Complements for the astroparticle & cosmology course (GraSPA 2026, P. Serpico) – Not for circulation!

Newtonian toy cosmology

Consider a uniform sphere of pressureless dust of radius a and mass M. Dust means matter whose pressure (in natural units) is negligible wrt energy density. For a thermal non-relativistic gas of particles of mass m and number density n, from the perfect gas EoS one has $P = nT \ll nm \simeq \rho$.

The total (conserved) energy writes

$$\frac{1}{2}m\dot{a}^2 - \frac{GMm}{a} = T + U = const. \equiv -\frac{1}{2}km \tag{1}$$

where the last definition is done for dimensional reasons (dimensionless k, mass of gas particle natural normalisation of particle energy) and the numerical pre-factor is for later convenience. From $M = 4\pi\rho a^3/3$ we obtain

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}\,. \tag{2}$$

This is equivalent to the the so-called *First Friedmann equation* derived for a metric (FLRW metric) describing a uniform and isotropic universe, where a is the "scale factor" (describing the stretching of the universe with time) and k is a constant related to the curvature (positive, null, or negative). Note the analogy of the latter with a recollapsing, asymptotically still or forever expanding gas cloud, respectively.

If we further apply the first law of thermodynamics for a perfect gas in the (adiabatically) expanding volume, we get

$$\mathrm{d}(\rho a^3) = -P\mathrm{d}(a^3) \Leftrightarrow \mathrm{d}U = -P\mathrm{d}V\,, \tag{3}$$

which yields

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H(\rho + P)\,. \tag{4}$$

This is equivalent to the second independent GR equation in the FLRW metric. Together with the first Friedman equation, it provides a closed system for ρ, P, a if an equation of state $P = P(\rho)$ is also provided. Indeed, although the derivation above implicitly or explicitly used $P \ll \rho$, the equations we obtained are true for a more general ideal fluid in GR.

We can easily solve this system for fluids whose equation of state is linear, $P = w\rho$. We find:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\rho(1 + w) = 0 \tag{5}$$

which, separating ρ from a, leads to

$$\frac{\rho(t)}{\rho_0} = \left[\frac{a(t)}{a_0}\right]^{-3(1+w)} \tag{6}$$

Check the solutions/behaviours reported in figure 1.

Note that matter scales as the number density $n \propto a^{-3}$, since $\rho = mn$ and the mass m of the dust particles is a constant. Radiation scales as $\propto a^{-4}$, with an additional power of the scale factor, since not only the number density of relativistic particles is diluted as a^{-3} with the increasing volume, but also the energy of radiation redshifts during expansion, thus bringing an additional factor of a: radiation dilutes faster! Eventually, a cosmological constant will dominate also over matter, since its density stays constant. It is therefore reasonable to assume that the very early Universe was dominated by radiation, then matter takes over and, eventually, Λ wins. We will call and denote these three regimes as: *radiation dominated (RD)*, *matter dominated (MD)* and Λ *dominated* (Λ*D)* Universe.

Exercise Prove that, in the case the equation of state of the fluid is not a constant, but the relation between pressure and energy density is still linear, i.e. $P = w(a)\,\rho$, Eq. 5 leads to a more general solution:

$$\frac{\rho(a)}{\rho_0} = \exp\left\{-3\int_{a_0}^{a}\frac{\mathrm{d}a'}{a'}[1 + w(a')]\right\}\,. \tag{7}$$

	Equation of State	Behaviour of ρ	Scale Factor
Matter	$P \simeq 0$ $(T \ll m)$	$\rho \propto a^{-3}$	$a \propto t^{2/3}$
Radiation	$P = \rho/3$	$\rho \propto a^{-4}$	$a \propto t^{1/2}$
Cosm. constant	$P = -\rho$	$\rho = \text{const.}$	$a \propto e^{H_0 t}$

FIG. 1: Simple solutions of the "toy cosmology" equations.

(Cosmological) Redshift

The notion of cosmological redshift can only be rigourously made sense of within GR. However, a heuristic justification can be given if you think that the fabric of the space-time through which waves propagate (and all particles are associated to a wave, even "matter" ones. . . remember De Broglie!) can expand/contract, as described by $a(t)$. In particular, expansion produces the physical effect that radiation emitted at some time in the past arrives to us shifted to lower frequencies. By defining the *redshift* as the relative change in wavelength λ (or frequency ν - for photons, proportional to energy E as well):

$$(1+z) \equiv \frac{\lambda_0}{\lambda} = \frac{\nu}{\nu_0} = \frac{E}{E_0}\,, \tag{8}$$

it can be shown that in a FLRW Universe (homogeneous, isotropic) the redshift is directly related to the scale factor:

$$(1+z) = \frac{a_0}{a(t)}\,. \tag{9}$$

The redshift corresponding to today is $z_0 = 0$. Anticipating that the scale factor in the past was smaller than what it is today, eq. (9) shows that redshifts corresponding to past events are positive, $z > 0$: the larger the redshift, the earlier in time t a photon emitted with wavelength λ reaches us with wavelength λ_0. Earlier in time also means that the photon was emitted farther in space. By measuring the redshift of radiation emitted from a known process, e.g. from an atomic line emission (for which we assume that the physical processes are the same everywhere and at anytime in the Universe!), we have direct information on its redshift, and therefore on the size of the change in the scale factor of the Universe between the instant of emission and today.

Newtonian perturbation theory

Let us start by recalling the key equations of Newtonian fluid mechanics in presence of gravity: Continuity eq. (mass conservation) Euler eq. (momentum eq.) and Poisson eq. (Gravity law);

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0\,, \tag{10}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\frac{\nabla P}{\rho} - \nabla\Phi\,, \tag{11}$$

$$\nabla^2\Phi = 4\pi G_N \rho\,. \tag{12}$$

Then we ask ourselves how small perturbations around a homogeneous and isotropic background evolve according to these equations, namely we set $\rho(t,\mathbf{x}) = \bar{\rho}(t) + \delta\rho(t,\mathbf{x})$ and similarly for velocity and pressure.

In a *static space in absence of gravity* the equation for perturbations around $\bar{\rho} = const.$, $\bar{P} = const.$, $\bar{\mathbf{v}} = 0$ is the well-known wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\delta\rho = 0 \tag{13}$$

where the speed of sound is defined by $c_s^2 \equiv \delta P/\delta\rho$. Its solution is oscillatory with constant amplitude, i.e. in Fourier space one has

$$\delta\rho(t,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\delta\tilde{\rho}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{14}$$

where

$$\delta\tilde{\rho}_{\mathbf{k}} = A_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t} + B_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}\,, \tag{15}$$

with dispersion relation

$$\omega_{\mathbf{k}}^2 = c_s^2\, k^2\,. \tag{16}$$

In a *static space, but with the gravitational potential due to a constant density* $\bar{\rho}$, the solution to the new equation

$$\left[\frac{\partial^2}{\partial t^2} - \left(c_s^2 \nabla^2 + 4\pi G_N \bar{\rho}\right)\right]\delta\rho = 0 \tag{17}$$

is still formally given by Eqs. (14,15), but Eq. (16) is replaced by

$$\omega_{\mathbf{k}}^2 = c_s^2\, k^2 - 4\pi G_N \bar{\rho}\,. \tag{18}$$

In turn, this implies oscillatory solutions for $k > k_J$, where we introduced the *Jeans wavenumber*

$$k_J^2 \equiv \frac{4\pi G_N \bar{\rho}}{c_s^2}\,, \tag{19}$$

while large-scale modes with $k < k_J$ *grow exponentially*, with rate given by the imaginary part of ω. It is also customary to associate to k_J a *Jeans length* $\lambda_J \equiv 2\pi/k_J$. This is an important qualitative lesson: in a "classical" cold fluid subject to gravity, we expect that small-scale (high-k) modes oscillate, but large scale perturbations grow and eventually collapse forming non-linear structures. (By the way, if you ever wondered, this is also the main piece of physics setting the "stellar" mass scale, starting from interstellar molecular clouds with densities of hundreds or thousands particles per cubic centimeter and sound speeds of hundreds of meters per second).

Let us come back to the previously introduced "Newtonian proxy" for the flat FLRW setting: an *expanding fluid* with scale factor $a(t)$ and expansion rate $H \equiv \dot{a}/a$. In this context, the relation between the physical position $\mathbf{x}$ and velocity $\mathbf{v}$ and the comoving coordinate $\mathbf{r}$ and proper velocity $\mathbf{u}$ are

$$\mathbf{x} = a\mathbf{r}\,, \quad \mathbf{v} = H\mathbf{x} + \mathbf{u} \tag{20}$$

with $\mathbf{u} = a\dot{\mathbf{r}}$ (the term $H\mathbf{x}$ describes thus the speed of the fabric of the space-time carrying e.g. the galaxy with it, while $\mathbf{u}$ its peculiar velocity with respect to that, e.g. due to pull of neighbour galaxies). Hence, the derivatives with respect to the two coordinates are linked via $\nabla_{\mathbf{x}} = a^{-1}\nabla_{\mathbf{r}}$, and the partial derivative with respect to time are linked by

$$\left.\frac{\partial}{\partial t}\right)_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right)_{\mathbf{r}} - H\mathbf{r}\cdot\nabla_{\mathbf{r}}\,. \tag{21}$$

Rewriting the above equation in terms of comoving variables, from the continuity equation we obtain the zeroth order equation

$$\frac{\partial}{\partial t}\bar{\rho} + 3H\bar{\rho} = 0 \Rightarrow \bar{\rho} \propto a^{-3}\,; \tag{22}$$

the first order equation reads instead

$$\frac{\partial}{\partial t}\delta + \frac{1}{a}\nabla_{\mathbf{r}}\cdot\mathbf{u} = 0\,, \tag{23}$$

where we introduced the relative density contrast $\delta \equiv \delta\rho/\bar{\rho}$. Similarly, the Euler equation yields

$$\frac{\partial \mathbf{u}}{\partial t} + H\mathbf{u} = -\frac{1}{a\,\bar{\rho}}\nabla_{\mathbf{r}}\delta P - \frac{1}{a}\nabla_{\mathbf{r}}\delta\Phi\,, \tag{24}$$

and the Poisson equation is

$$\nabla^2\delta\Phi = 4\pi G_N\, a^2\,\bar{\rho}\,\delta\,. \tag{25}$$

The (partial) time derivative of Eq. (23), combined with the divergence of Eq. (24) and Eq. (25) leads to the (generalized) Jeans equation

$$\boxed{\frac{\partial^2\delta_m}{\partial t^2}+2H\frac{\partial\delta_m}{\partial t}-\left(\frac{c_s(t)^2}{a^2}\nabla^2_{\mathbf{r}}+4\pi G_N\bar{\rho}_m(t)\right)\delta_m=0\,.} \tag{26}$$

where, with the use of $_m$, we specify that we refer to matter densities : this formalism is largely insufficient to describe the evolution of other types of fluctuations; physically, you can think that the only thing that clusters is matter, not radiation or "dark energy". This is similar to Eq. (18), but for two differences: The Jeans scale Eq. (19) is now a function of time, via $c_s(t)$ and $\bar{\rho}_m(t)$; there is a "friction" term, proportional to the expansion rate H. We can anticipate that this leads to a slower than exponential growth of the instability for $k>k_J$.

Let us now study some *benchmark evolutions* over different phases of evolution for the smooth universe. Henceforth, we neglect the c_s^2 term (i.e. pressureless matter): This is ok for the largest structures (why not for the smaller ones? Go to Fourier space to see explicitly!).

During matter era we have $a\propto t^{2/3}$, $H=2/(3\,t)$, and $4\pi G_N\bar{\rho}_m(t)=3H/2$:

$$\frac{\partial^2\delta_m}{\partial t^2}+\frac{4}{3\,t}\frac{\partial\delta_m}{\partial t}-\frac{2\delta_m}{3\,t^2}=0\,. \tag{27}$$

Besides a decreasing mode, this equation admits a *growing mode* $\delta_m\propto t^{2/3}\propto a$, as it can be checked by seeking solutions of the form $\delta_m\propto t^b$.

In the other epoch of interest, when either radiation or dark energy is dominant, one can easily convince oneself that evolution on cosmological timescales implies $[\dot{\delta}_m]\sim H\delta_m$ and $[\ddot{\delta}_m]\sim H^2\delta_m$, hence the first two terms in Eq. (26) are of order $H^2\delta$, vs. the last one which is of order $4\pi G_N\bar{\rho}_m(t)\delta_m$. Since

$$H^2=\frac{8\pi G_N}{3}\bar{\rho}\gg 4\pi G_N\bar{\rho}_m(t)\,, \tag{28}$$

where at the LHS is the first Friedmann equation and the inequality follows from the hypothesis that either radiation or dark energy dominates the expansion, we can approximate Eq. (26) as

$$\frac{\partial^2\delta_m}{\partial t^2}+2H\frac{\partial\delta_m}{\partial t}\simeq 0\,. \tag{29}$$

When dark energy is dominant, Eq. (29) admits either a constant solution, or a solution $\delta_m\propto a^{-2}$. We conclude that the perturbations stop growing during dark energy dominance (If you wonder, it turns out that the same freezing happens, qualitatively, if there is a phase dominated by curvature).

In radiation epoch, by using $a\propto t^{1/2}$, $H=(2t)^{-1}$, the relevant equation for the perturbations reduces to

$$\frac{\partial^2\delta_m}{\partial t^2}+\frac{1}{t}\frac{\partial\delta_m}{\partial t}\approx 0\,, \tag{30}$$

which admits a (slowly) growing mode solution $\delta\propto\ln t$ (in addition to the constant solution).

We conclude that the *growth of matter perturbations in an expanding universe, at the linear level, can only proceed very slowly in the radiation epoch, goes linearly with a in the matter epoch, and freezes at late time when dark energy dominates.* Reinstating pressure, i.e. the c_s^2 term, can only make things worse (at small scales only), since it counteracts gravity.

Exercise: The CMB formed at at redshift of about $z\simeq 1100$. Consider matter-dominated expansion of the universe down to $z\simeq 1$. How large should the initial perturbations of the matter density field be at $z\simeq 1100$, in order for them to have grown to O(1) by redshift ~ 10 (roughly the epoch of the most ancient galaxies we see) according to the Newtonian proxy prediction for the gravitational instability given by Eq. (27)?