Constructing the Brownian sphere from a random unicycle

Mathieu Mourichoux

Journée carte à Marne-la-vallée

Brownian geometry

Goal : Construct and study some "canonical random surfaces"

Brownian geometry

Goal : Construct and study some "canonical random surfaces"

More precisely, we want to consider random surfaces:

- that arise as the scaling limit of discrete models, i.e. planar maps
- that have some universality property

Plan of the talk

1. **A quick history of the Brownian sphere**

2. **Labelled unicycles and planar maps**

3. **Toward a new construction**

A bit of history

A quick history of the Brownian sphere

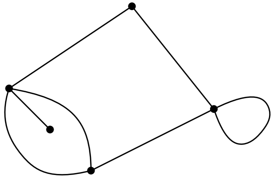
Planar maps

Definition

A planar map is the embedding of a finite graph $G = (V, E)$ in the sphere $\mathbb{S}^2$ so that :

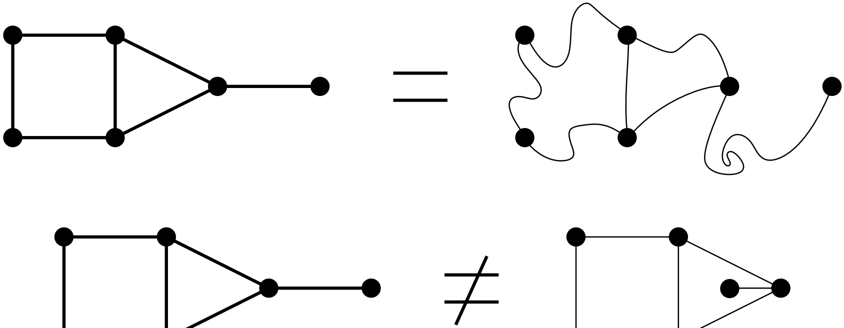
- The edges do not cross
- Each face is homeomorphic to the unit disk

where faces are the complements of the connected components of the embedding in $\mathbb{S}^2$.

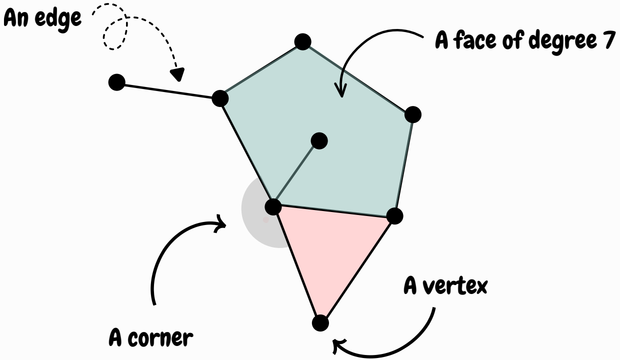


We consider planar maps up to homeomorphism of the sphere, and as metric spaces (equipped with the graph distance).

Planar maps



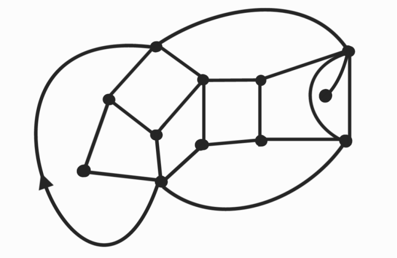
Planar maps



Quadrangulations

Definition

A quadrangulation is a planar map where each face has degree 4. We denote by $\mathcal{Q}_n$ the set of quadrangulations with n faces, and $\mathcal{Q}_n^\bullet$ the set of pointed quadrangulations with n faces.



General question

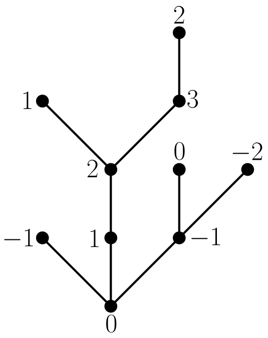
What is the typical behavior of a large planar map ?

Labelled tree

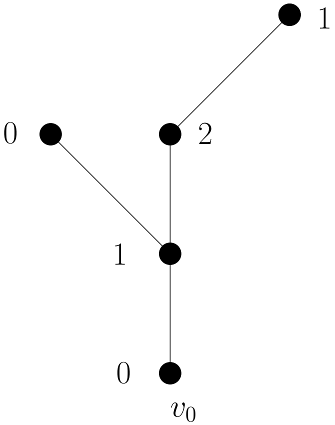
Fix a rooted planar tree $\mathbf{t}$ with root vertex u_0. An admissible labelling function on $\mathbf{t}$ is a function $\ell : V(\mathbf{t}) \to \mathbb{Z}$ such that

- $\ell(u_0) = 0$.
- $|\ell(u) - \ell(v)| \leq 1$ if u and v are neighbors.

Let $\mathbb{T}_n$ be the set of all pairs $(\mathbf{t}, \ell)$ of labelled tree with n edges.



The CVS bijection



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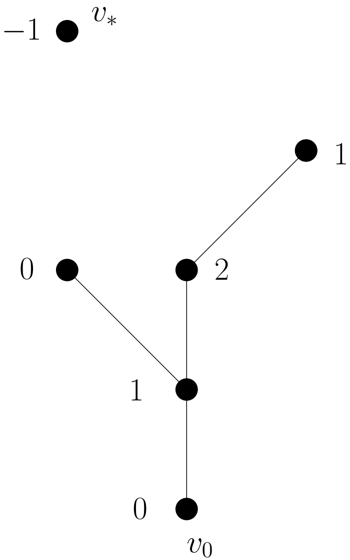

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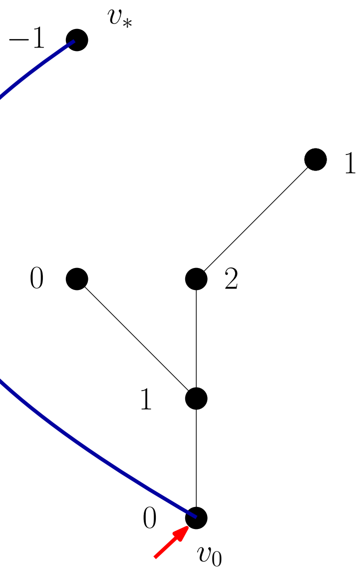

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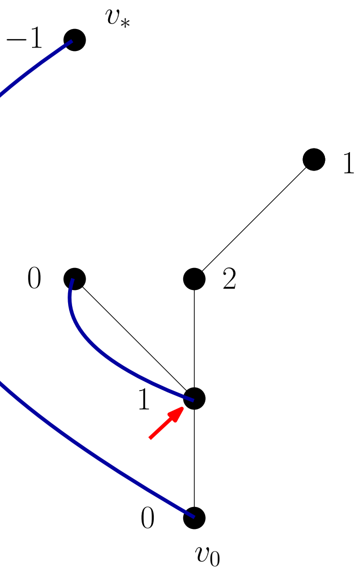

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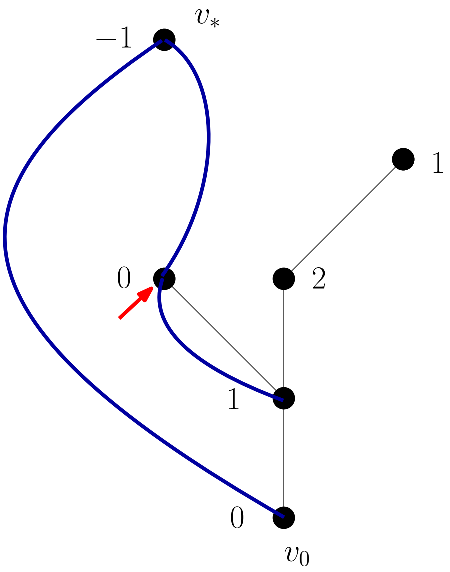

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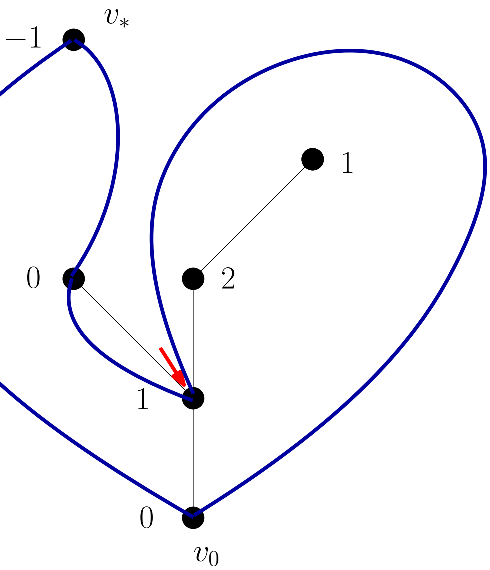

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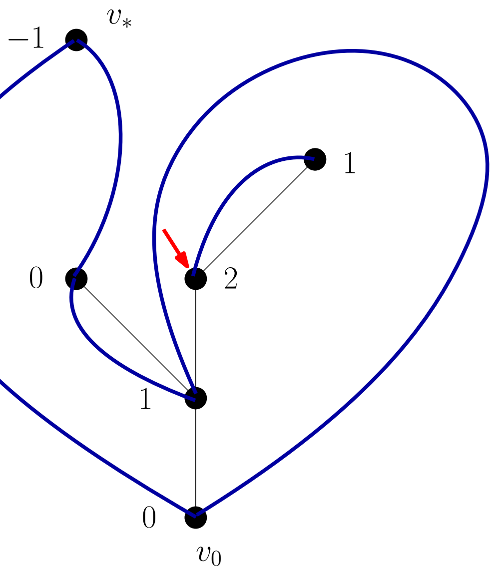

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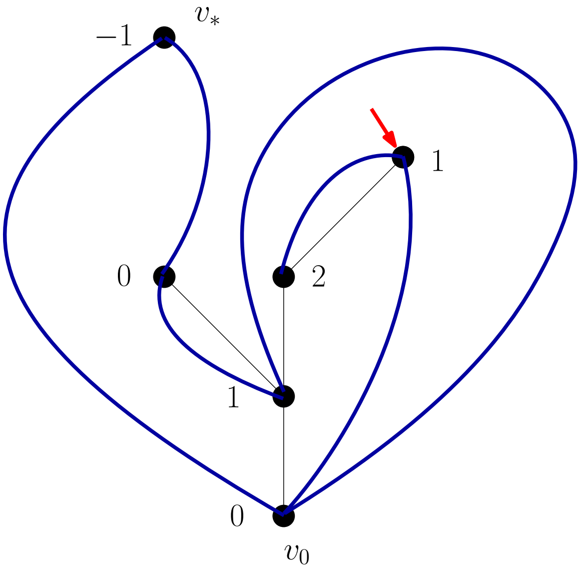

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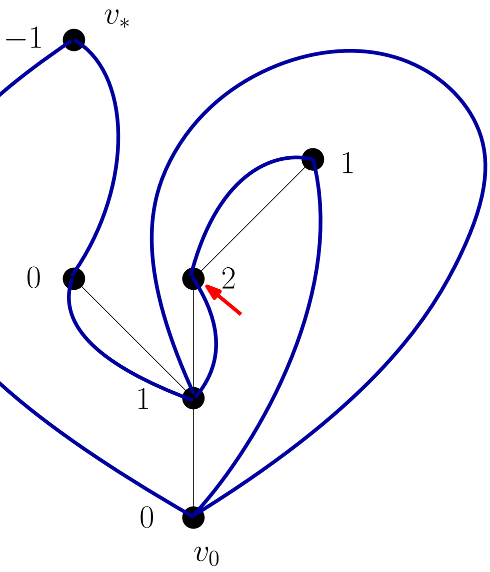

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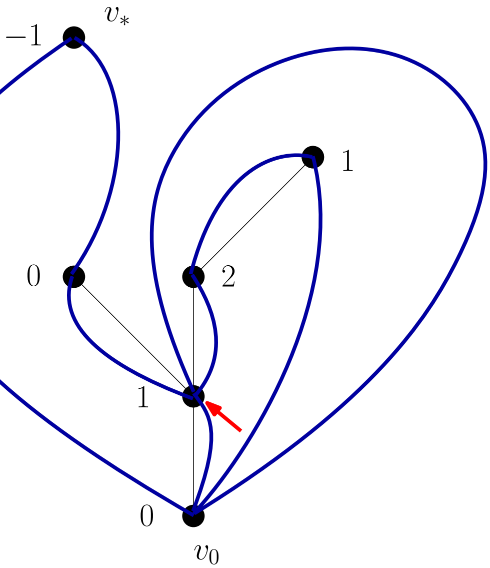

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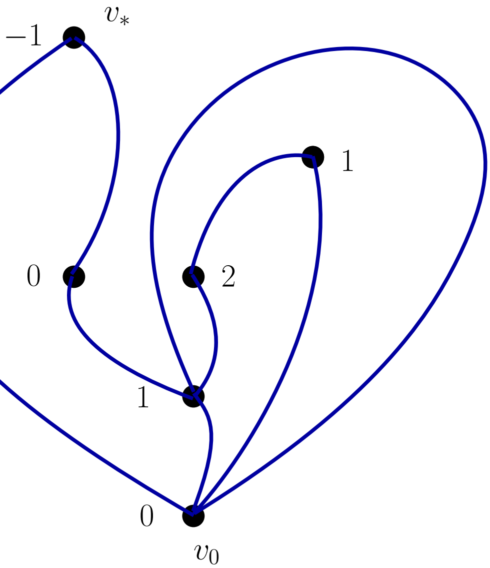

The CVS bijection

Theorem (Cori, Vauquelin, 1981 and Schaeffer, 1998)

The pointed graph $(\mathbf{q}, v_*)$ obtained by this procedure is a pointed quadrangulation with n faces. Moreover, it gives a bijection between $\mathbb{T}_n \times \{-1, 1\}$ and $\mathcal{Q}_n^\bullet$.

The CVS bijection

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The pointed graph $(\mathbf{q}, v_*)$ obtained by this procedure is a pointed quadrangulation with n faces. Moreover, it gives a bijection between $\mathbb{T}_n \times \{-1, 1\}$ and $Q_n^\bullet$.

To study random quadrangulations :

- Choose a random labelled tree with n edges and a uniform labelling (T_n, ℓ_n)
- Apply the CVS bijection
- Study the properties of the quadrangulation through the bijection

Some properties of the CVS bijection

Proposition

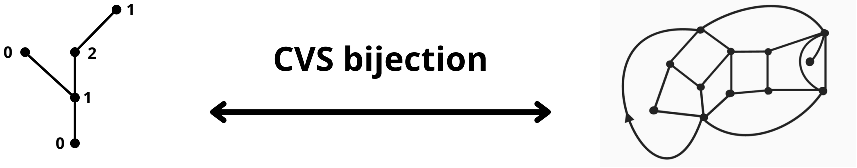
For every $v \in \mathbf{q}$, we have

$$d_{\mathbf{q}}(v, v_*) = \ell(v) - \ell(v_*).$$

Moreover, for every $u, v \in \mathbf{q}$,

$$d_{\mathbf{q}}(u, v) \geq |\ell(u) - \ell(v)|.$$

Towards a scaling limit



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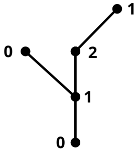
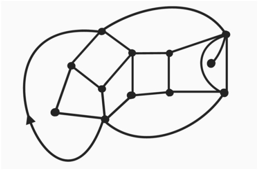
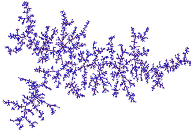


CVS bijection



Scaling limit



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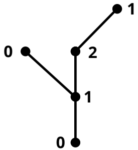
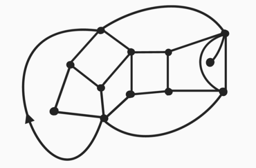
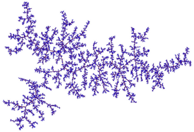


CVS bijection

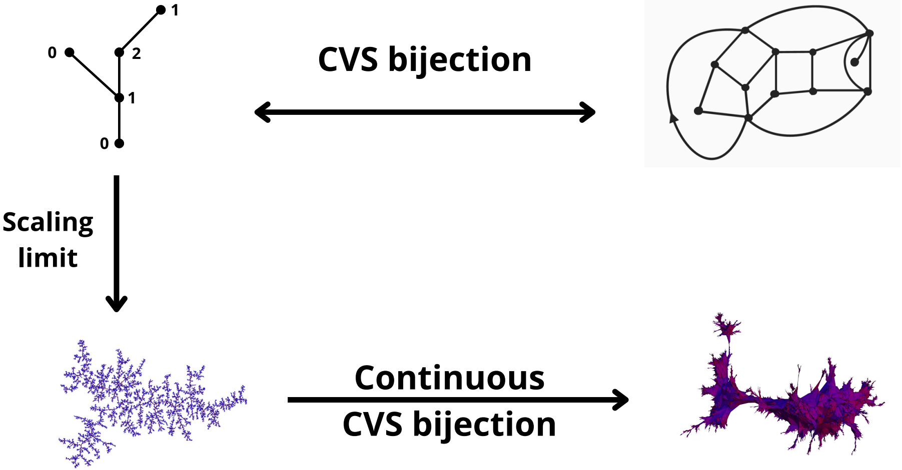


Scaling limit

(Aldous, Le Gall, Chassaing, Schaeffer...)



Towards a scaling limit



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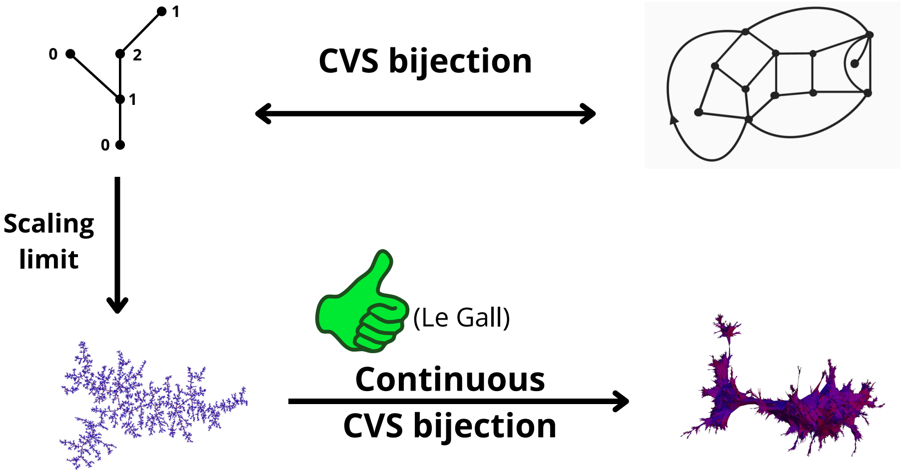

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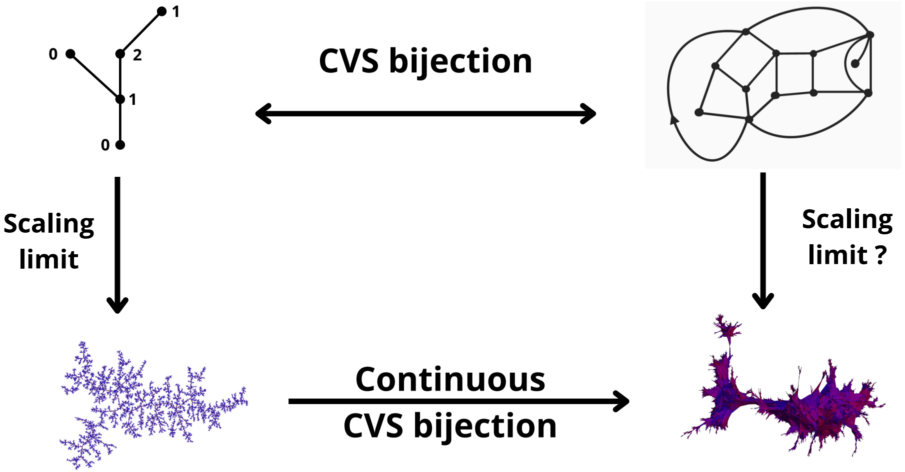

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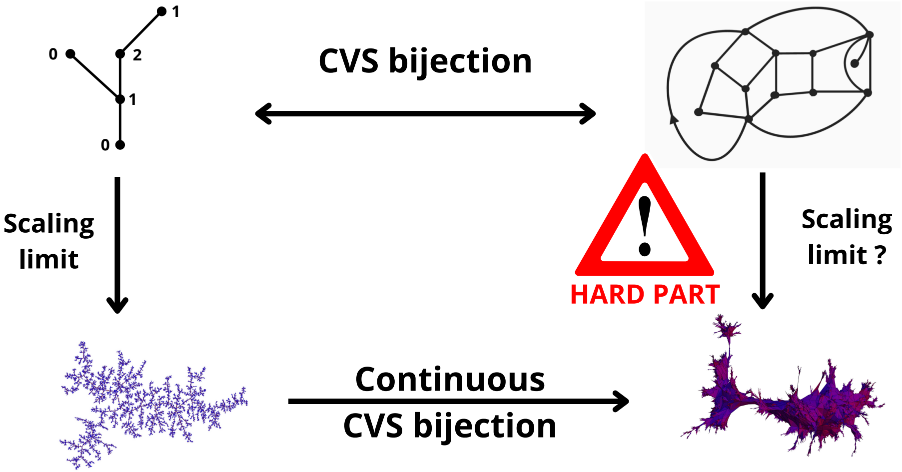


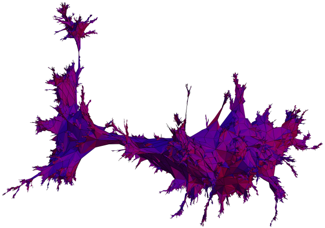
The Brownian sphere

Theorem (Le Gall, Miermont, 2011)

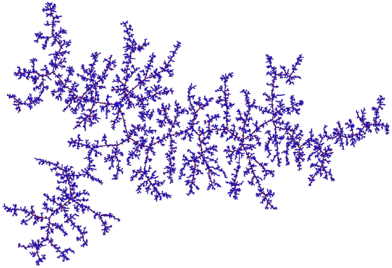
Let Q_n be a uniform random quadrangulation with n faces. There exists a random metric space $(\mathcal{S}, D)$ called the Brownian sphere such that

$$\left(Q_n, \left(\frac{9}{8n}\right)^{1/4} d_{gr}\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{S}, D)$$

for the Gromov-Hausdorff topology.



How to build the Brownian sphere



The Brownian sphere $(\mathcal{S}, D)$ is constructed as a quotient of a CRT (Continuum random tree) $\mathcal{T}$ with "uniform" labels, obtained with a "Continuous CVS bijection". This tree is obtained under a probability measure $\mathbb{N}_0^{(1)}$. It is also naturally equipped with a volume measure μ.

How to build the Brownian sphere

Let ℓ_* be the minimal label on the tree $\mathcal{T}$, and x_* the corresponding point in the Brownian sphere. For every $x \in \mathcal{S}$, we have

$$D(x, x_*) = \ell_x - \ell_*$$

Proposition (Le Gall-Paulin, Miermont, 2008)

Almost surely, the Brownian sphere is homeomorphic to the sphere $\mathbb{S}^2$ and has Hausdorff dimension 4.

Many other properties (Le Gall, Miermont, Curien, Bettinelli, Riera, Gwynne, Miller, Metz-Donnadieu ...)

Some problems

Very useful to :

- Do 1-point estimates
- Study geodesics toward a point

Not very useful to :

- Study distances toward several points simultaneously

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Question

Can we obtain a new construction of the Brownian sphere ?

Already obtained for other Brownian surfaces (Bettinelli-Miermont, Le Gall, Caraceni-Curien, Le Gall-Riera ...)

Here we go again

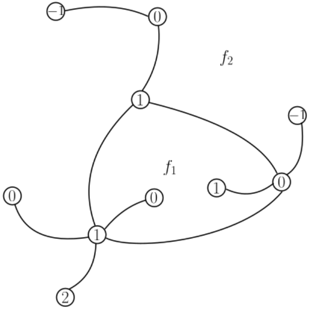
Labelled unicycles

and planar maps

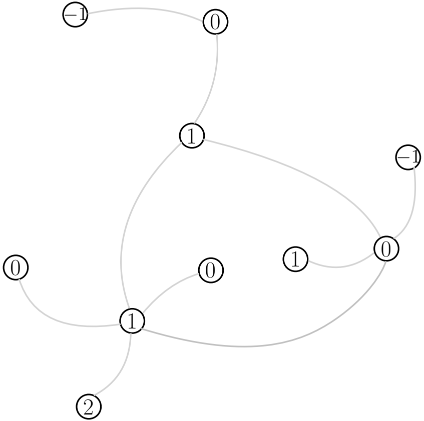
The unicycles appear

Definition

A **unicycle** is a planar map with two faces (labelled f_1 and f_2). A **well-labelled unicycle** is a unicycle equipped with a labelling function.



Miermont bijection



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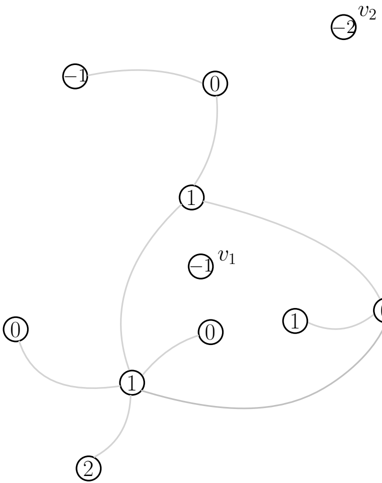

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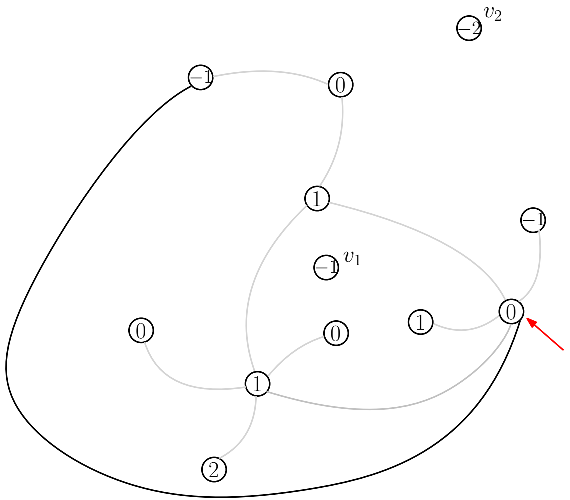

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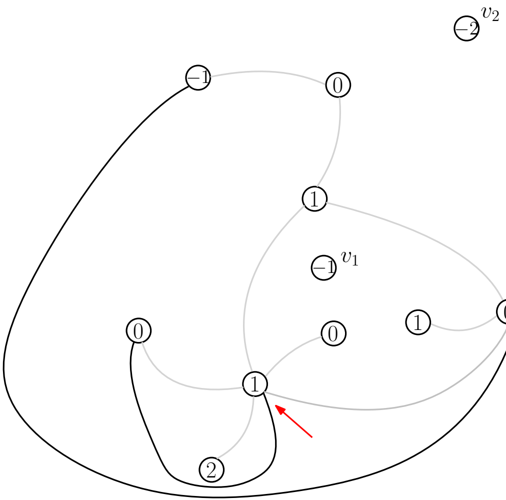

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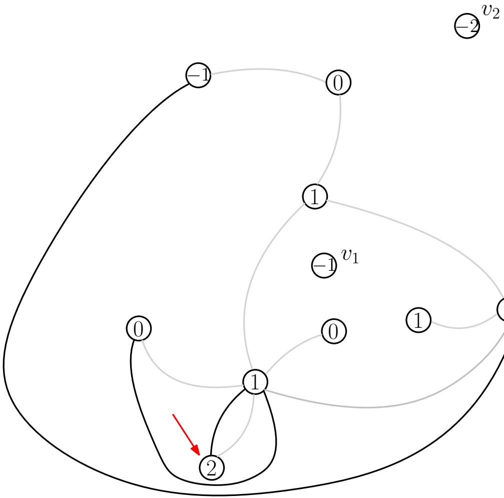

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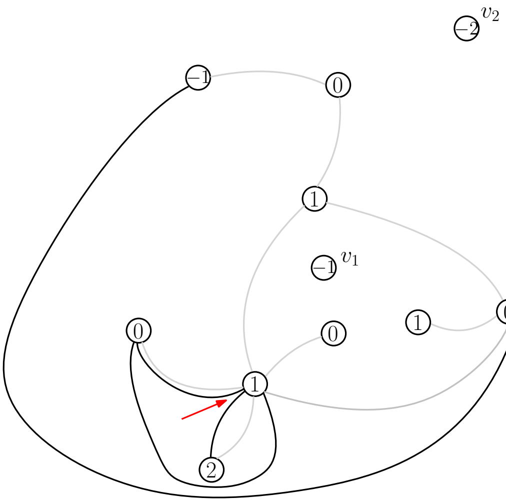

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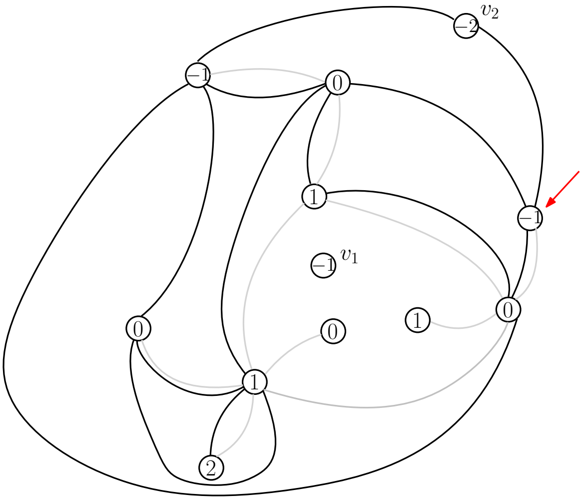

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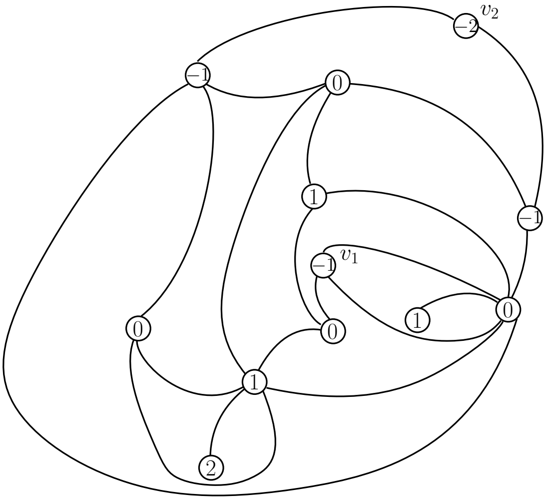
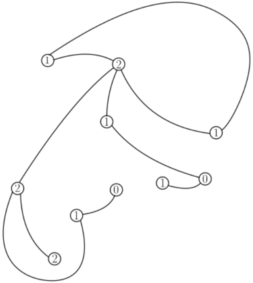
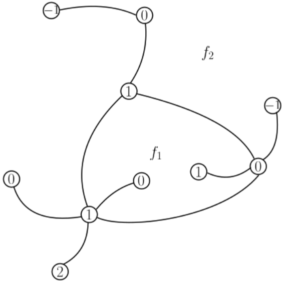


Sadness

Not a bijection
with quadrangulations





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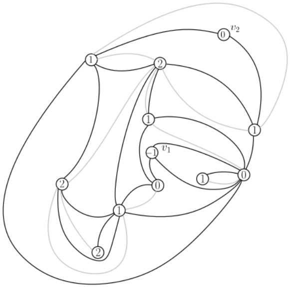
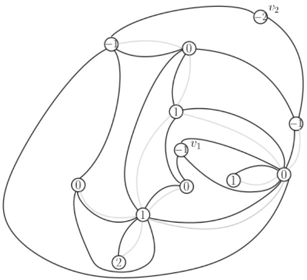




Keep track of the delay

Then for every $u \in \mathbf{q}$, we have

$$\ell_u = \min(d_{\mathbf{q}}(u, v_1) + \ell_{v_1}, d_{\mathbf{q}}(u, v_2) + \ell_{v_2}).$$

Therefore, we are interested in $\delta = \ell_{v_2} - \ell_{v_1}$.

Definition

A **delayed quadrangulation** is a tuple $(\mathbf{q}, v_0, v_1, \delta)$, where :

- $\mathbf{q}$ is a rooted quadrangulations,
- v_1 and v_2 are two distinct vertices of $\mathbf{q}$,
- δ is an integer with the same parity as $d_{\mathbf{q}}(v_1, v_2)$ such that $|\delta| < d_{\mathbf{q}}(v_1, v_2)$.

It's a bijection !

Theorem (Miermont, 2007)

The previous procedure is a bijection between the set of well-labelled unicycle with n edges and the set of delayed quadrangulations with n faces.

Remark : this is a particular case of a more general result.

Properties of the Miermont bijection

Proposition

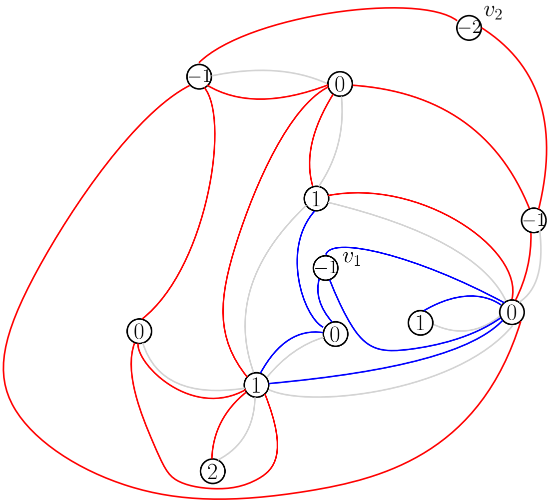
For every $u \in f_1$, we have

$$d_{\mathbf{q}}(u, v_1) = \ell_u - \ell_{v_1}$$

whereas for every $u \in f_2$, we have

$$d_{\mathbf{q}}(u, v_2) = \ell_u - \ell_{v_2}.$$

Properties of the Miermont bijection

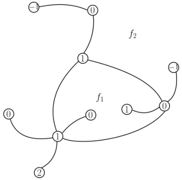


Last but not least

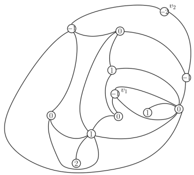
Toward a

new construction

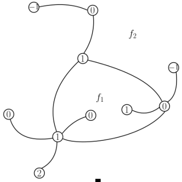
Towards a scaling limit ... again



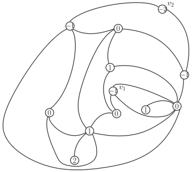
Miermont bijection



Towards a scaling limit ... again



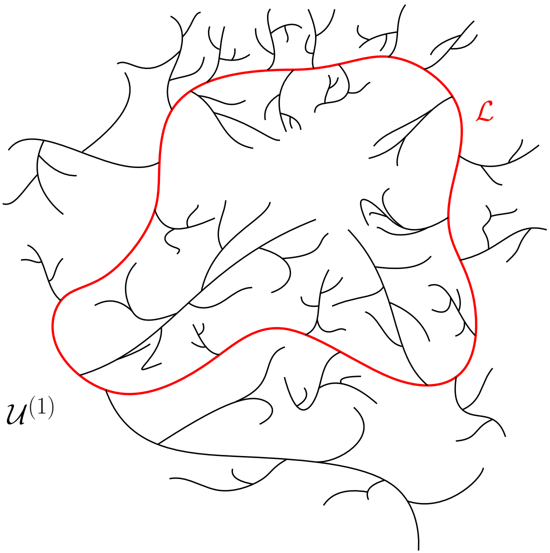
Miermont bijection



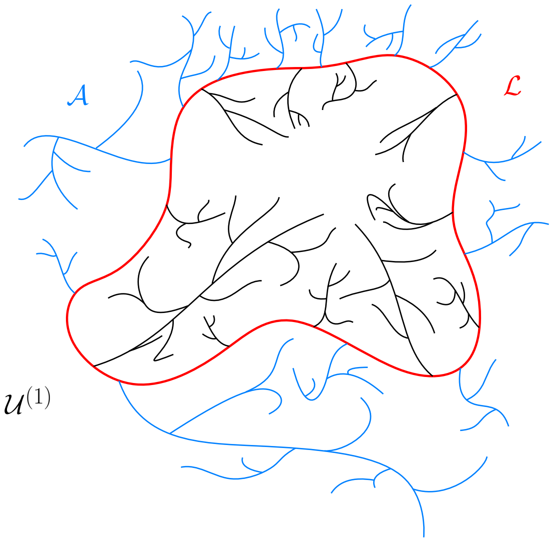
Scaling limit

The continuous labelled unicycle

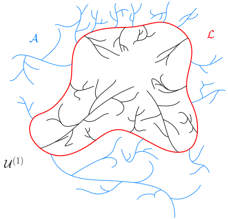
The continuous labelled unicycle



The continuous labelled unicycle



An explicit density



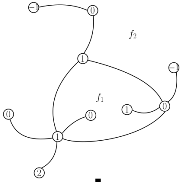
Proposition

The joint law of $(\mathcal{L}, \mathcal{A})$ *has density*

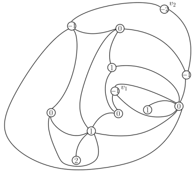
$$\mathbf{1}_{\{(x,y)\in\mathbb{R}_+\times[0,1]\}}\frac{1}{2^{1/4}\Gamma(1/4)\sqrt{\pi}}\frac{x^{1/2}}{(y(1-y))^{3/2}}\exp\left(-\frac{x^2}{2y(1-y)}\right).$$

- The labels on the cycle evolve as a Brownian excursion of duration $\mathcal{L}$
- Every subtree is a labelled CRT with a random volume

Towards a scaling limit … again



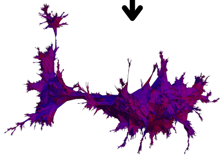
Miermont bijection ⟷



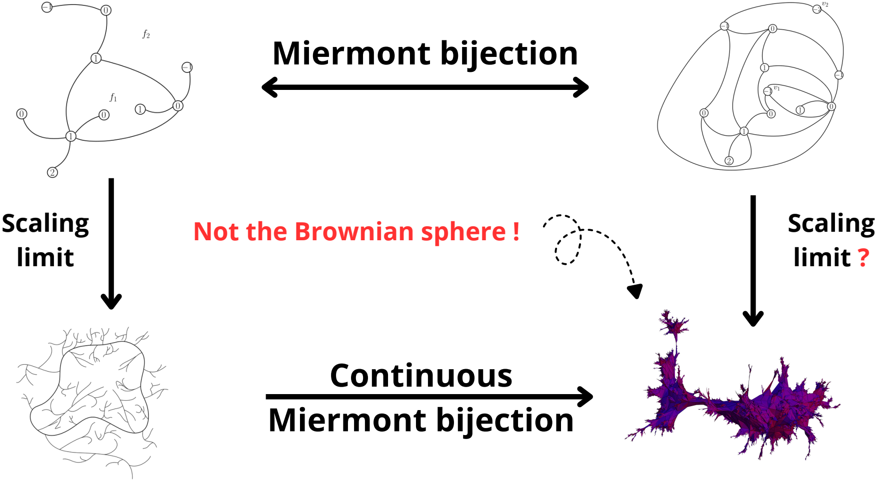
Scaling limit ↓

Scaling limit ? ↓

Continuous Miermont bijection ⟶



Towards a scaling limit ... again



The biased Brownian sphere

Let $(\mathcal{S}, D, x_*, \overline{x}_*, \mu)$ be a standard Brownian sphere $(\mathcal{S}, D, \mu)$ with two distinguished points sampled according to the measure μ.

Definition

The biased Brownian sphere $(\mathcal{S}_b, D, x_*, \overline{x}_*, \mu, \Delta)$ is defined by the following formula, valid for every non-negative measurable function f :

$$\mathbb{E}[f(\mathcal{S}_b, D, x_*, \overline{x}_*, \mu, \Delta)] = \frac{1}{2\mathbb{E}[D(x_*, \overline{x}_*)]}\mathbb{E}\left[\int_{-D(x_*,\overline{x}_*)}^{D(x_*,\overline{x}_*)} f(\mathcal{S}, D, x_*, \overline{x}_*, \mu, t)dt\right].$$

- The marginal $(\mathcal{S}_b, x_*, \overline{x}_*)$ has the law of the standard Brownian sphere biased by the distance between x_* and $\overline{x}_*$
- Conditionally on $(\mathcal{S}_b, x_*, \overline{x}_*)$, the random variable Δ is uniform on $[-D(x_*, \overline{x}_*), D(x_*, \overline{x}_*)]$

Constructing the biased Brownian sphere

Let $(Q_n^{(b)}, d_{gr}, x_n^{(b)}, y_n^{(b)}, \Delta_n)$ be uniform random delayed quadrangulation with n faces.

Theorem (M, 2025)

We have the convergence

$$\left(Q_n^{(b)}, \left(\frac{9}{8n}\right)^{1/4} d_{gr}, x_n^{(b)}, y_n^{(b)}, \left(\frac{9}{8n}\right)^{1/4} \Delta_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{S}_b, D, x_*, \overline{x}_*, \Delta)\,.$$

Moreover, the biased Brownian sphere has an explicit construction as a quotient of the labelled unicycle $\mathcal{U}^{(1)}$.

More precisely, we can define a pseudo-distance D on $\mathcal{U}^{(1)}$ such that

$$\mathcal{S}_b = \mathcal{U}^{(1)}/\{D = 0\}.$$

Where is the delay ?

Properties of the distance

Proposition

For every $x \in f_1$, we have

$$D(x, x_*) = \ell_x - \ell_*,$$

whereas for every $x \in f_2$, we have

$$D(x, \overline{x}_*) = \ell_x - \overline{\ell}_*.$$

In particular,

$$D(x_*, \overline{x}_*) = -(\ell_* + \overline{\ell}_*).$$

Moreover, for every x, y, we have

$$D(x, y) \geq |\ell_x - \ell_y|.$$

Sketch of the proof

Step 1 : Use the convergence of uniform quadrangulations towards the Brownian sphere.

For every $n \in \mathbb{N}$

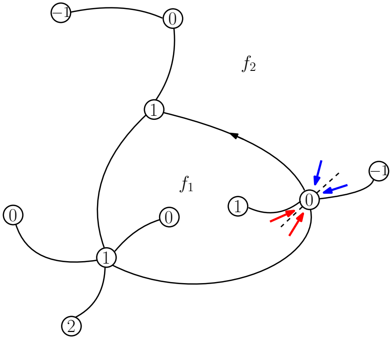
$$\mathbb{E}\left[F(Q_n^{(b)}, x_n^{(b)}, y_n^{(b)}, \Delta_n)\right] = \frac{\mathbb{E}\left[\sum_{\delta_n \in A(Q_n, x_n, y_n)} F(Q_n, d_{gr}, x_n, y_n, \delta_n)\right]}{\mathbb{E}\left[\left(d_{Q_n}(x_n, y_n) - 1\right)_+\right]},$$

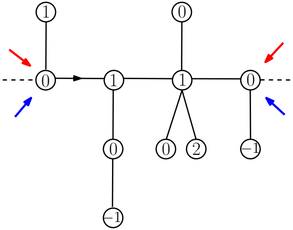
where (Q_n, x_n, y_n) is uniformly distributed on the set of bi-pointed quadrangulations with n faces and $A(Q, x, y)$ is the set of admissible delays of (Q, x, y).

$\longrightarrow$ We just need to control the Radon-Nikodym derivative !

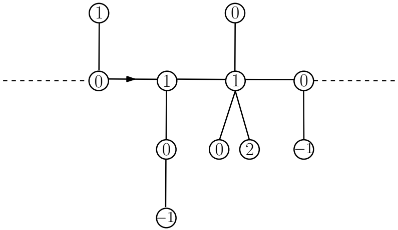
Sketch of the proof

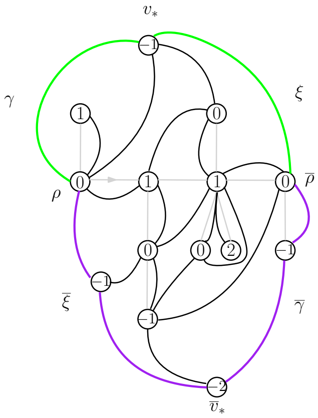
Step 2 : Look for a link with another model for which the scaling limit is known.





Sketch of the proof

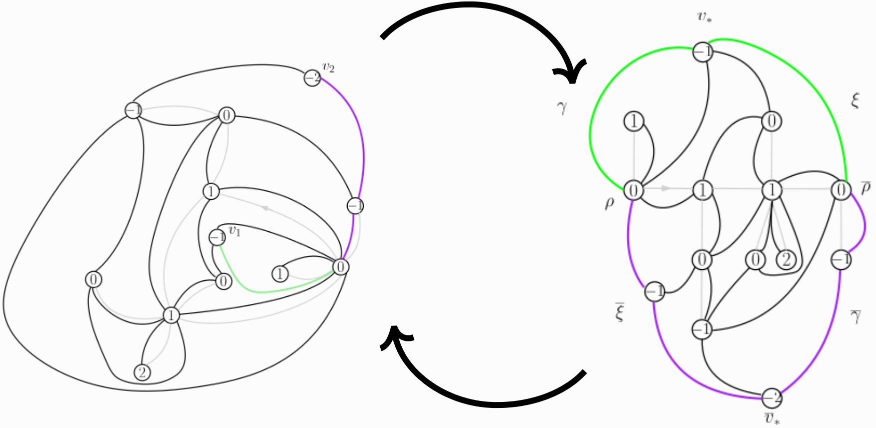




⟶ The scaling limit of quadrangle with geodesic sides Q is known !
(Bettinelli-Miermont)

How to relate delayed quadrangulations and quadrangles

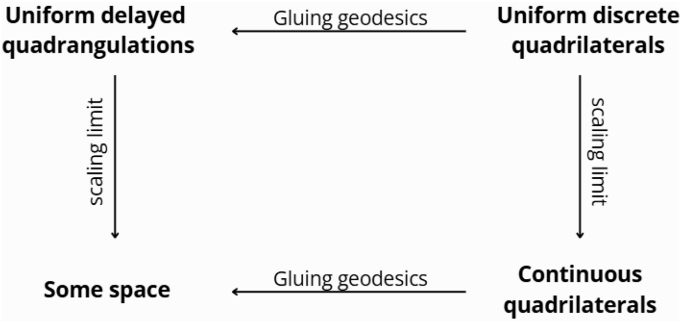
Cut alongside both geodesics



Glue the geodesics by pair

Another diagram

Step 3 : use this link to obtain a scaling limit, he showed that it can be constructed as a quotient of $\mathcal{U}^{(1)}$.



The diagram commutes because the gluing of geodesics is a continuous operation ! (Bettinelli-Miermont)

Delayed Voronoï cells

For any metric space (X, d) with two distinguished points x and y, and any parameter $\delta \in (-d(x, y), d(x, y))$, we define

$$\Theta_\delta := \{z \in X : d(x, z) \leq d(y, z) + \delta\},$$
$$\overline{\Theta}_\delta := \{z \in X : d(x, z) \geq d(y, z) + \delta\}.$$

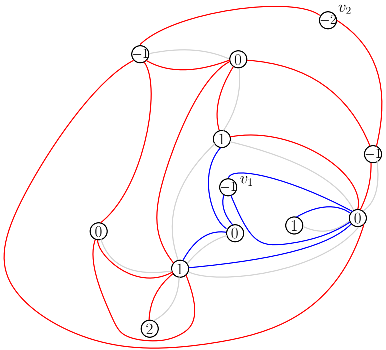
Definition

The sets Θ_δ and $\overline{\Theta}_\delta$ are called the δ-delayed Voronoï cells of X with respect to x and y.

Delayed Voronoï cells in the Brownian sphere

Proposition

The faces of $\mathcal{U}^{(1)}$ exactly correspond to the Δ-Voronoï cells of $\mathcal{S}_b$, and the cycle C correspond to the boundary of these cells.



Delayed Voronoï cells in the Brownian sphere

Proposition

The faces of $\mathcal{U}^{(1)}$ *exactly correspond to the* Δ*-Voronoï cells of* $\mathcal{S}_b$*, and the cycle* C *correspond to the boundary of these cells.*

Corollary

Let $\mu(\Theta_\Delta)$ *and* $\mathcal{P}(\Theta_\Delta)$ *stand for the volume and the perimeter of* Θ_Δ*. Then, the law of* $(\mathcal{P}(\Theta_\Delta), \mu(\Theta_\Delta))$ *has density*

$$\mathbf{1}_{\{(x,y)\in\mathbb{R}_+\times[0,1]\}} \frac{1}{2^{1/4}\Gamma(1/4)\sqrt{\pi}} \frac{x^{1/2}}{(y(1-y))^{3/2}} \exp\left(-\frac{x^2}{2y(1-y)}\right).$$

In particular, $\mu(\Theta_\Delta)$ *is a* $\mathrm{Beta}(1/4, 1/4)$ *random variable.*

Getting rid of the bias

Recall that

$$D(x_*, \overline{x}_*) = -(\ell_* + \overline{\ell}_*) \quad \text{and} \quad \Delta = \overline{\ell}_* - \ell_*.$$

Therefore,

$$\text{Conditioning on } \Delta \text{ and } D(x_*, \overline{x}_*) \longleftrightarrow \text{Conditioning on } \ell_* \text{ and } \overline{\ell}_*$$

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Therefore,

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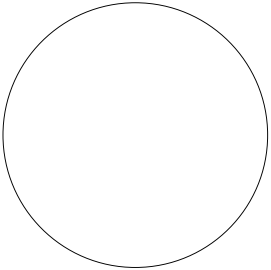
Problem : I cannot even compute $\mathbb{P}(\ell_* < -a, \overline{\ell}_* < -b)$ ...

Unicycle with free volume

- Sample the length of the cycle with the infinite measure $ct^{-3/2}dt$
- The labels on the cycle evolve as a Brownian excursion **e** with the appropriate duration

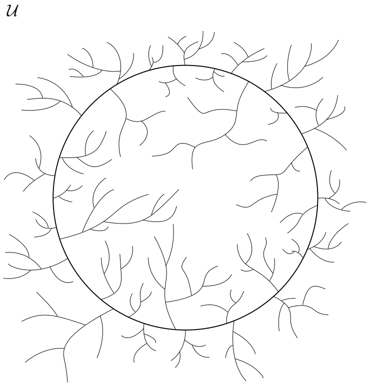
$\mathcal{U}$



Unicycle with free volume

- Conditionnaly on **e**, the forests on each side are distributed as independent Poisson point measures with intensity

$$2\mathbf{1}_{[0,\sigma]}(t)dt\mathbb{N}_{\mathbf{e}_t}(dW).$$



Properties of the free biased Brownian sphere

Let $\mathbb{U}$ be the (infinite) measure used to construct $\mathcal{U}$.

Proposition

For every $a > 0$, we have

$$\mathbb{U}(\mu(\mathcal{U}) > a) = \frac{\Gamma(1/4)}{(2a)^{1/4}\pi}.$$

Moreover, the unicycle $\mathcal{U}$ ***conditioned to have*** *$\mu(\mathcal{U}) = 1$ has the law of $\mathcal{U}^{(1)}$.*

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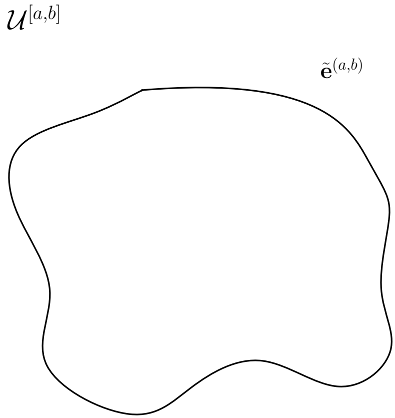
Proposition

For every $a, b > 0$, we have

$$\mathbb{U}(\ell_* < -a, \overline{\ell}_* < -b) = \frac{1}{a+b}.$$

Fixing ℓ_* and $\overline{\ell}_*$



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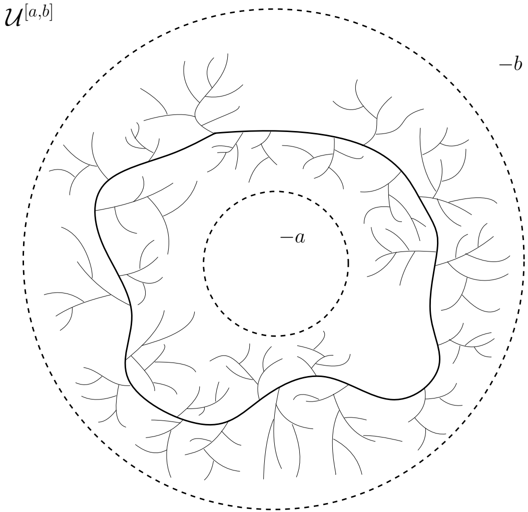
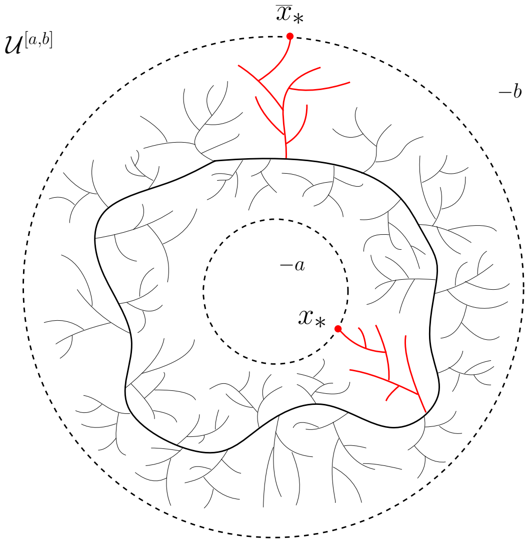


Fixing ℓ_* and $\overline{\ell}_*$



Weird but explicit processes

The law of the random variable $\tilde{\mathbf{e}}^{(a,b)}$ is absolutely continuous with respect to Ito's excursion measure $\mathbf{n}(d\mathbf{e})$, with density

$$18(a+b)^3\left(\int_0^\sigma\int_0^\sigma \frac{dsdt}{(\mathbf{e}_s+a)^3(\mathbf{e}_t+b)^3}\right)\exp\left(-3\int_0^\sigma\left(\frac{1}{(\mathbf{e}_u+a)^2}+\frac{1}{(\mathbf{e}_u+b)^2}\right)du\right).$$

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$$18(a+b)^3 \underbrace{\left(\int_0^\sigma \int_0^\sigma \frac{dsdt}{(\mathbf{e}_s+a)^3(\mathbf{e}_t+b)^3}\right)}_{\text{where to graft the two atoms}} \underbrace{\exp\left(-3\int_0^\sigma \left(\frac{1}{(\mathbf{e}_u+a)^2}+\frac{1}{(\mathbf{e}_u+b)^2}\right)du\right)}_{\text{conditioning the two Poisson point measures}}.$$

A disintegration formula

Proposition

Let F and G be measurable non-negative functions. Then

$$\mathbb{U}\left(F(-\ell_*, -\overline{\ell}_*)G(\mathcal{U})\right) = \int_{\mathbb{R}_+\times\mathbb{R}_+} F(a,b)\frac{2}{(a+b)^3}\mathbb{E}\left[G(\mathcal{U}^{(a,b)})\right]\mathrm{d}a\mathrm{d}b.$$

Hence, the unicycle $\mathcal{U}$ ***conditioned to have*** *$\ell_* = -a$* ***and*** *$\overline{\ell}_* = -b$ has the law of $\mathcal{U}^{(a,b)}$.*

The new construction, finally

Theorem (M, 2025)

For every $a, b > 0$, the random surface associated to $\mathcal{U}^{(a,b)}$ is distributed as a free Brownian sphere $\mathcal{S}$ with two distinguished points at distance $a + b$. Moreover, the faces of $\mathcal{U}^{(a,b)}$ correspond to the $(a - b)$-delayed Voronoï cells of $\mathcal{S}$.

Applications and perspectives

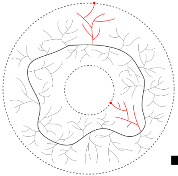
A couple of applications :

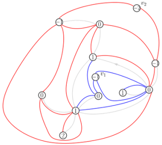
- Explicit distributions of several quantities related to Voronoï cells
- Study of the local behaviour around a geodesic

In the future :

- More results about Voronoï cells
- Explicit distributions for the Brownian annulus ?
- Study of exceptional delays ?

The End





Thank you for your attention !