Amplitude Factorization Prescription for Exclusive Processes and NNLO Application to Quarkonium $\to 3\gamma$ Decay

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Outline

Heavy Quarkonium Decay and production as QCD Laboratory

- Heavy quarkonium decay and production: crucial proving ground for QCD
  - NRQCD factorization: systematic separation of short/long-distance physics [2]
- $J/\psi \to 3\gamma$
  - Sensitive to both perturbative expansion on QCD coupling constant α_s and v^2 (v is velocity of heavy quark at Quarkonium rest frame)
- Experimental status:
  - CLEO-c (2008): $\mathrm{Br}(J/\psi \to 3\gamma) = (1.2 \pm 0.3 \pm 0.2) \times 10^{-5}$
  - BESIII (2013): $\mathrm{Br}(J/\psi \to 3\gamma) = (11.3 \pm 1.8 \pm 2.0) \times 10^{-6}$

The Long-standing "Sign Crisis" — A 40-Year Puzzle

  - CLEO-c (2008): $\mathrm{Br}(J/\psi \to 3\gamma) = (1.2 \pm 0.3 \pm 0.2) \times 10^{-5}$
  - BESIII (2013): $\mathrm{Br}(J/\psi \to 3\gamma) = (11.3 \pm 1.8 \pm 2.0) \times 10^{-6}$
- Traditional perturbative expansion for $J/\psi \to 3\gamma$:

$$\Gamma = \Gamma_{LO}\left(1 - 12.6\frac{\alpha_s}{\pi} - 8.23\langle v^2\rangle + 68.9\frac{\alpha_s}{\pi}\langle v^2\rangle - 28.7\frac{\alpha_s^2}{\pi^2}\right)$$

- The convergence pattern is clearly pathological:
  - LO: too large compared with experiment
  - NLO (α_s, 1981): large and negative
  - $\mathcal{O}(v^2)$ (1983): large and negative
  - $\mathcal{O}(\alpha_s v^2)$ (2012): large and positive — accidentally "rescues" the result
  - **NNLO (α_s^2, this work): large and negative again!**
- Similar sign crisis in $e^+e^- \to J/\psi + J/\psi$ at NLO [9] and NNLO [10]
- This 40-year puzzle signals a **fundamental issue**: so far.

Traditional perturbative expansion, it have been used more than 50 years

Perturbative expansion truncated at amplitude-square level

$$|M|^2 = |M_0 + M_1 + M_2|^2 = \underbrace{\underbrace{\underbrace{|M_0|^2}_{\text{LO}} + 2\mathrm{Re}(M_0 M_1^*)}_{\text{NLO}} + |M_1|^2 + 2\mathrm{Re}(M_0 M_2^*)}_{\text{NNLO}} + (\text{Higher-order terms})$$

$\Rightarrow$ NLO and NNLO are not a modulus squared, decay rate can be negative results at any order.

This Work: A Systematic Solution

Two key contributions

1. First complete NNLO QCD correction to $J/\psi \to 3\gamma$ (48 years after NLO!)
2. **Amplitude-level NRQCD factorization** is proposed as a systematic prescription for exclusive processes

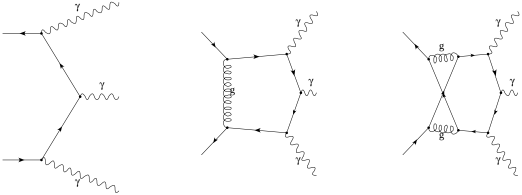
Amplitude factorization: perturbative expand at amplitude level, then square

$$\Gamma_M = \int d\phi \sum_{\text{pol}} \Big| \sum_{l=0}^{M} \mathcal{A}^{(l)} \Big|^2, \quad M = 0, 1, 2, \ldots$$

$\Rightarrow |M_0 + M_1 + \cdots + M_M|^2 \geq 0$ guaranteed at every order

- The prescription guarantees positive decay rates at any order
- Extended to $\Upsilon \to 3\gamma$ — where the problem is even more striking

Calculation Overview



Typical Feynman diagrams: 6 (tree) + 48 (1-loop) + 894 (2-loop)

Complete NNLO Result for $J/\psi \to 3\gamma$

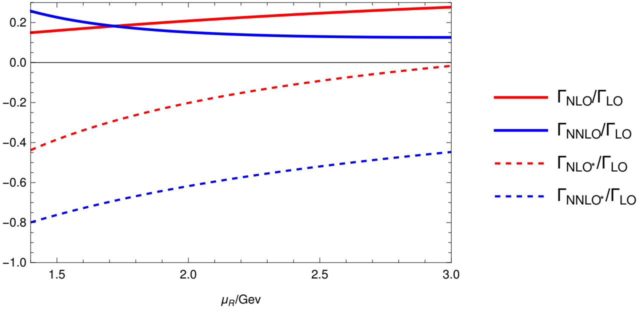
$$\Gamma_{LO} = \frac{16(\pi^2-9)q_c^6\alpha^3|R_{J/\psi}|^2}{3\pi M_{J/\psi}^2}, \quad \Gamma_{NLO} = \Gamma_{LO}\Big[1 - 12.63(\alpha_s/\pi) + 45.28(\alpha_s/\pi)^2\Big],$$

$$\begin{aligned}\Gamma_{NNLO} = \Gamma_{LO}\Big\{&1 - 12.63(\alpha_s/\pi) + (\alpha_s/\pi)^2\left[5.369 n_l - 1.986\overline{q}^2 - 41.77 + (4.210 n_l - 65.26) l_R - 51.18 l_\Lambda\right] \\ &+ (\alpha_s/\pi)^3\left[(467.4 - 30.16 n_l) l_R + 323.2 l_\Lambda - 42.27 n_l + 10.10\overline{q}^2 + 647.4\right] + (\alpha_s/\pi)^4\left[654.7 l_\Lambda^2\right. \\ &+ (1670. - 107.7 n_l) l_R l_\Lambda + (1208. - 155.9 n_l + 5.028 n_l^2) l_R^2 \\ &+ (52.21\overline{q}^2 - 3.368 n_l\overline{q}^2 + 3346. - 434.3 n_l + 14.10 n_l^2) l_R + (50.82\overline{q}^2 + 2228. - 137.4 n_l) l_\Lambda \\ &\left.+ 4.018\overline{q}^4 + 63.97\overline{q}^2 - 3.448 n_l\overline{q}^2 + 2367. - 314.0 n_l + 10.65 n_l^2\right]\Big\}.\end{aligned}$$

where $l_R = \ln\frac{\mu_R}{m_c}$, $l_\Lambda = \ln\frac{\mu_\Lambda}{m_c}$, $q_c = \frac{2}{3}$, $\overline{q}^2 = (q_u^2 + q_d^2 + q_s^2)/q_c^2$, $n_l = 3$.

- Positive decay widths at LO, NLO, and NNLO (amplitude expansion)
- $\Gamma_{NLO}/\Gamma_{LO} = 0.277, \quad \Gamma_{NNLO}/\Gamma_{LO} = 0.127 \quad \Rightarrow$ **good convergence**

Renormalization Scale Dependence: J/ψ



- Solid lines (amplitude expansion): positive, much weaker scale dependence
- Dashed lines (traditional expansion): NLO and NNLO go negative
- NNLO band overlaps with NLO — perturbative stability

Numerical Results for $J/\psi \to 3\gamma$

Theoretical predictions for $\Gamma(J/\psi \to 3\gamma)$ (eV)					
Contribution	Central Value	$\Delta\mu_R$	Δm_c	$\Delta\mu_\Lambda$	$\Delta\lvert R(0)\rvert^2$
Γ_{NNLO}	0.6659	$^{+0.5181}_{-0}$	$^{+0.1053}_{-0}$	$^{+0.2299}_{-0}$	$^{+0.3421}_{-0.0884}$
Γ_{Rel}	0.2970	$^{+2.474}_{-0}$	$^{+0.2502}_{-0}$	$^{-0.2346}_{+0}$	$^{+0.1653}_{-0.0394}$
Γ_{Full}	0.9629	$^{+2.992}_{-0}$	$^{+0.3555}_{-0}$	$^{-0.0047}_{+0}$	$^{+0.5074}_{-0.1278}$
BESIII (2013)	$1.046 \pm 0.167 \pm 0.185$				

- $\Gamma_{\text{Full}} = \Gamma_{NNLO} + \Gamma_{Rel}$ (with known $\mathcal{O}(\alpha_s v^2)$ correction from Feng, Jia, Sang 2013)
- Final prediction: $\Gamma(J/\psi \to 3\gamma) = 0.96^{+4.32}_{-0.13}$ eV
- Markedly improved agreement with BESIII data

$\Upsilon \to 3\gamma$: A Cleaner System, A Sharper Problem

- Υ system: $\alpha_s(m_b) \approx 0.18$, much smaller than $\alpha_s(m_c) \approx 0.25$
- Perturbation theory expected to work **better** for Υ than J/ψ
- **But:** traditional expansion at NNLO gives **negative rate over the entire scale range!**
- Our amplitude factorization result (with $n_l = 4$, $m_b = 4.75$ GeV, $\mu_R = 2m_b$):

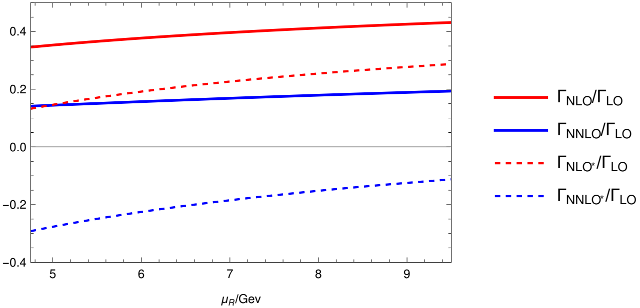
$$\Gamma_{NLO}(\Upsilon)/\Gamma_{LO} = 0.437$$
$$\Gamma_{NNLO}(\Upsilon)/\Gamma_{LO} = 0.355$$

- Combined with relativistic correction $\Gamma_{Rel}(\Upsilon)$:

$$\Gamma_{\mathrm{Full}}(\Upsilon \to 3\gamma) = 0.0086^{+0.0028}_{-0.0006}\ \mathrm{eV}$$

- Much smaller uncertainties than J/ψ — as expected for a more perturbative system

Renormalization Scale Dependence: Υ



- Amplitude expansion (solid): positive, weak scale dependence
- Traditional expansion (dashed): **NNLO negative everywhere!**
- Υ confirms: the sign crisis is **not** specific to J/ψ — it's a systematic problem of the traditional expansion

First-Principles Justification

The First talk of this work was presented at a workshop in May,2026. Our expansion scheme was criticized by many colleagues

"Why does amplitude Factorization Work? What is the physics reason behind?"
This is the tough question asked by the referee

I prepared answer to referee, but no idea about this tough question, then send our paper and referee reports to AI agent, ask AI Agent to draft the reply

The reply have been finished. To the tough question, AI agent also prepared answer "..infrared divergent are cancelled at amplitude level, so the IR structure supply ...."

First-Principles Justification

Why amplitude-level factorization is the natural prescription

- For **exclusive** processes: all initial and final particles are color-singlets and electrically neutral
- All IR divergences cancel at the **amplitude level** — no real emission needed
- $\Rightarrow$ The amplitude is finite and well-defined at any fixed order
- The full modulus squared $|\sum_l \mathcal{A}^{(l)}|^2$ is the proper definition of a probability
- For **inclusive** processes: IR cancellation only after summing over unobserved states $\Rightarrow$ perturbative expansion must expand and truncate at squared level and probability definition has to be broken.

Exclusive processes $\neq$ Inclusive processes
Amplitude-level truncation is the correct first-principles prescription for exclusive processes

Summary

- **A 40-year puzzle resolved:** the sign crisis in exclusive quarkonium processes is a consequence of organizing the perturbative expansion at the amplitude-square level
- **Amplitude-level NRQCD factorization** proposed as the systematic, first-principles prescription:
  - Guarantees positive rates at any order by construction
  - First complete NNLO QCD correction to $J/\psi \to 3\gamma$ (48 years after NLO)
  - Substantially improves perturbative convergence and scale dependence
  - Generalizable to any exclusive process
- **Both J/ψ and Υ** results validate this prescription
  - J/ψ: $\Gamma_{\rm Full} = 0.96^{+4.32}_{-0.13}$ eV, in good agreement with BESIII ($1.046 \pm 0.167 \pm 0.185$ eV)
  - Υ: traditional expansion gives negative NNLO; amplitude factorization restores positivity with much smaller uncertainties

Thank you for your attention!

Questions & Discussion

Input Parameters

Parameter	Central Value
m_c	1.5 GeV
$M_{J/\psi}$	$2m_c$
$\lvert R_{J/\psi}(0)\rvert^2$	$0.934\ \mathrm{GeV}^3$
$\langle v^2\rangle_{J/\psi}$	0.225
μ_R	$2m_c$
μ_Λ	m_c
$\alpha_s^{(3)}(\mu_R)$	0.25406 (3-loop)
m_b	4.75 GeV
M_Υ	9.46 GeV
$\lvert R_\Upsilon(0)\rvert^2$	$6.477\ \mathrm{GeV}^3$

Technical Details: Amplitude Projection

- Complete basis of 14 Lorentz structures for $J/\psi(P) \to \gamma(k_1)\gamma(k_2)\gamma(k_3)$:

$$
\begin{aligned}
\epsilon_{1,2,3} &= (\varepsilon \cdot \varepsilon_1)(\varepsilon_2 \cdot \varepsilon_3),\ (\varepsilon \cdot \varepsilon_2)(\varepsilon_1 \cdot \varepsilon_3),\ (\varepsilon \cdot \varepsilon_3)(\varepsilon_1 \cdot \varepsilon_2) \\
\epsilon_{4,5,6} &= (\varepsilon \cdot \varepsilon_1)(k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_3),\ \ldots \\
\epsilon_{7-12} &= (k_{1,2} \cdot \varepsilon)(\varepsilon_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_3),\ \ldots \\
\epsilon_{13,14} &= (k_{1,2} \cdot \varepsilon)(k_3 \cdot \varepsilon_1)(k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_3)
\end{aligned}
$$

- Gauge choice: $k_2 \cdot \varepsilon_1 = k_3 \cdot \varepsilon_2 = k_1 \cdot \varepsilon_3 = 0$
- $\det(G_{ij}) \propto \varepsilon^2 \Rightarrow$ first 12 structures form complete basis in $D = 4$

Computational Workflow

FeynArts $\rightarrow$ 6 tree + 48 one-loop + 894 two-loop diagrams

$\downarrow$

Dirac & color trace (private code)

$\downarrow$

CalcLoop $\rightarrow$ 6 + 96 integral families

$\downarrow$

Kira (IBP reduction) $\rightarrow$ Master Integrals

$\downarrow$

AMFlow (differential equation) $\rightarrow$ numerical MIs

$\downarrow$

Gauss-Kronrod G7-K15: $15 \times 15 = 225$ phase-space points

- Differential equations for MIs set up symbolically in s_1, s_2
- One boundary point computed with AMFlow, rest propagated via ODE solver
- Phase-space uncertainty $\sim 0.6\%$, validated against known LO and NLO results

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