $\frac{x}{c} = km_\phi r$, where $k = \pi^2/6$, so:

$$\chi_l(x) \underset{x\to 0}{\sim} (km_\phi r)^{l+1}\,\frac{1}{2}\left|\frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l+\lambda_-^l-\omega)}\right|.$$

# Hulthén potential: Wavefunction Behavior as $x \to 0$

Using the parameter definitions again, it can be seen that:

$$\kappa r = \frac{m_x}{2} v r \quad , \quad k m_\phi r = k m_\phi \frac{m_x v}{2} \frac{2}{m_x v} r = \frac{2 k m_\phi}{m_x v} \kappa r = \frac{\kappa r}{ac}$$

allowing us to rewrite:

$$\begin{aligned}
\chi_l(x) &\sim \frac{(k m_\phi)^{l+1}}{(\kappa)^{l+1}} (\kappa r)^{l+1} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \\
&\sim \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \frac{(\kappa r)^{l+1}}{(2l+1)!!}.
\end{aligned}$$

Comparing:

$$\begin{aligned}
\chi_l(x) &\sim \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \frac{(\kappa r)^{l+1}}{(2l+1)!!} \\
\chi_l(r) &\to B_l \frac{(\kappa r)^{l+1}}{(2l+1)!!}
\end{aligned}$$

# Hulthén potential: Wavefunction Behavior as $x \to 0$

it follows that:

$$B_l = \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|.$$

To determine the Sommerfeld factor:

$$S = \frac{|B_l|^2}{|B_l(c \to 0, ac = u)|^2}$$

we need to compute $B_l(c \to 0, ac = u) = B_l(V \to 0)$, where we have set $ac = \frac{m_x v}{2km_\phi} = u$. We have:

$$\begin{aligned} B_l &= \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right| \\ &= \frac{(2l+1)!!}{2u^{l+1}} \left| \frac{\Gamma(1+l+iu+ac\sqrt{\frac{1}{a^2c}-1})\Gamma(1+l+iu-ac\sqrt{\frac{1}{a^2c}-1})}{\Gamma(2l+2)\Gamma(2iu)} \right| \end{aligned}$$

# Hulthén potential: Wavefunction Behavior as $x \to 0$

taking the limit $c \to 0$ and $a \to \infty$:

$$B_l = \frac{(2l+1)!!}{2u^{l+1}} \left| \frac{\Gamma(1+l+iu+iu)\Gamma(1+l+iu-iu)}{\Gamma(2l+2)\Gamma(2iu)} \right|$$

which ultimately yields:

$$B_l(c \to 0, ac = u) = \frac{(2l+1)!!}{2(ac)^{l+1}} \left| \frac{\Gamma(1+l+2iac)\Gamma(1+l)}{\Gamma(\omega)\Gamma(2iac)} \right|.$$

# Hulthén potential: the Sommerfeld factor

Having determined the two distortion coefficients:

$$B_l = \frac{(2l+1)!!}{(ac)^{l+1}} \frac{1}{2} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(\lambda_+^l + \lambda_-^l - \omega)} \right|$$

$$B_l(c \to 0, ac = u) = \frac{(2l+1)!!}{2(ac)^{l+1}} \left| \frac{\Gamma(1+l+2iac)\Gamma(1+l)}{\Gamma(\omega)\Gamma(2iac)} \right|$$

the Sommerfeld factor results in:

$$\begin{aligned} S_l &= \frac{|B_l|^2}{|B_l(c \to 0, ac = u)|^2} \\ &= \left( \frac{(2l+1)!!}{2(ac)^{l+1}} \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(\omega)\Gamma(2iac)} \right| \right)^2 \left( \frac{2(ac)^{l+1}}{(2l+1)!!} \left| \frac{\Gamma(\omega)\Gamma(2iac)}{\Gamma(1+l+2iac)\Gamma(1+l)} \right| \right)^2 \\ &= \left| \frac{\Gamma(\lambda_+^l)\Gamma(\lambda_-^l)}{\Gamma(1+l)\Gamma(1+l+2iac)} \right|^2 . \end{aligned}$$

# Hulthén potential: the Sommerfeld factor

which, by utilizing:

$$|\Gamma(i\alpha)|^2 = \frac{\pi}{\alpha \sinh(\pi\alpha)} \quad , \quad |\Gamma(1+i\alpha)|^2 = \frac{\pi\alpha}{\sinh(\pi\alpha)} \quad , \quad \alpha \in \mathbb{R}$$

can be rewritten as:

$$S_l = \left| \frac{(1+iac+\sqrt{c-a^2c^2})_l \; (1+iac-\sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times$$
$$\times \left| \frac{\Gamma(1+iac+i\sqrt{a^2c^2-c})\Gamma(1+iac-i\sqrt{a^2c^2-c})}{\Gamma(1+2iac)} \right|^2 .$$

$$S_l = \left| \frac{(1+iac+\sqrt{c-a^2c^2})_l \; (1+iac-\sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times$$
$$\times \frac{\pi(ac+\sqrt{a^2c^2-c})\pi(ac-\sqrt{a^2c^2-c})}{\sinh(\pi(ac+\sqrt{a^2c^2-c}))\sinh(\pi(ac-\sqrt{a^2c^2-c}))} \cdot \frac{\sinh(2\pi ac)}{\pi(2ac)} .$$

# Hulthén potential: the Sommerfeld factor

We can make use of:

$$\sinh x \sinh y = \frac{1}{2}\left(\cosh(x+y) - \cosh(x-y)\right)$$

to obtain the final result:

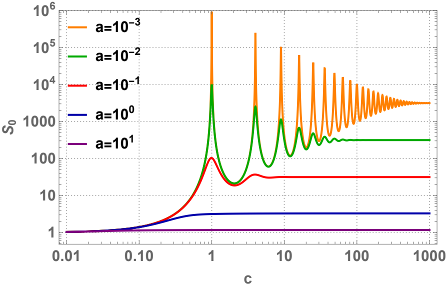
$$S_l = \left| \frac{(1+iac+\sqrt{c-a^2c^2})_l \; (1+iac-\sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times$$

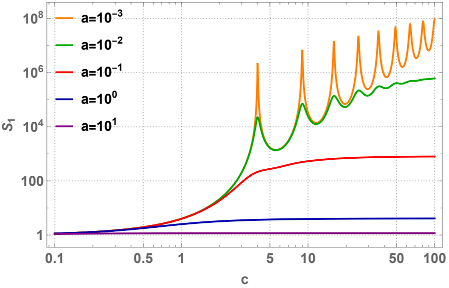
$$\times \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi\sqrt{a^2c^2-c})}.$$

The cases for $l=0$ and $l=1$ are:

$$S_0 = \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi\sqrt{a^2c^2-c}))} \quad , \quad S_1 = \frac{(1-c)^2+4a^2c^2}{1+4a^2c^2} S_0$$

# Hulthén potential: the Sommerfeld factor




The resonant values for $S_l$ can be determined:

$$S_l = \left| \frac{(1+iac+\sqrt{c-a^2c^2})_l \; (1+iac-\sqrt{c-a^2c^2})_l}{\Gamma(1+l)(1+2iac)_l} \right|^2 \times$$

$$\times \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cosh(2\pi\sqrt{a^2c^2-c})}$$

# Hulthén potential: the Sommerfeld factor

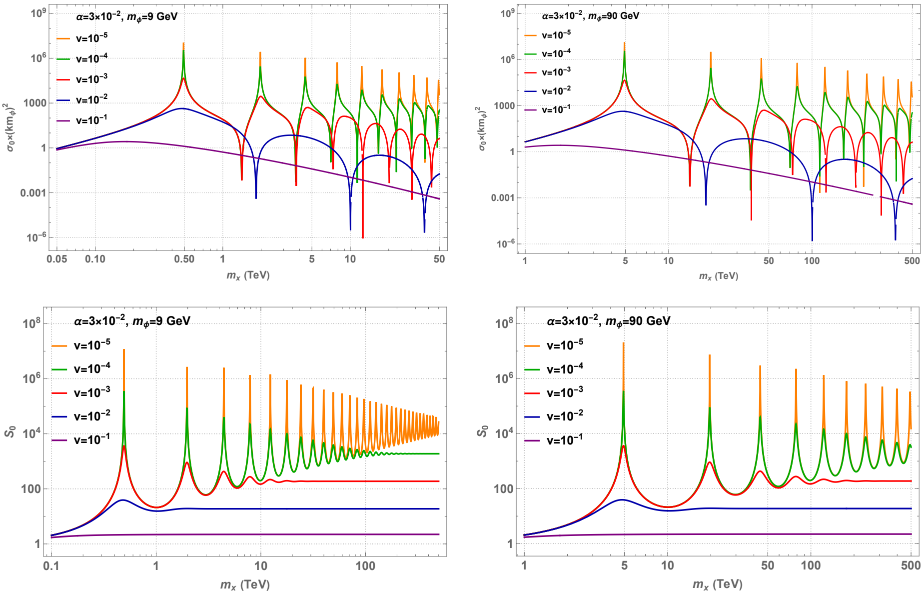
considering the low-velocity limit $v \to 0$ ($a \to 0$). Resonance occurs for:

$$m_x = \frac{\pi^2 m_\phi}{6\alpha_x} n^2 \qquad , \qquad n \in \mathbb{N}.$$

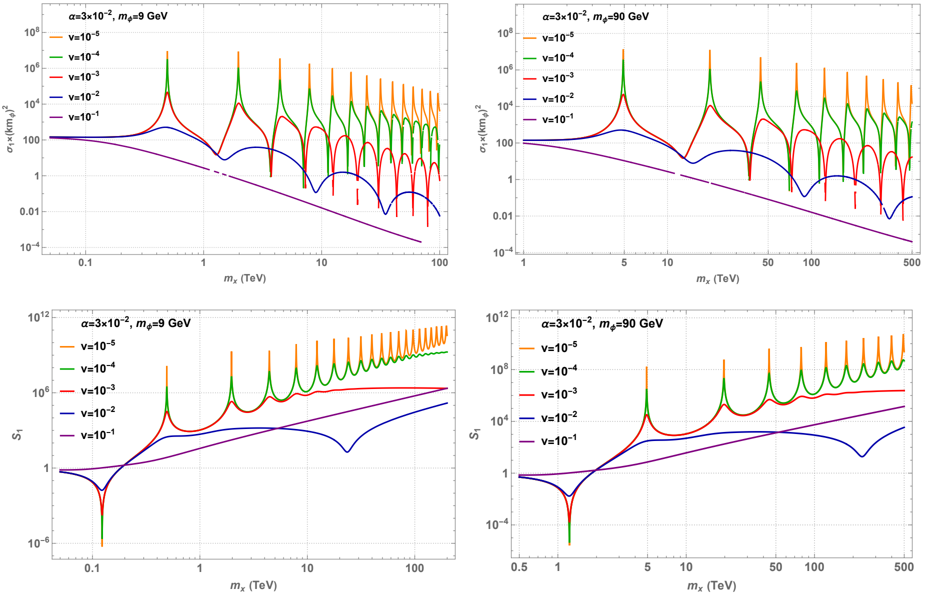
around which the behavior of $S$ is:

$$S = \frac{4\,\alpha_x k\, m_\phi}{m_x v^2}.$$

# Hulthén potential: plots of the $S$ factor and $\sigma$ for $l = 0$



# Hulthén potential: plots of the $S$ factor and $\sigma$ for $l = 1$



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