Crossings numbers.

Alfredo Hubard

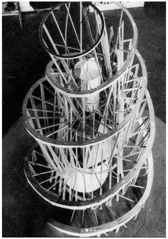
Université Gustave Eiffel

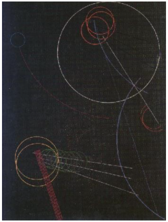
2026

Anthony Hill (1930–2020)

Anthony Hill (circa 1962): highly regarded by connoisseurs of abstraction Credit: ©Anthony Hill Estate

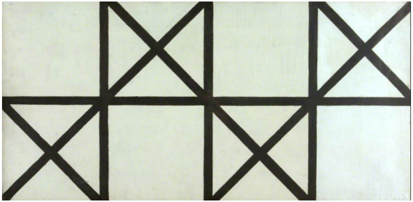
Anthony Hill was the founder of British Constructivism that built on the legacy of the Russian avant-garde.

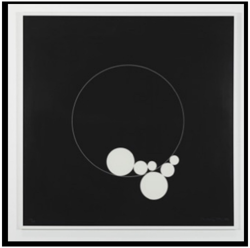




Hill wanted to counter the subjective and emotional aesthetics dominant in postwar British art,

with rationality, objectivity, economy of means, and rigorous constructions.





One day he showed up at University College London with a problem and a conjectural solution:

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \sim \frac{n^4}{64}$$

336 F. HARARY AND A. HILL

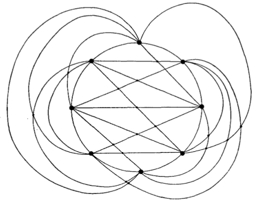


FIG. 4

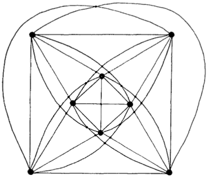
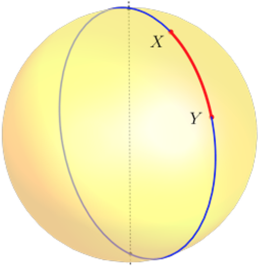
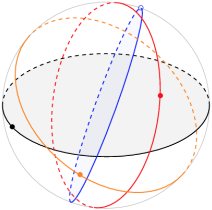
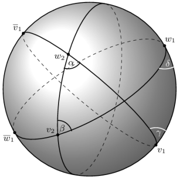


FIG. 5

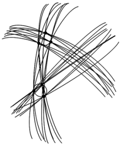


X
Y





$\overline{v}_1$
w_1
w_2
α
δ
$\overline{w}_1$
v_2
β
γ
v_1



Theorem

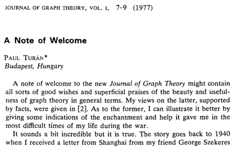
Hill's conjecture holds for the following families of drawings:

1. *Bi-shellable (Abrego, Aichholzer, Fernández-Merchant, McQuillan, Mohar, Mutzel, Ramos, Richter, and Vogtenhuber in 2018)*
2. *Spherically geodesic (Streltsova and Wagner in 2025)*
3. *Spherically antipodal (Fradelizi, H, Ndiaye and Sole-Pi)*

Bi-shellable generalizes shellable, 2-page,monotone and rectilinear. Our result generalizes to maps from the d-skeleton of an $(n-1)$-simplex to $\mathbb{S}^{2d}$

The Harary-Hill paper was not the first one on crossing numbers. In a labor camp, working at a brick factory, Paul Turan had come up with the bipartite version and Zarankiewicz with a construction and an incorrect proof of its optimality. Now known as Zarankiewicz conjecture:

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$



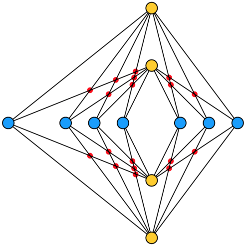
JOURNAL OF GRAPH THEORY, VOL. I, 7–9 (1977)

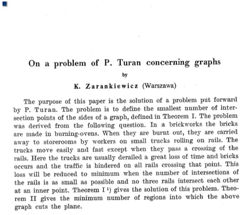
A Note of Welcome

PAUL TURÁN*
Budapest, Hungary

A note of welcome to the new *Journal of Graph Theory* might contain all sorts of good wishes and superficial praises of the beauty and usefulness of graph theory in general terms. My views on the latter, supported by facts, were given in [2]. As to the former, I can illustrate it better by giving some indications of the enchantment and help it gave me in the most difficult times of my life during the war.

It sounds a bit incredible but it is true. The story goes back to 1940 when I received a letter from Shanghai from my friend George Szekeres





On a problem of P. Turan concerning graphs

by

K. Zarankiewicz (Warszawa)

The purpose of this paper is the solution of a problem put forward by P. Turan. The problem is to define the smallest number of intersection points of the sides of a graph, defined in Theorem I. The problem was derived from the following question. In a brickworks the bricks are made in burning-ovens. When they are burnt out, they are carried away to storerooms by workers on small trucks rolling on rails. The trucks move easily and fast except when they pass a crossing of the rails. Here the trucks are usually derailed a great loss of time and bricks occurs and the traffic is hindered on all rails crossing that point. This loss will be reduced to minimum when the number of intersections of the rails is as small as possible and no three rails intersect each other at an inner point. Theorem I 1) gives the solution of this problem. Theorem II gives the minimum number of regions into which the above graph cuts the plane.

A third beggining for crossing numbers was Frank Leighton's PhD thesis:

Complexity Issues in Very Large Scale Integration (VLSI) Optimal Layouts for the Shuffle-Exchange Graph and Other Networks

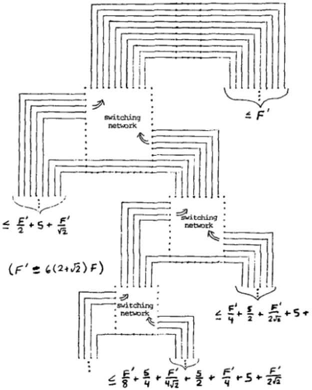
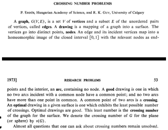


Figure 5: *Embedding an arbitrary network in the tree of meshes.*

Erdős-Guy conjectures


CROSSING NUMBER PROBLEMS

P. Erdős, Hungarian Academy of Science, and R. K. Guy, University of Calgary

A **graph**, $G(V,E)$, is a set V of **vertices** and a subset E of the unordered pairs of vertices, called **edges**. A **drawing** is a mapping of a graph into a surface. The vertices go into distinct points, **nodes**. An edge and its incident vertices map into a homeomorphic image of the closed interval $[0,1]$ with the relevant nodes as end-

1973] RESEARCH PROBLEMS 53

points and the interior, an **arc**, containing no node. A **good** drawing is one in which no two arcs incident with a common node have a common point; and no two arcs have more than one point in common. A common point of two arcs is a **crossing**. An **optimal** drawing in a given surface is one which exhibits the least possible number of crossings. Optimal drawings are good. This least number is the **crossing number** of the graph for the surface. We denote the crossing number of G for the plane (or sphere) by $\nu(G)$.

Almost all questions that one can ask about crossing numbers remain unsolved.

c

Conjecture

1. *There exist constants $a, b > 0$ such that every graph with $e \geq an$ edges satisfies*

$$\mathrm{cr}(G) \geq b\frac{e^3}{n^2}$$

2. *There exist a constant c such that if $n = o(e)$ and $e = o(n^2)$ then*

$$\lim_{n\to\infty} \min_{G:n,e} (\mathrm{cr}(G))\frac{n^2}{e^3} = c$$

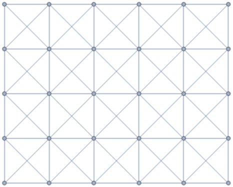
Ajtai-Chvátal-Newborn-Szemerédi and Leighton confirmed the first. Pach-Spencer-Toth confirmed the second.

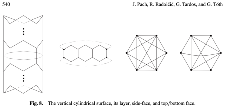
The crossing number inequality

Theorem

- *Every graph with* $e \geq 4n$ *edges satisfies* $\mathrm{cr}(G) \geq \frac{1}{64}\frac{e^3}{n^2}$
- *Every graph with* $e \geq 4.5n$ *edges satisfies* $\mathrm{cr}(G) \geq \frac{1}{61}\frac{e^3}{n^2}$
- *Every graph with* $e \geq 6.7n$ *edges satisfies* $\mathrm{cr}(X) \geq \frac{1}{27.48}\frac{e^3}{n^2}$ *(Büngener and Kaufmann, 2024).*

This is why is sensible to look at the limit when $n \to \infty$ and $n = o(e)$.





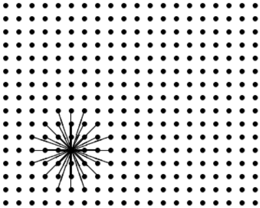
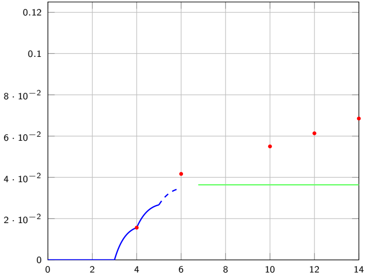
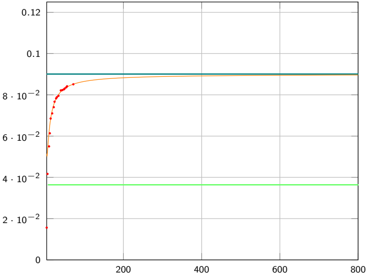


Figure 4.

$$b(a) = \lim_{n \to \infty} \min_{\substack{G:\,|V(G)|=n \\ |E(G)| \geq an}} \operatorname{cr}(G)\,\frac{n^2}{e^3}$$





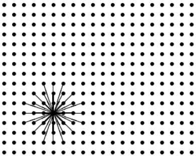
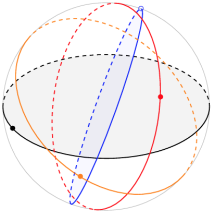
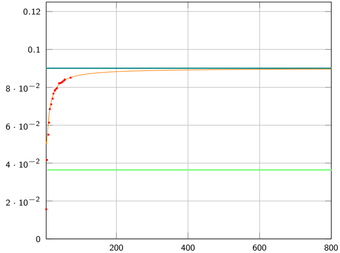
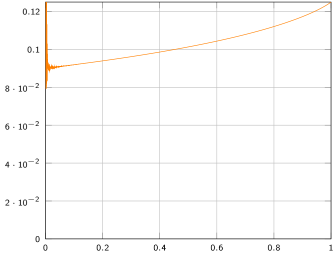


Figure 4.



$$b^{+}(\alpha) = \lim_{n\to\infty} \min_{\substack{G:\,|V(G)|=n \\ |E(G)|\geq\alpha\binom{n}{2}}} \mathrm{cr}(G)\,\frac{n^2}{e^3}$$





Czabarka, Singih, Székely, and Wang showed $\lim_{\alpha\to 0} b(a) = \frac{8}{9\pi^2}$

Theorem

For every 1-dimensional simplicial complex with n vertices and $e \geq an$ edges, any continuous map into $\mathbb{S}^2$ incurs into $b\frac{e^3}{n^2}$ crossings.

- What happens if we consider more general surfaces as target space?
- What happens if we go up in dimension?
- What happens if we consider more general graphs, i.e. not simplicial complexes (multi-graphs) in the domain?

What if we draw in a surface of genus g?

Theorem (Shahrokhi, Sýkora, Székely and Vr'to 1990s)

- $\mathrm{cr}_g(G) \geq c_g \frac{e^3}{n^2}$ *provided* $\max(64g, 8n) \leq e \leq \frac{n^2}{g}$.
- $\mathrm{cr}_g(G) \geq c_g \frac{e^2}{g}$ *provided* $e \geq \max(64g, 8n, \frac{n^2}{g})$

For the complete graph they showed that if $n^2 > 128g$ then

$$c\frac{n^4}{g} \leq \mathrm{cr}_g(K_n) \leq c'\frac{n^4}{g}\log^2 g$$

and conjectured the lower bound to be correct.

Joint with Arnaud de Mesmay and Hugo Parlier(2025)

For the complete graph they showed:

$$c\frac{n^4}{g} \leq \mathrm{cr}_g(K_n) \leq c'\frac{n^4}{g}\log^2 g$$

and conjectured the lower bound to be the correct order of growth.

Theorem

Let G be a graph with n vertices and e edges and let g be an integer larger than 2. For any $\varepsilon > 0$, there exists a constant $B(\varepsilon)$ such that if $e \geq 10^6 \max(n^{3/2}, ng, \frac{n^2}{g^{1/2-\varepsilon}})$, then

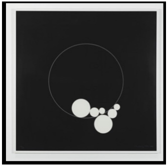
$$\mathrm{cr}_g(G) \geq B(\varepsilon)e^2\frac{\log^2 g}{g}.$$

Joint with Arnaud de Mesmay and Hugo Parlier(2025)

Theorem

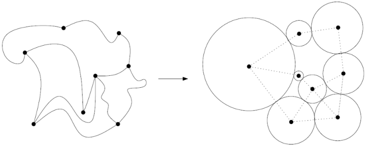
There exists a family of hyperbolic surfaces S_k of genus $g_k \to \infty$ such that for any $n > 0$, sampling n points uniformly at random on S_k and connecting them with shortest paths yields a drawing of K_n on S_k with $O(n^4 \frac{\log^2 g_k}{g_k})$ crossings in expectation.

Joint with Arnaud de Mesmay and Hugo Parlier(2025)



Let us show that for g large and n much larger

$$\mathrm{cr}_g(K_n) \geq c\frac{n^4}{g}\log^2 g$$



Joint with Hugo Parlier(2024)

What happens if we consider non-simplicial graphs (multi-graphs)?
A:There exist multi-graphs with $n = 2$, e arbitrary and $\mathrm{cr}(G) = 0$.

Pach-Toth 2018 idea: consider only drawings in which no two edges are homotopic.

A family of **simple** arcs (or closed curves) is a k-system they are pairwise non-homotopic and intersect pairwise at most k-times.

Theorem

For each $k > 0$ there exists c_k, such that if $\mathcal{A}$ is a k-system of e simple arcs on an n-punctured sphere with $e > 4n$, then

$$\mathrm{cr}(\mathcal{A}) \geq c_k \frac{e^{2+1/k}}{n^{1+1/k}}.$$

Joint with Hugo Parlier(2024)

Q: What happens for systems of curves?

Theorem

For each $\kappa > 0$ there exists c_κ, such that if Γ is a family of e simple closed curves which are pairwise homotopically distinct, on a surface of curve complexity $\kappa = 3g - 3 + n$ (where g is the genus and n the number of punctures), we have

$$\mathrm{cr}(\Gamma) \geq c_\kappa e^{2+\frac{1}{\kappa+1}}.$$

There exists families for which

$$\mathrm{cr}(\Gamma) \leq c'_\kappa e^{2+\frac{2}{\kappa+1}}.$$

Joint with Hugo Parlier(2025)

Pach-Tardos-Toth 2021: Non-simple edges.

Theorem

Let Γ be a collection of m distinct homotopy classes of non-trivial closed and primitive curves on a surface Σ of Euler characteristic χ. Then, for all $m \geq 3^6(|\chi|+1)$, we have

$$\mathrm{cr}(\Gamma) > \frac{1}{128|\chi|}\left(m \log\left(\frac{m}{(|\chi|+1)3^6}\right)\right)^2 .$$

Theorem

Let G be a graph with n vertices and m edges drawn on a closed surface of genus $g \geq 0$ such that no two edges are homotopic. Then, for $m \geq 3^6(|\chi|+1)$, its crossing number satisfies

$$\mathrm{cr}(G) \geq \frac{1}{512|\chi|} m^2 \left(\log^2 \frac{m}{3^6(|\chi|+1)} - 256|\chi|\right).$$

Joint with Hugo Parlier(2025)

Theorem

Let G be a graph with n vertices and m edges drawn on a closed surface of genus $g \geq 0$ such that no two edges are homotopic. Then, for $m \geq 3^6(|\chi| + 1)$, its crossing number satisfies

$$\mathrm{cr}(G) \geq \frac{1}{512|\chi|} m^2 \left(\log^2 \frac{m}{3^6(|\chi| + 1)} - 256|\chi| \right).$$

Corollary

Let $k \geq 1$. If $\mathcal{A}$ is family of m distinct arcs on Σ, such that $i(a, b) \leq k$ for all $a, b \in \Gamma$, then

$$m \leq exp(24\sqrt{(k+1)|\chi|} + \log(|\chi| + 1) + 6).$$