Slice decomposition of hypermaps

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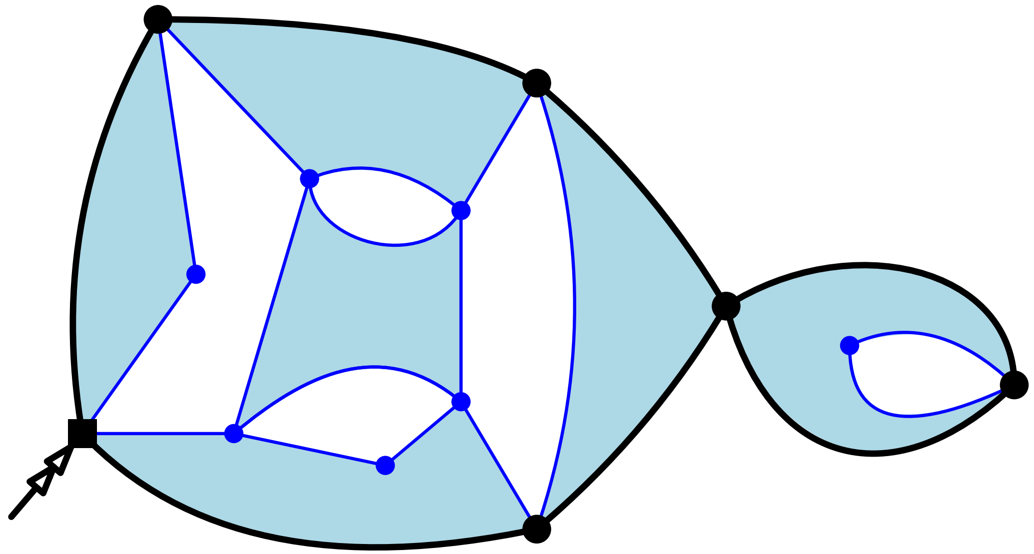
joint work with **Jérémie Bouttier** (IMJ, Sorbonne Université)

Journées cartes, Janvier 2026

Hypermaps



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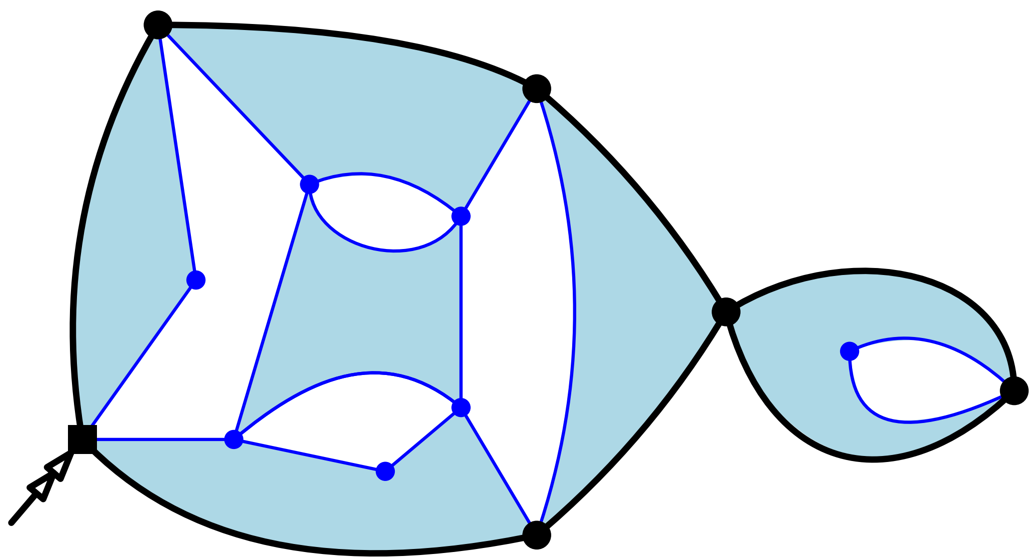
Why “hypermap” ?
→ Extend the notion of hypergraphs to maps.
→ Blue faces can been as **hyper-edges** which connect several vertices.

Hypermaps



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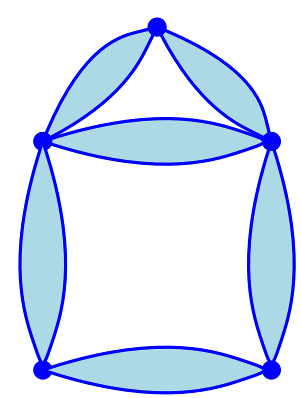
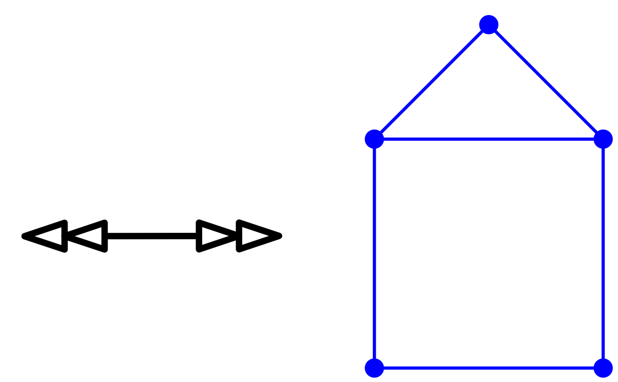
Hypermaps are a generalization of general maps:





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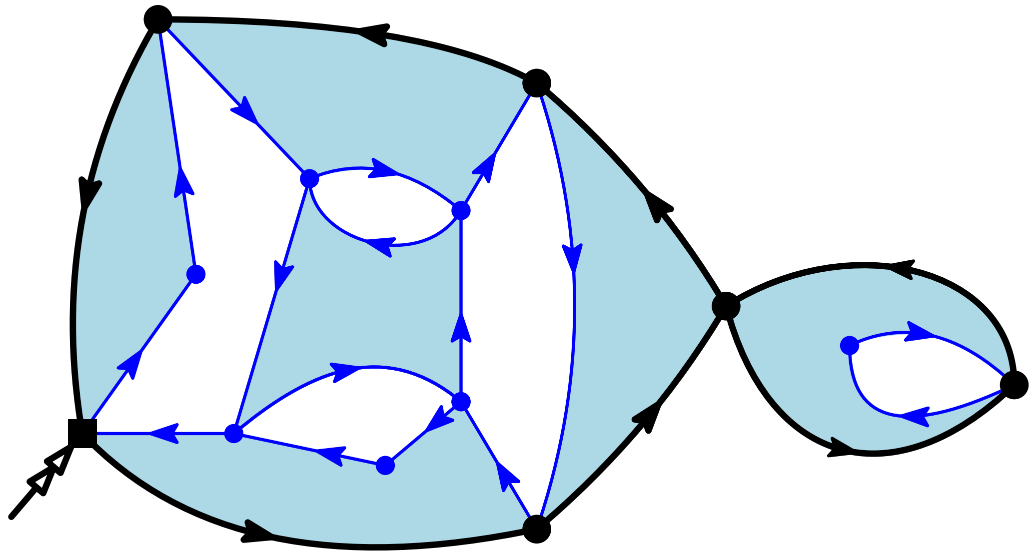

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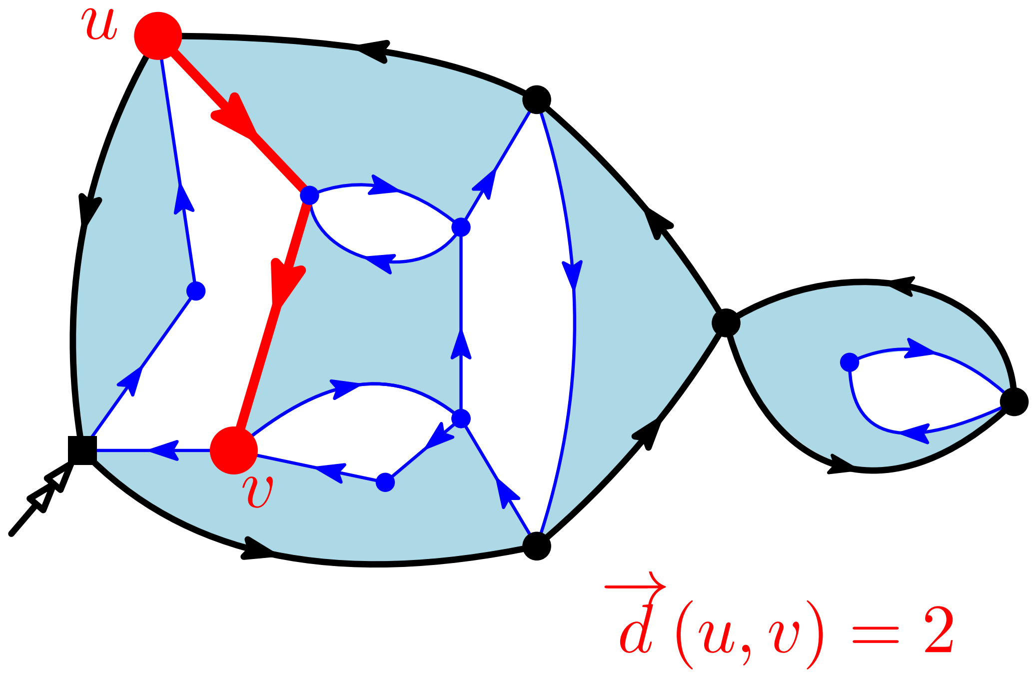

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Oriented (pseudo)-distance on the hypermap: **oriented graph distance**.

Motivations and existing litterature

Hypermaps generalize maps, also additional motivations from **theoretical physics**:

- **2-matrix models** ([Itzykson-Zuber 1980], [Eynard et al. 2000's])
- **Ising model on maps** ([Kazakov 1986])
- **Integrability** in the context of the 2-Toda hierarchy

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- Bijections with **blossoming trees**, [Bousquet-Mélou - Schaeffer 2002]
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To do that, we extend the **slice decomposition** of [Bouttier-Guitter] to hypermaps.

Hypermaps with boundaries: enumeration

A map **with boundaries** is a map where some faces are marked (and rooted).
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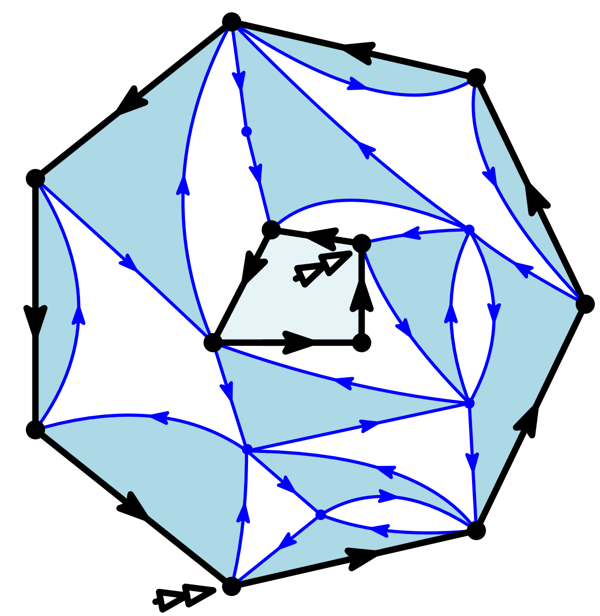
- Hypermap **with monochromatic boundaries**:

  All faces (inner and boundaries) are colored.

$\Leftrightarrow$ The contour of all faces are directed cycles.

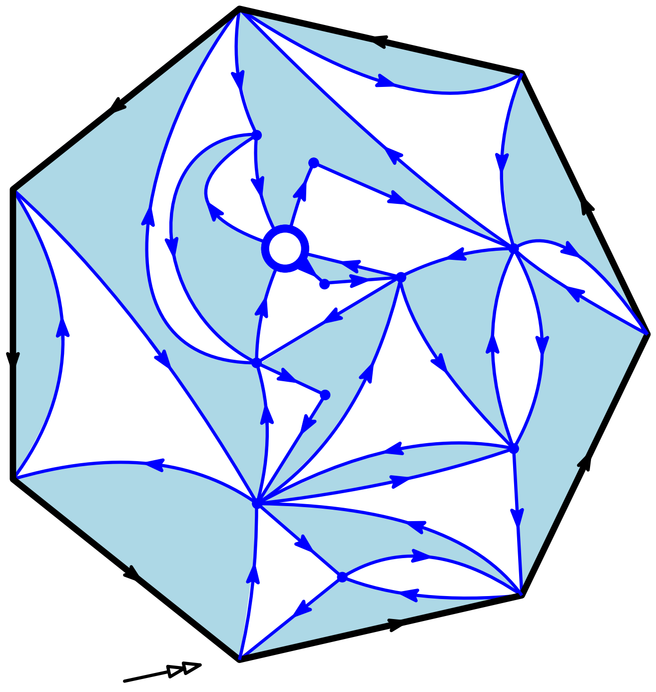


The weight of an hypermap m is defined by:

$$w(\mathrm{m}) := t^{|\text{vertices of } \mathrm{m}|} \prod_{f \in F^{\circ}_{\mathrm{inn}}} t^{\circ}_{\deg(f)} \prod_{f \in F^{\bullet}_{\mathrm{inn}}} t^{\bullet}_{\deg(f)}$$

where t, $t^{\bullet}_1$, $t^{\bullet}_2$, ..., t°_1, t°_2 are formal variables.

First example slice decomposition on pointed disks



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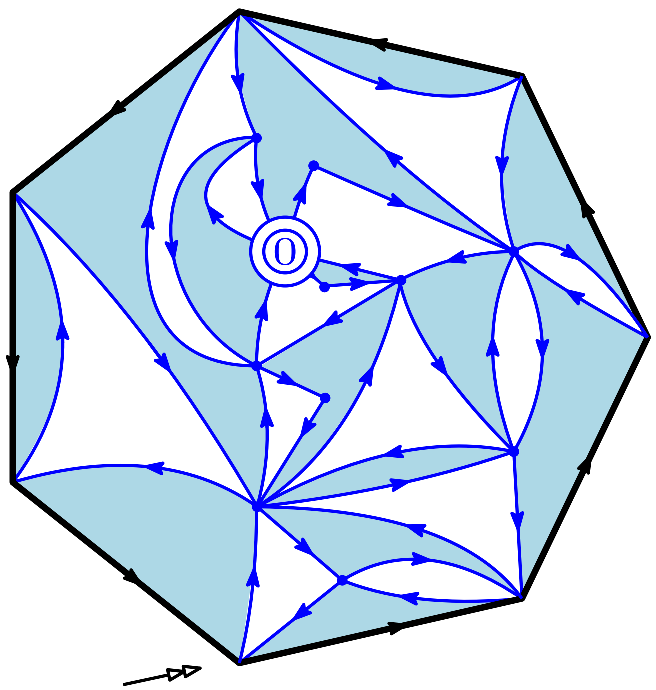

① Label every vertices by their **oriented distance** to the pointed vertex.

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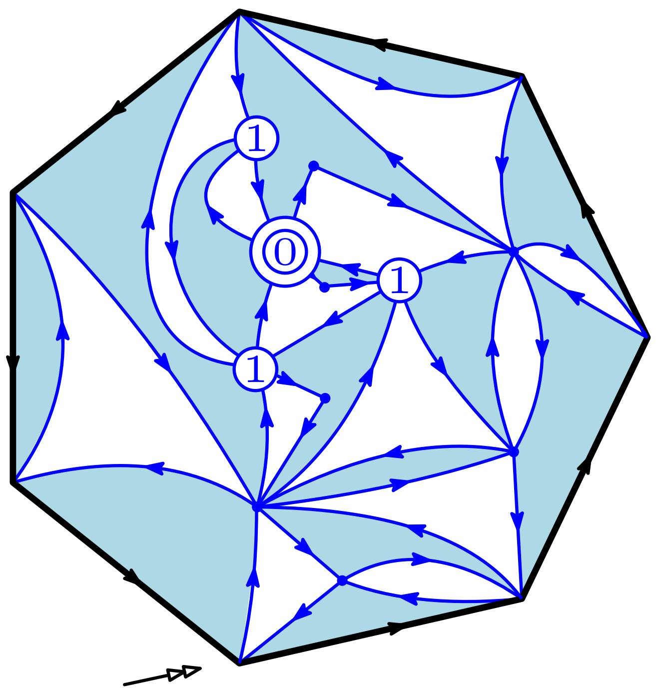

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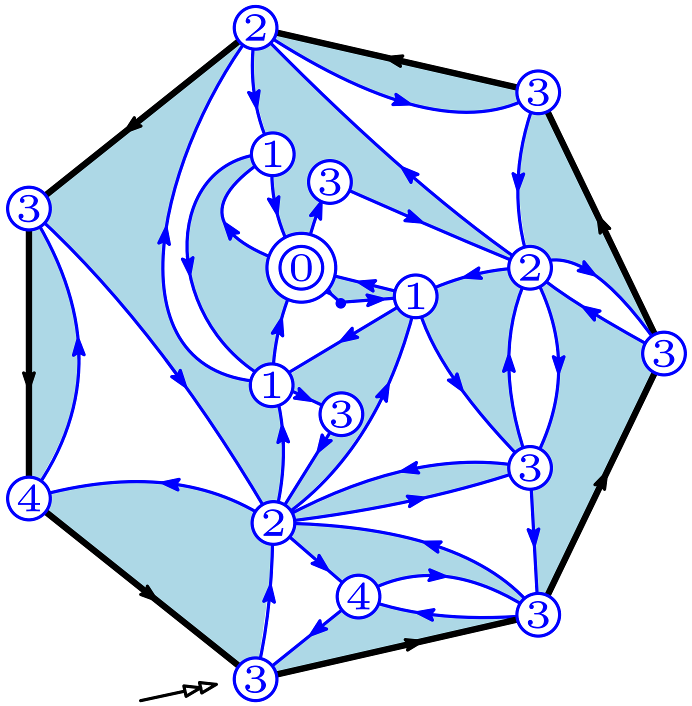

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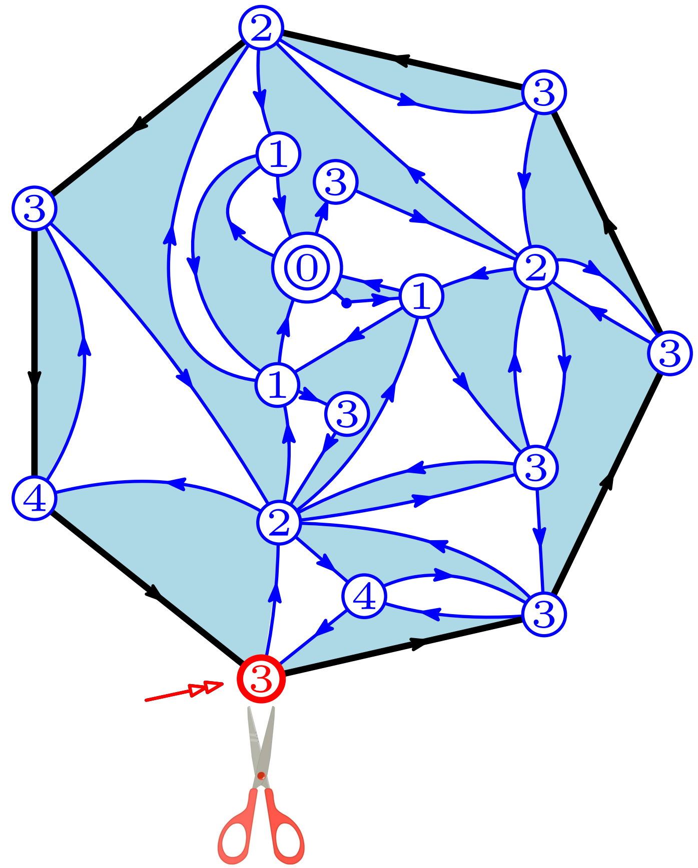

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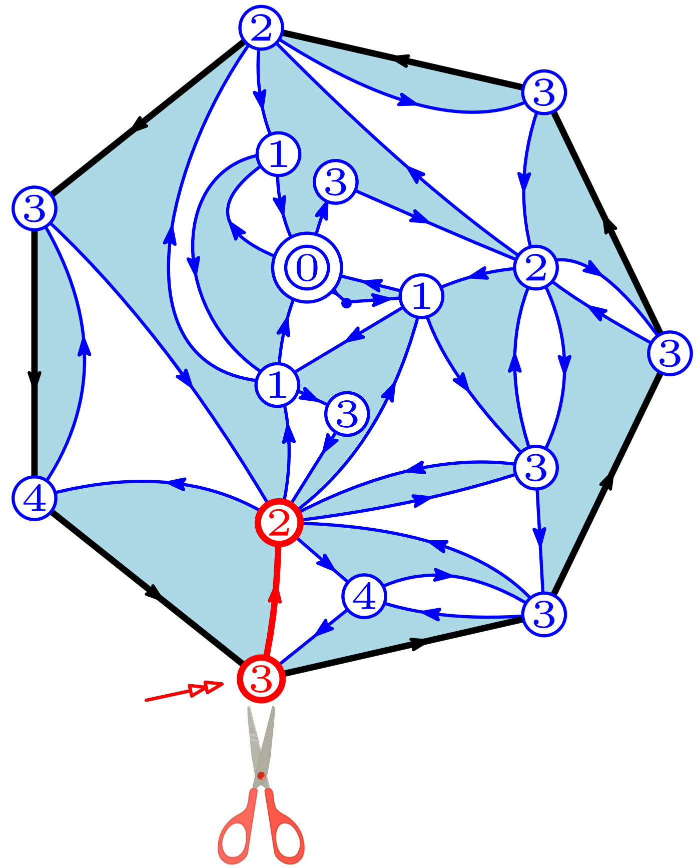

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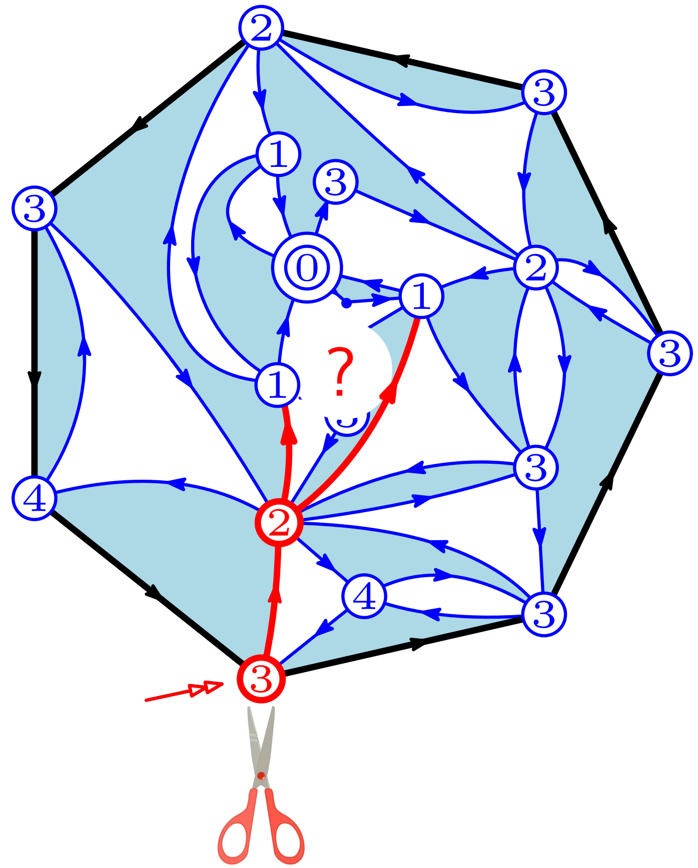

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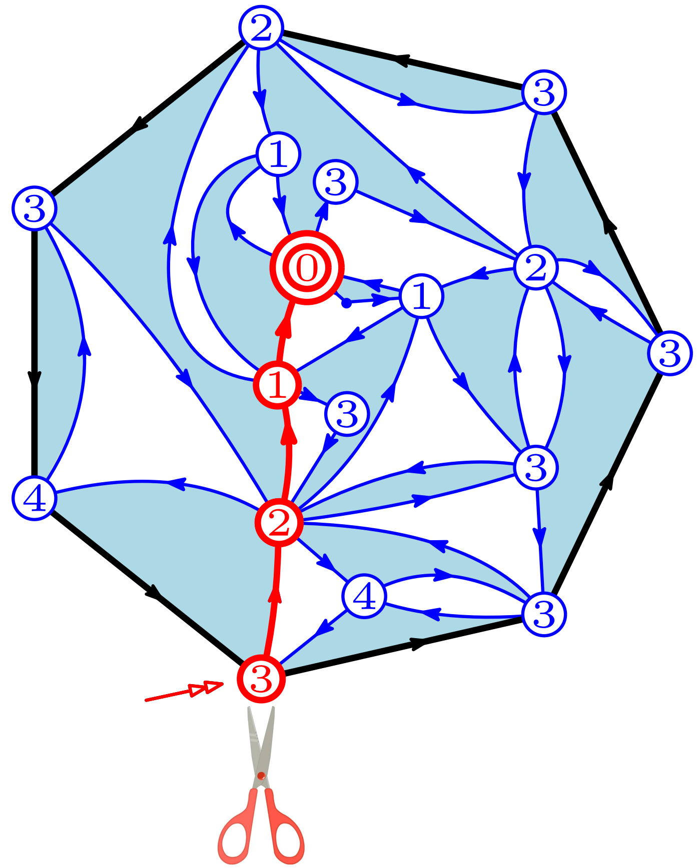

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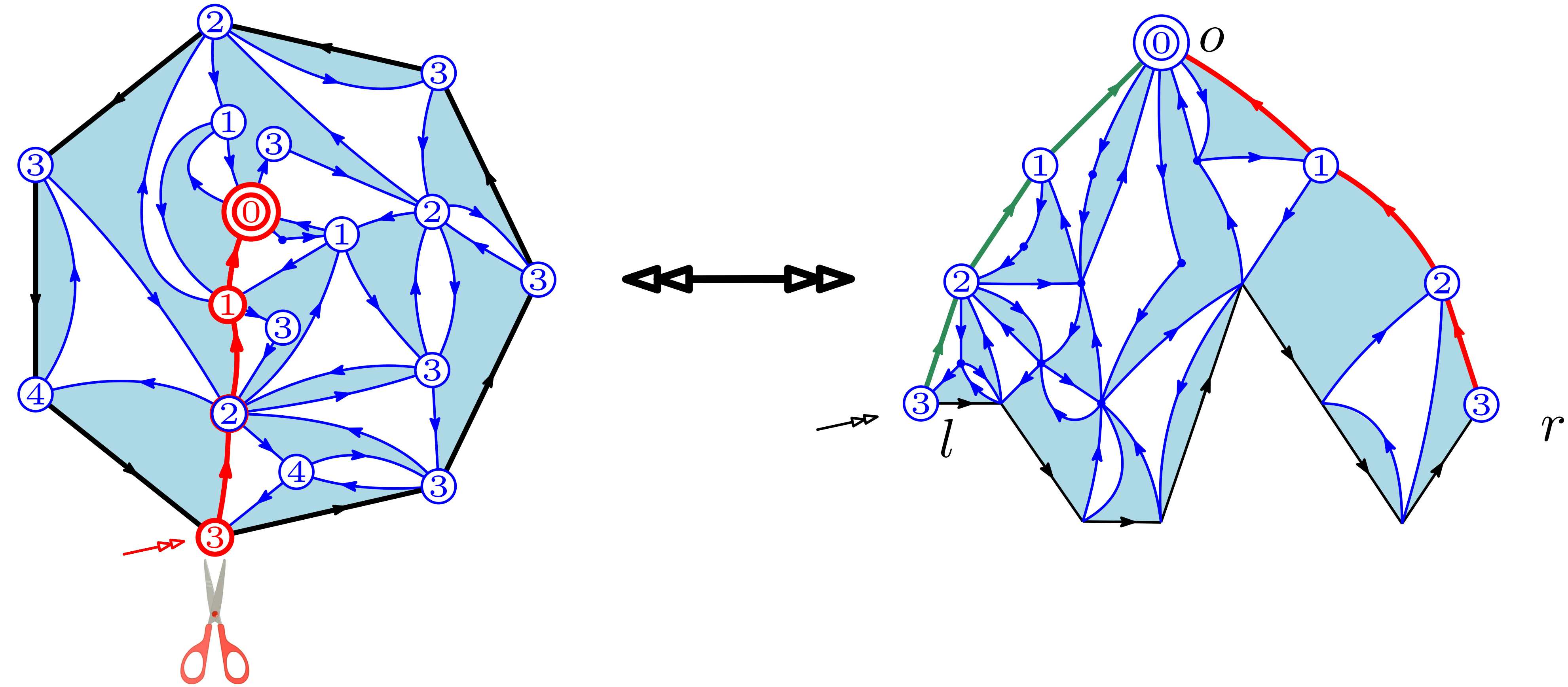

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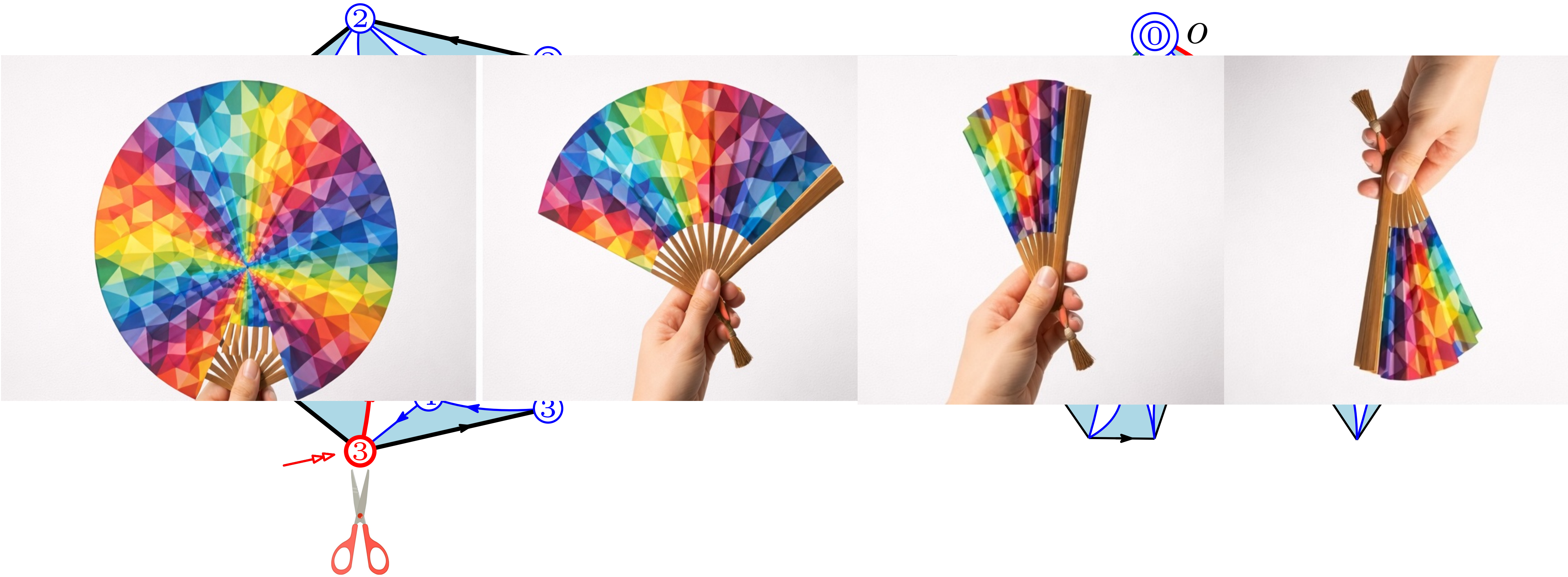

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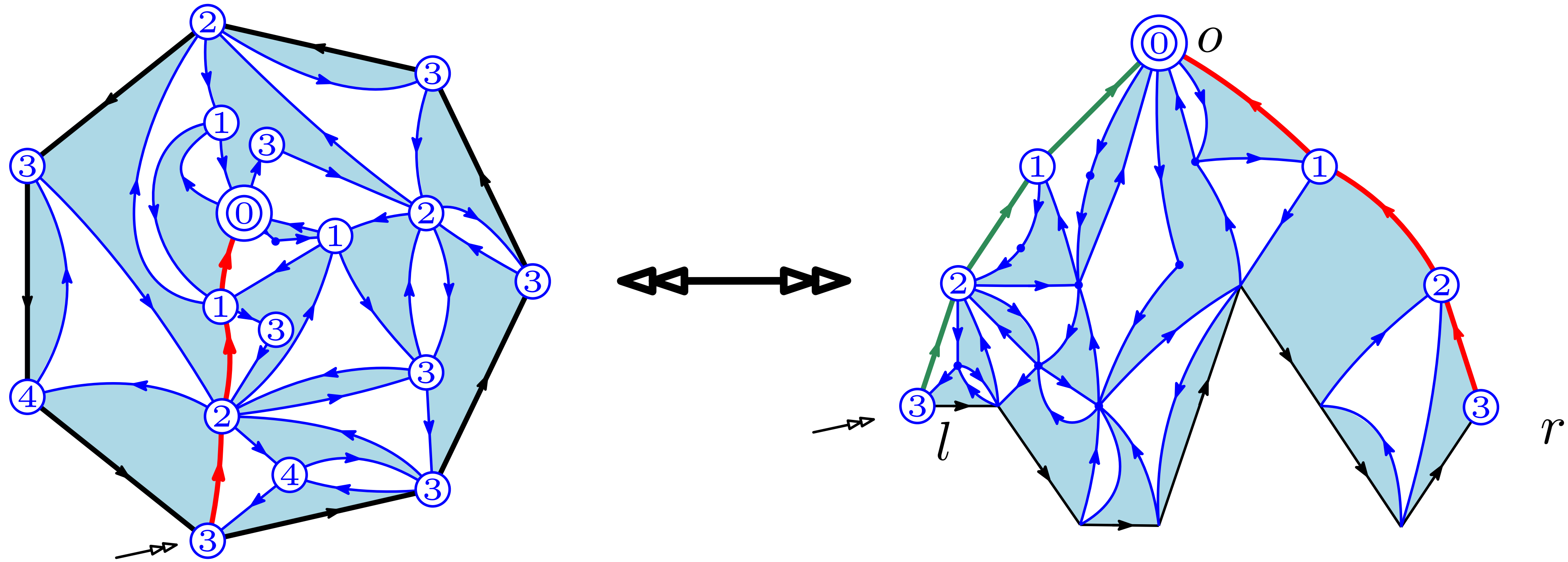

A **(hyper)-slice** is an hypermap with a boundary and 3 marked corners l, r and o such that:

- the **left boundary** from l to o is a **geodesic** (green edges)
- the **right boundary** from r to o is the **unique geodesic** (red edges),
- the **base** (black edges) is either oriented from l to r (="type A")
  or from r to l (="type B"),
- and the left and the right boundaries intersect only at o.

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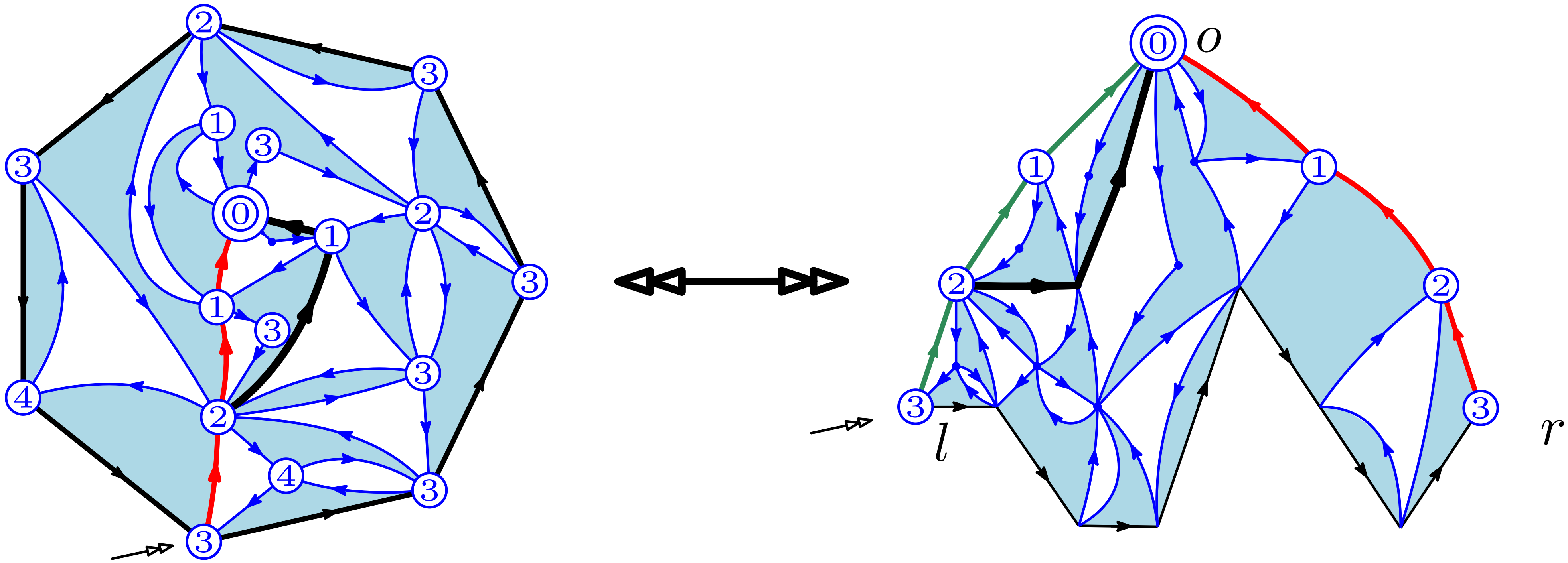

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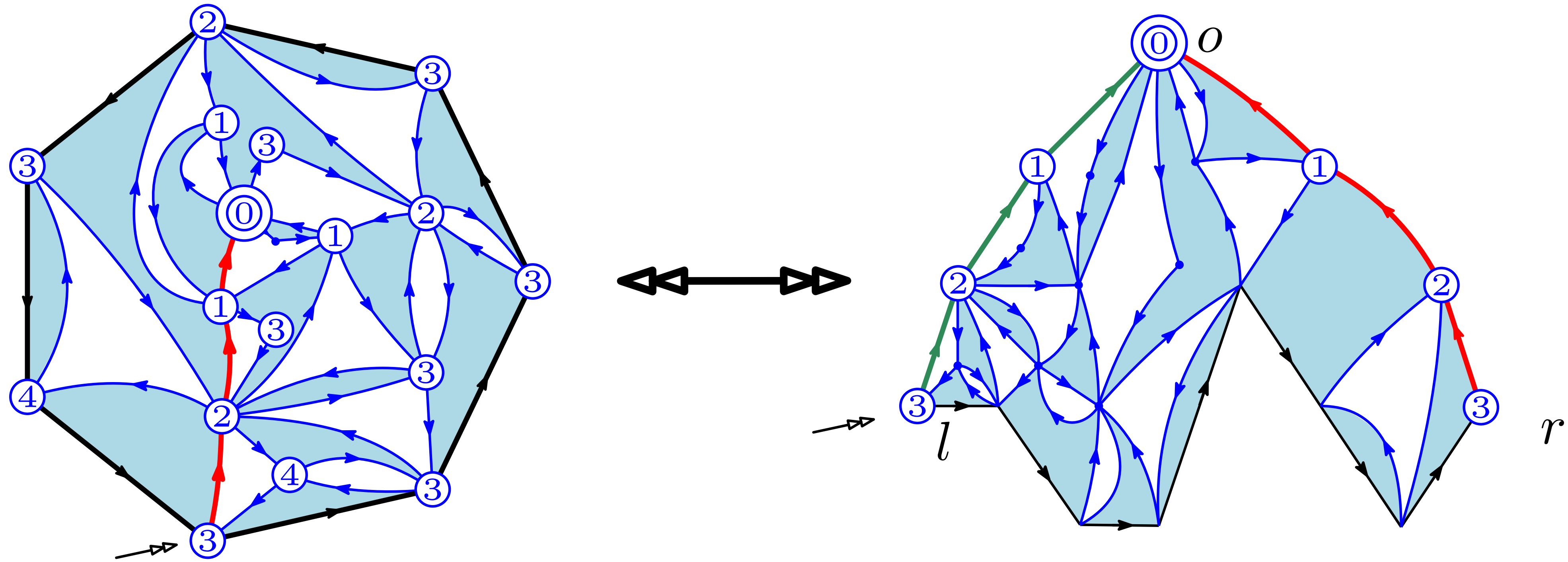

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The **increment** of a A-slice (resp. B-slice) is a difference between the labels of r and of l (resp. l and r).

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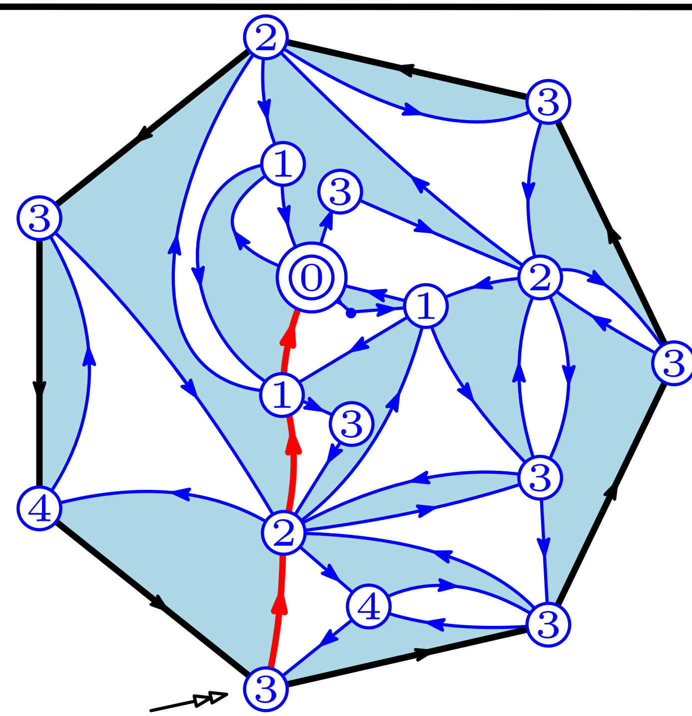

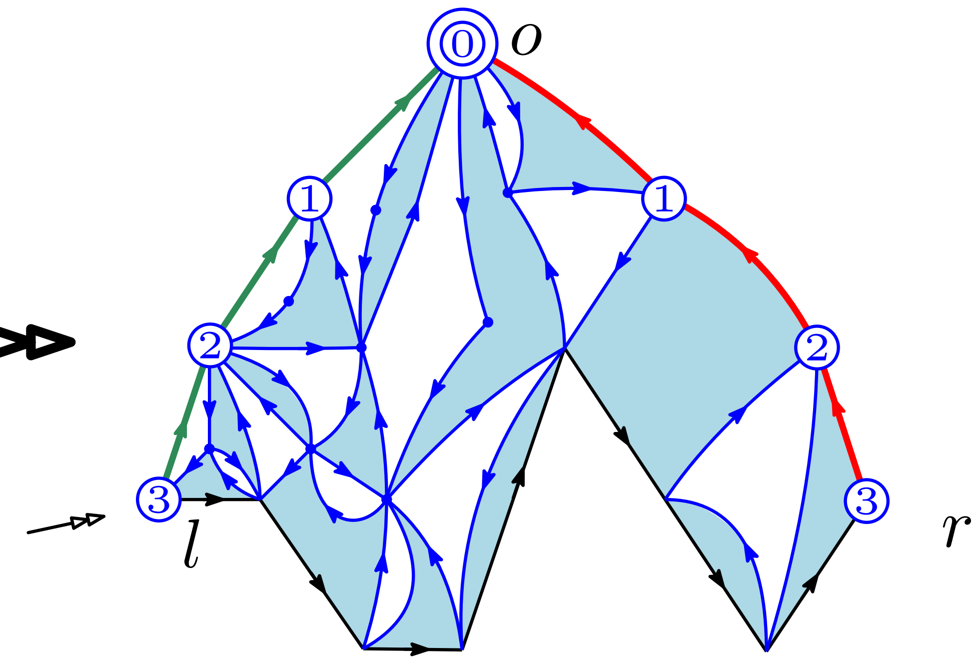

Pointed Disks
white / black root face

weight-preserving
bijection

Slice with 0 increment
type A / type B

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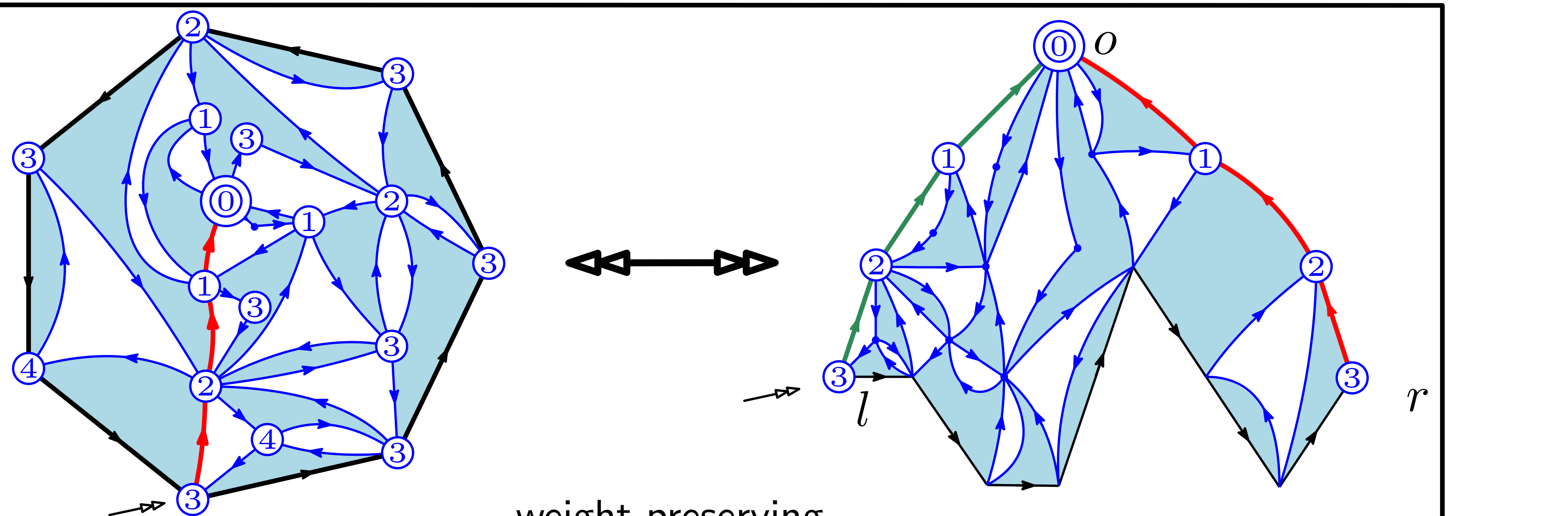

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no weight given to the vertices incident to
the right boundary of a slice.

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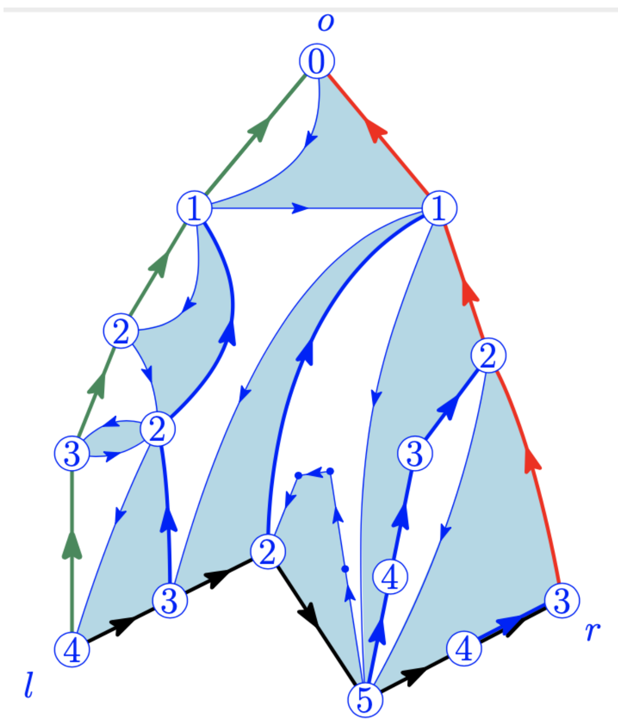
Why does this help ? Decomposition of slices

Slices can be further decomposed into “elementary slices”:



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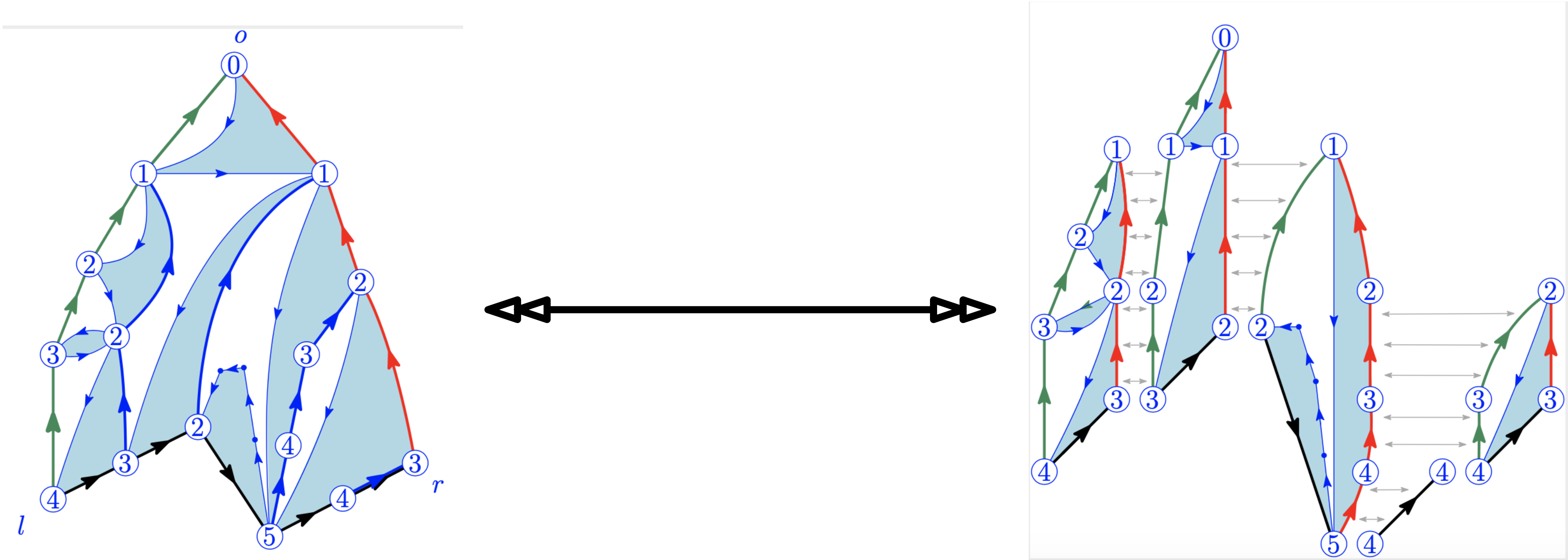

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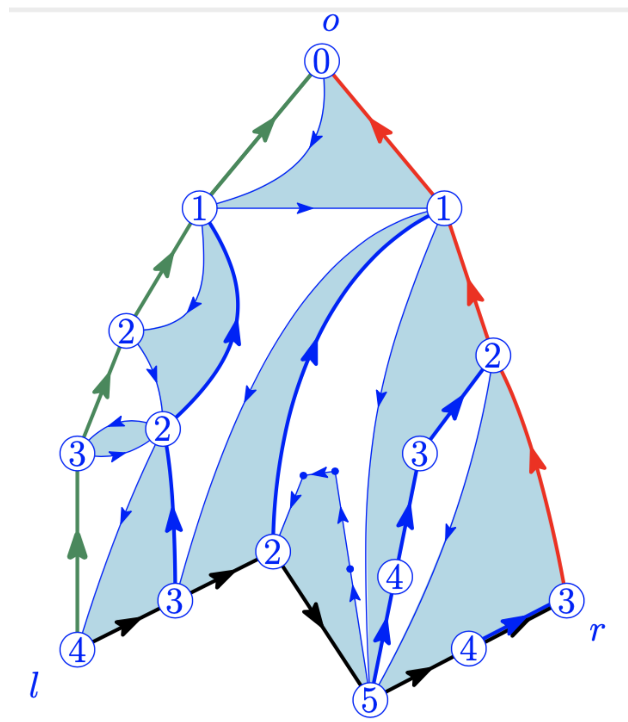

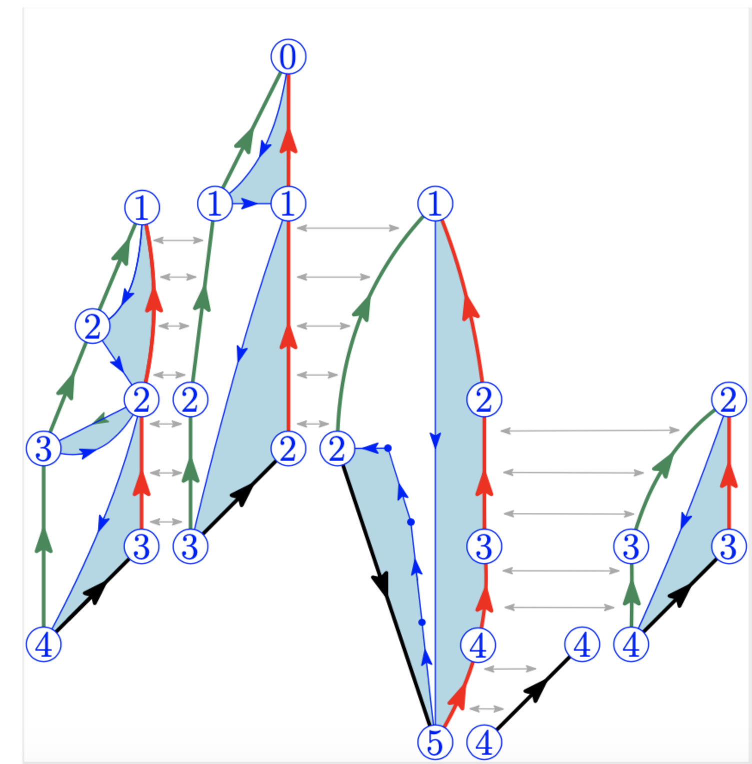

weight-preserving bijection

Type A / B slice with base of length p and increment k ⟺ $p-$tuple of type A/B **elementary** slices s.t. sum of increment = k

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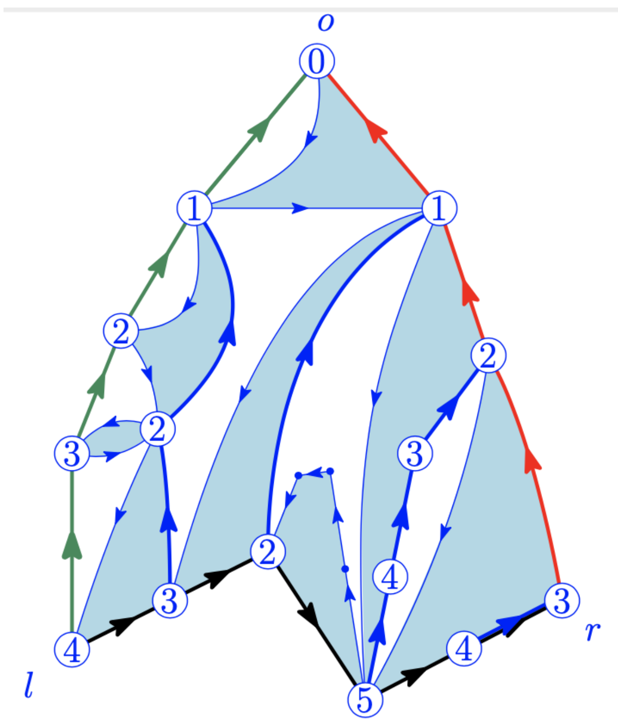

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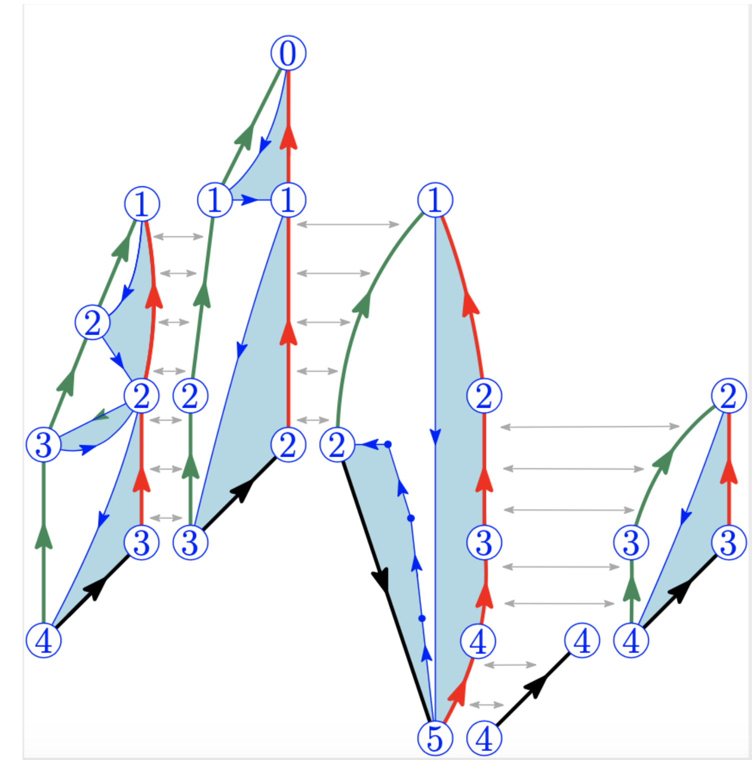
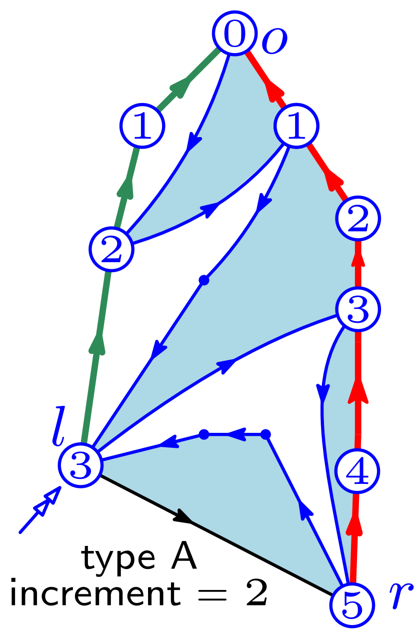
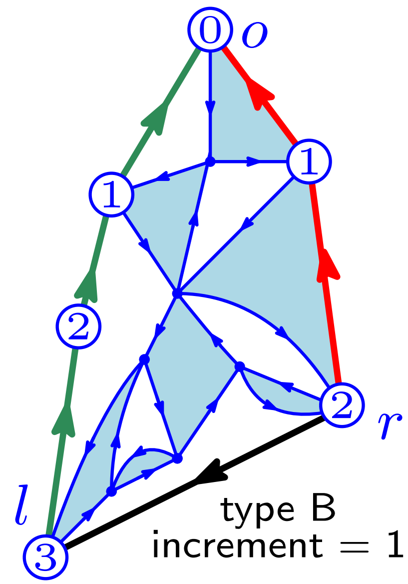


$p-$tuple of type A/B **elementary** slices
s.t. sum of increment = k

Elementary slice: slice with a base of length 1.

Why does this help ?? Enumeration of elementary slices





For $k \in \mathbb{Z}$, $\quad a_k, b_k :=$ generating series of elementary slices of type A/B and increment k.

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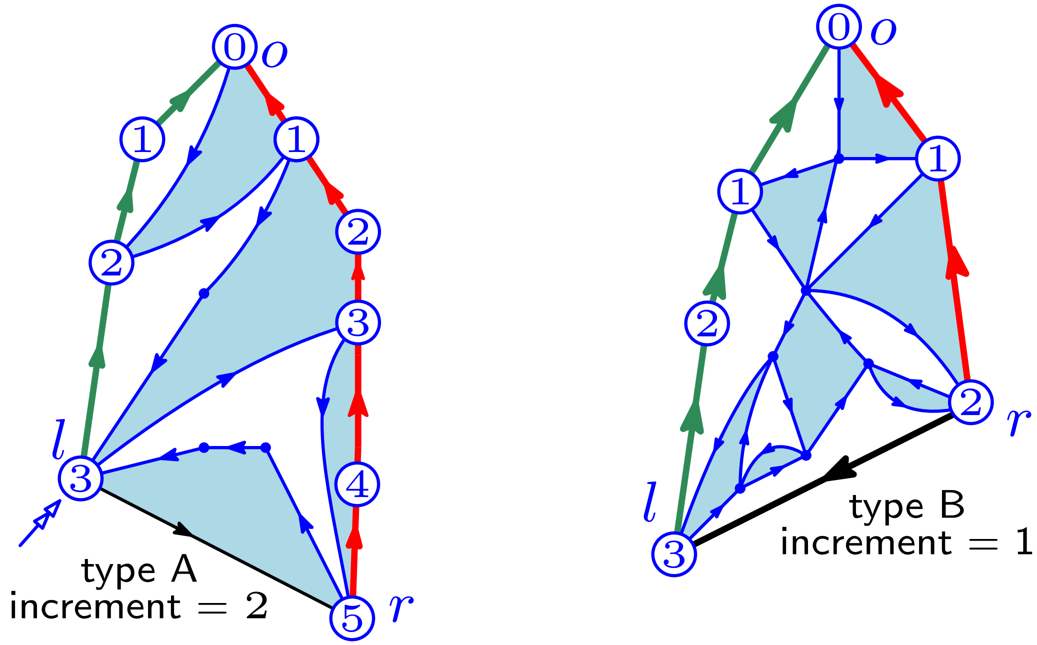

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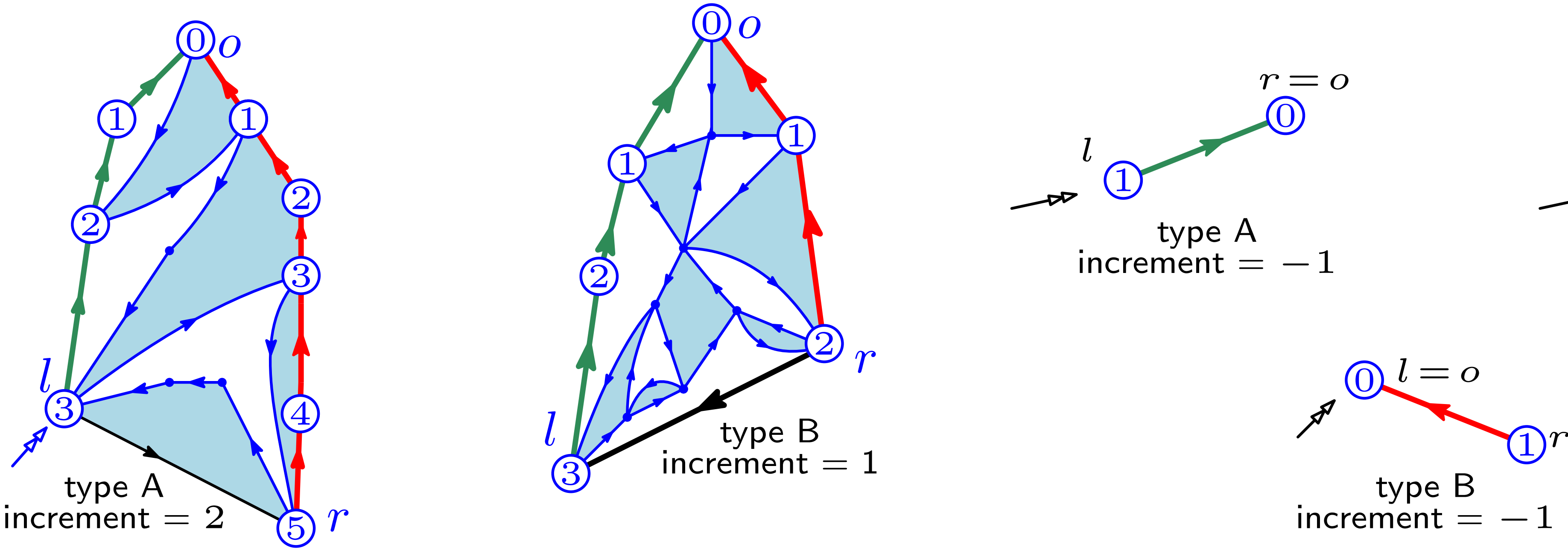

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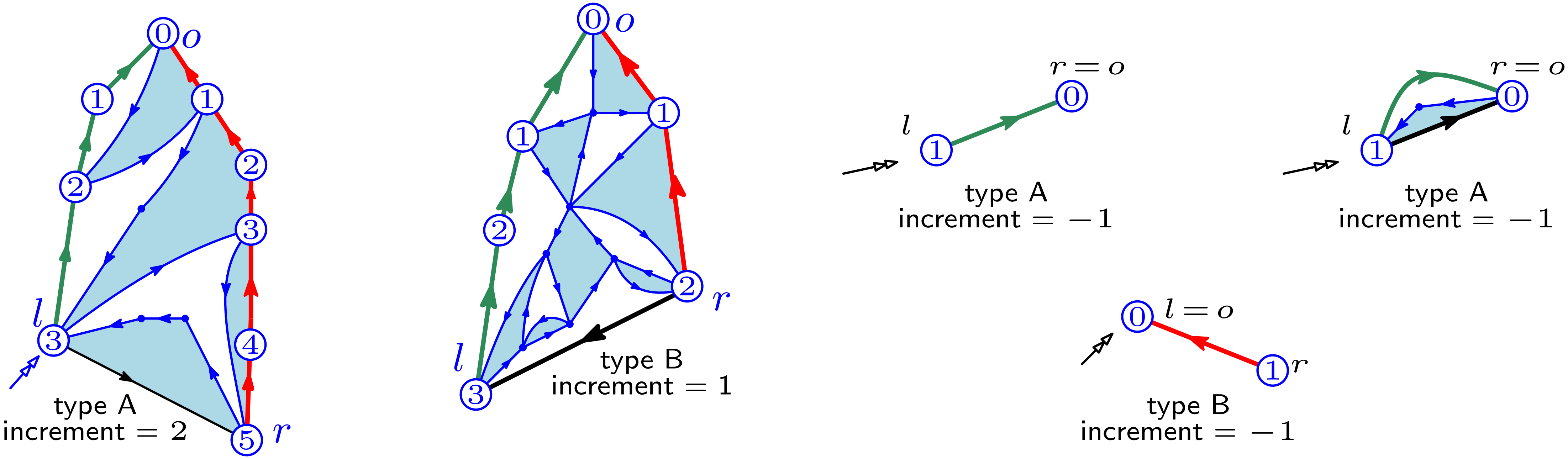

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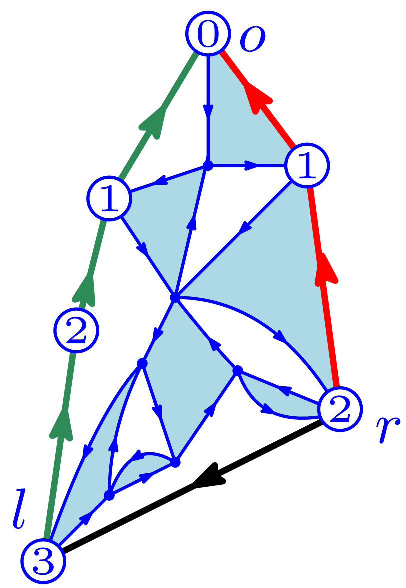
We combine all these quantities into two Laurent series:

$$x(z) := \sum_{k \geq -1} a_k z^{-k}, \qquad y(z) := \sum_{k \geq -1} b_k z^k.$$

Main result:
All "natural" generating series of hypermaps can be expressed in terms of $x(z)$ and $y(z)$ = "spectral curve".

Why does this help ???? Decomposition of elementary slices



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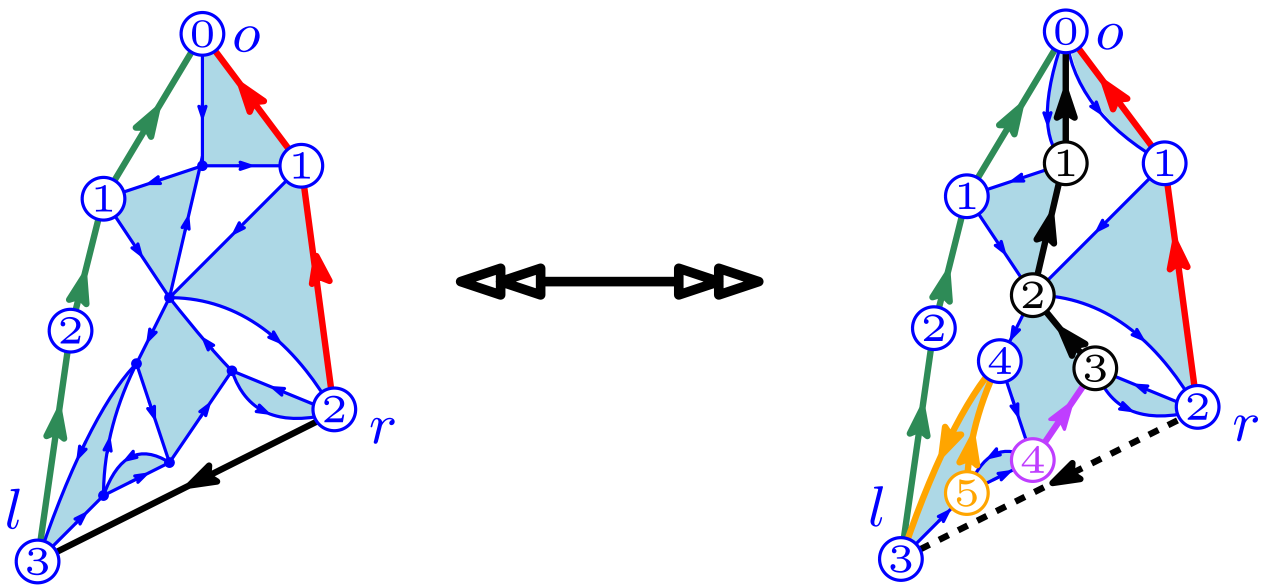

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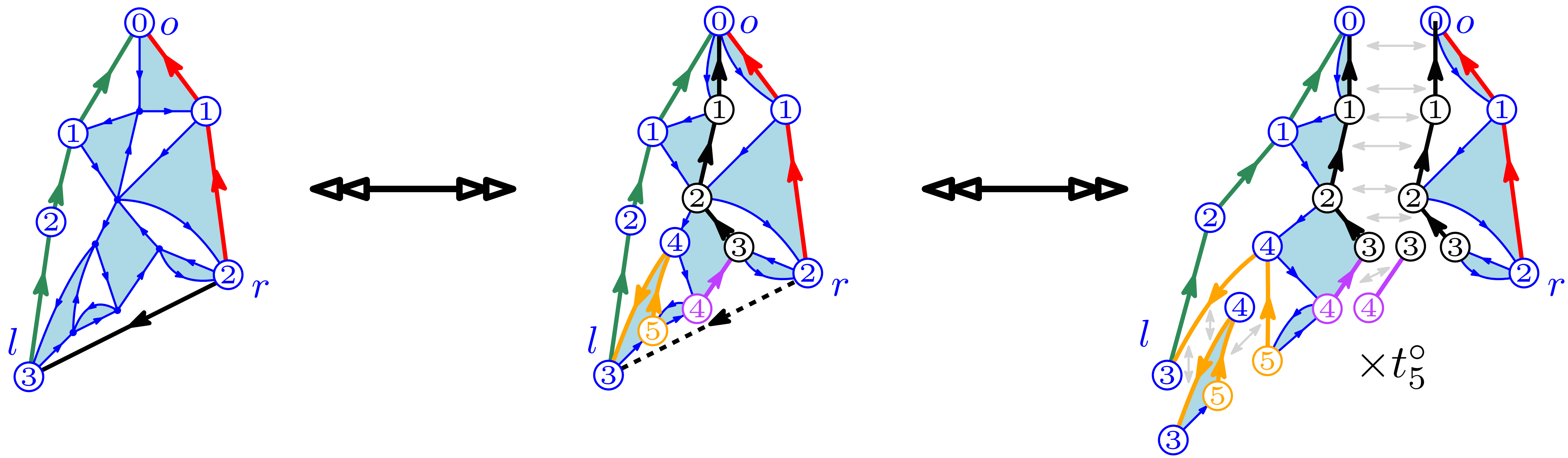

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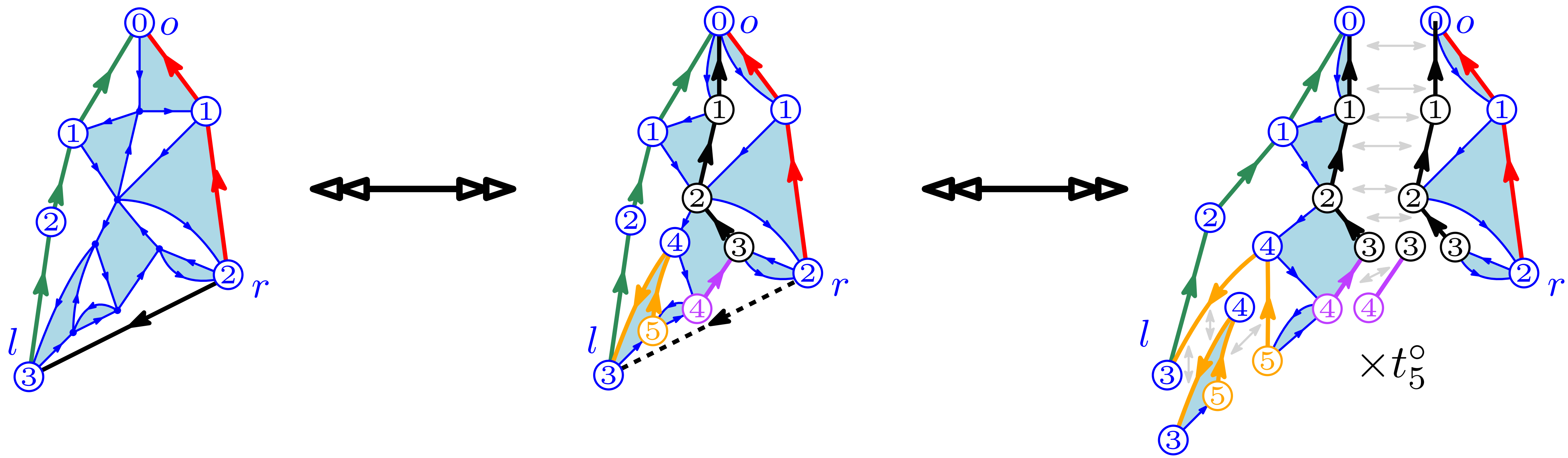

The generating series of elementary slices are uniquely determined by the following recursive system of equations:

$$a_k = t\delta_{k,-1} + \sum_{d\geq 1} t_d^\bullet [z^k] y(z)^{d-1} \quad \text{for } k \geq -1$$

$$b_{-1} = 1 \text{ and } b_k = \sum_{d\geq 1} t_d^\circ [z^{-k}] x(z)^{d-1} \quad \text{for } k \geq 0$$

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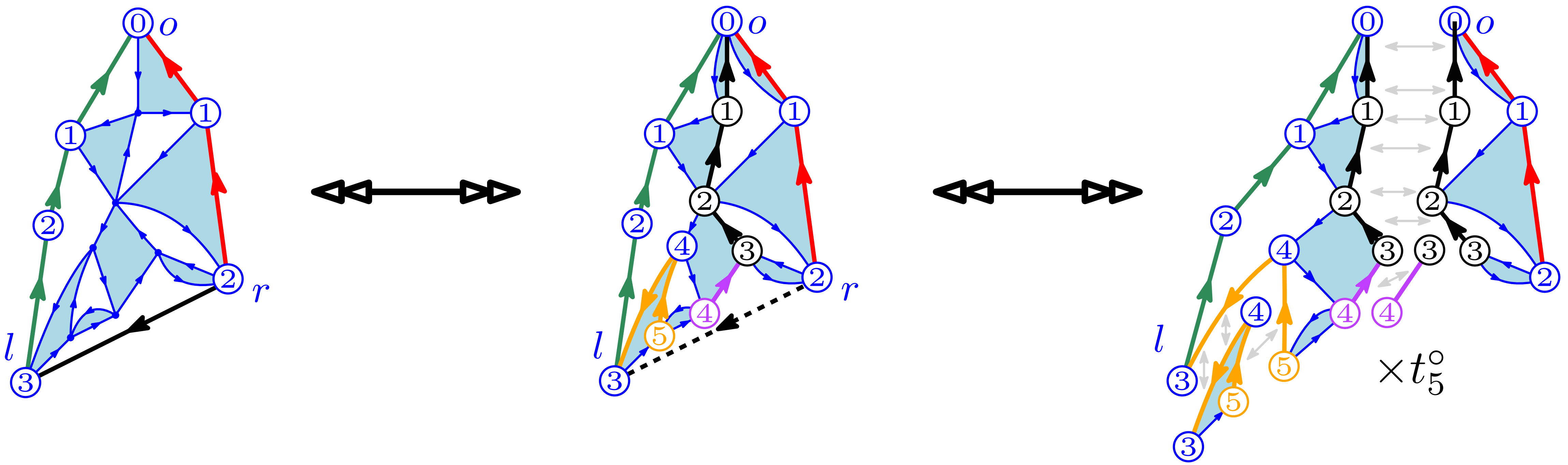

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→ This system is algebraic when the degree of the faces are assumed to be bounded.

→ Same system of equations as [Bousquet-Mélou, Schaeffer 02] + the system of [Bouttier, Di Francesco, Guitter 04] can be recovered using an additional combinatorial construction.

Elementary slices for Eulerian triangulations

The generating series of elementary slices are uniquely determined by the following recursive system of equations:

$$a_k = t\delta_{k,1} + \sum_{d\geq 1} t_d^{\bullet}[z^k]y(z)^{d-1} \quad \text{for } k \leq 1$$

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Eulerian triangulations:

$t_3^{\circ} = t_3^{\bullet} = 1$ and $t_k^{\circ} = t_k^{\bullet} = 0$ for $k \neq 3$:

Along an edge labels either decrease by 1 or increase by 2:

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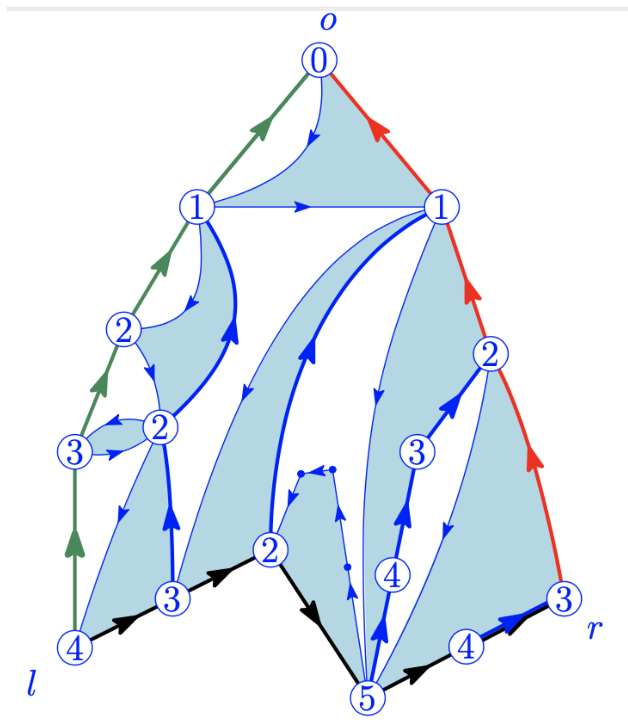
We get the following system of equations:
$$\begin{cases} x(z) = a_{-1} z + \dfrac{1}{z^2} \\ y(z) = \dfrac{1}{z} + a_{-1}^2 z^2 \\ a_{-1} = t + 2a_{-1}^2 \end{cases}$$

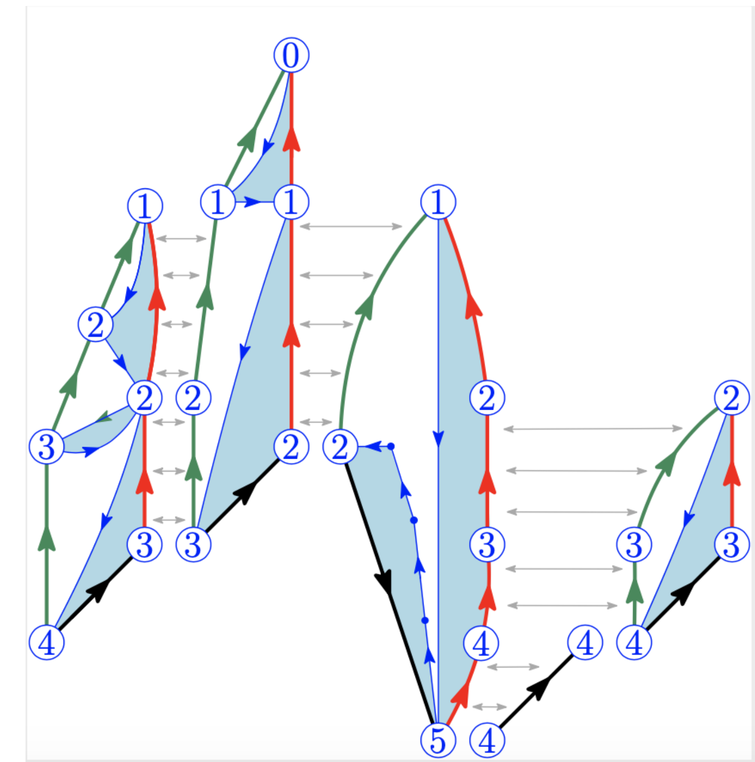
So that $a_{-1} = t + 2t^2 + 8t^3 + 40t^4 + 224t^5 + 1344t^6 + 8448t^7 + o(t^7)$

Generating series of slices



Type A / B slice with
base of length p and increment k

weight-preserving
bijection

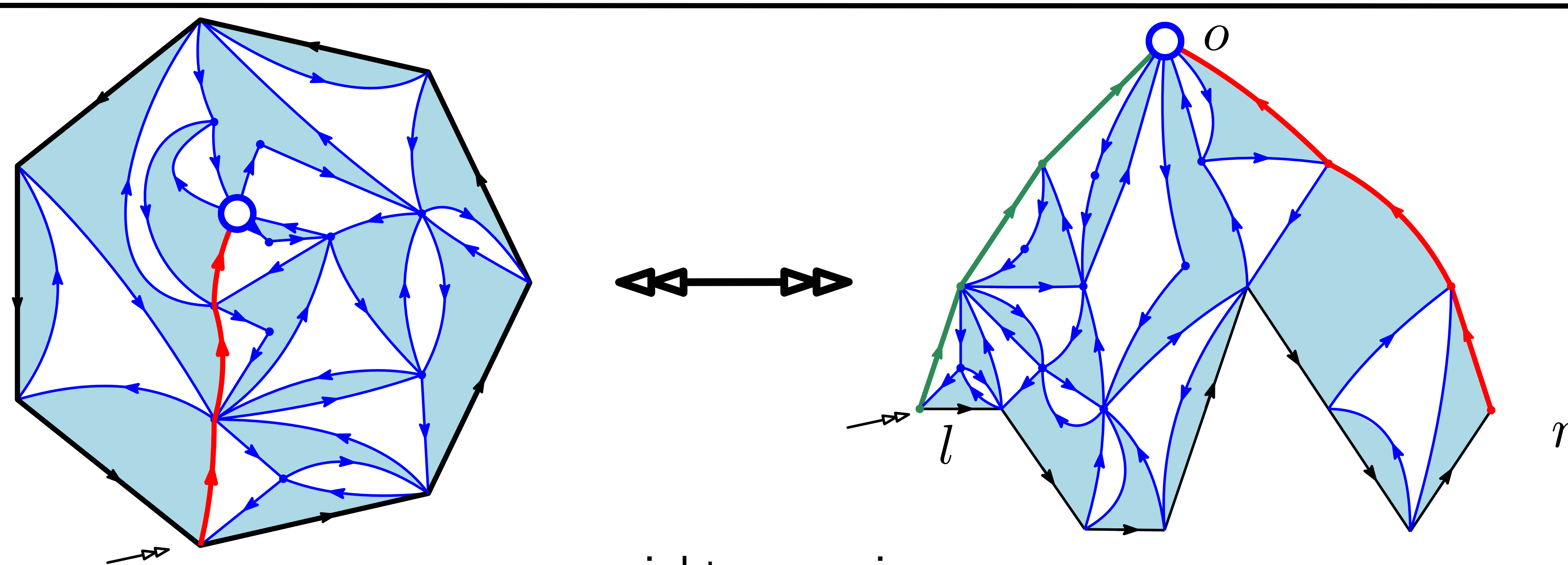


$p-$tuple of type A/B **elementary** slices
s.t. sum of increment = k

The generating series of slices with base of length p and increment k is given by:

$[z^{-k}]x(z)^p$ for type A, and $[z^k]y(z)^p$ for type B.

Coming back to pointed disks



Pointed Disks
white / black root face

weight-preserving
bijection

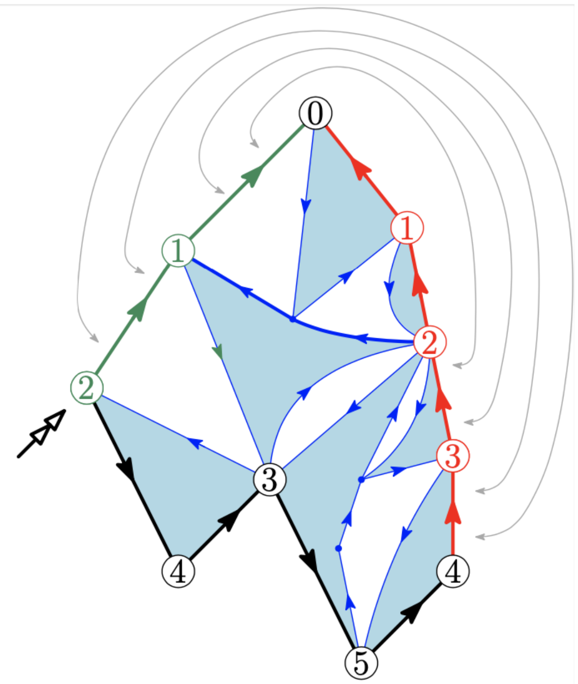
Slice with 0 increment
type A / type B

$F_p^\circ, F_p^\bullet$:= generating series of hypermaps with a monochromatic white (resp. black) boundary of degree p.

We have:

$$\frac{d}{dt}F_p^\circ = [z^0]x(z)^p, \qquad \text{resp. } \frac{d}{dt}F_p^\bullet = [z^0]y(z)^p.$$

Two boundaries: trumpets and slices with increment $\neq 0$



Slice with increment > 0.

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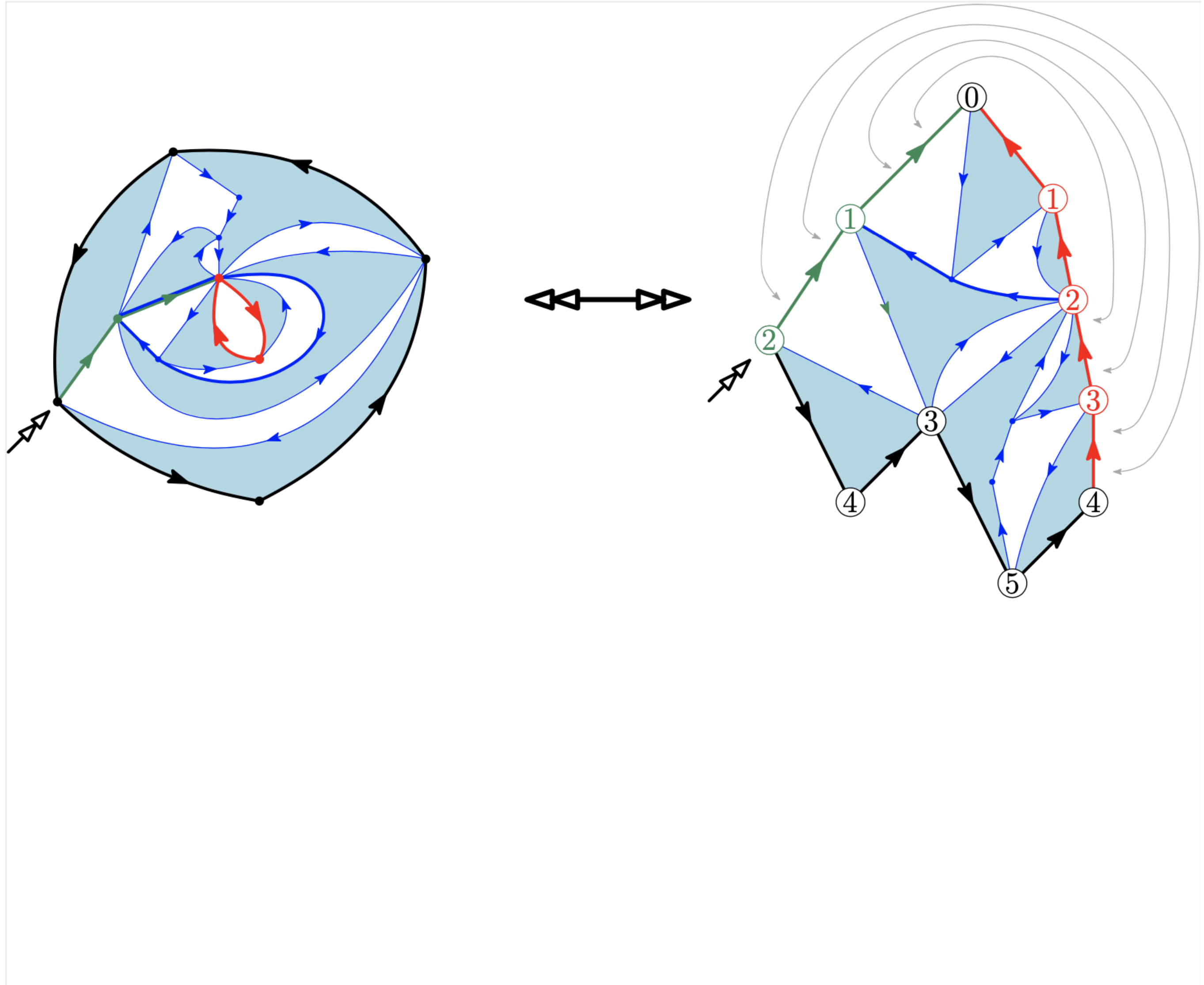

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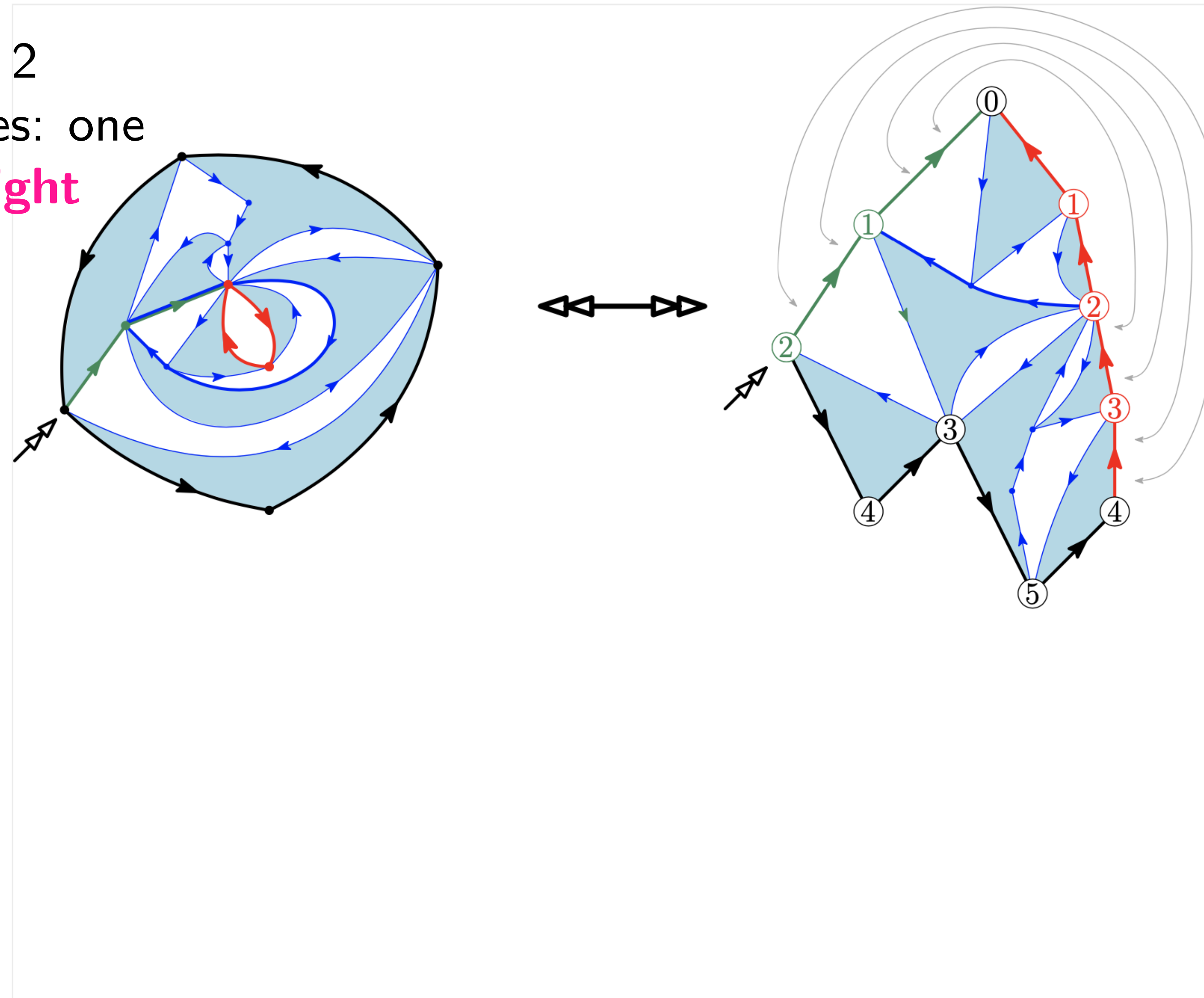
Cornet : Hypermap with 2 monochromatic boundaries: one rooted and one **strictly tight**

:= The boundary of the tight face is the unique shortest separating cycle between both boundaries.



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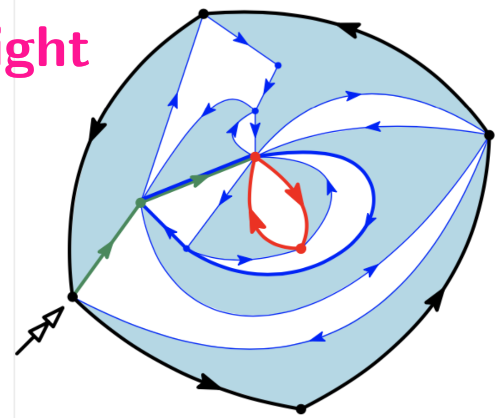

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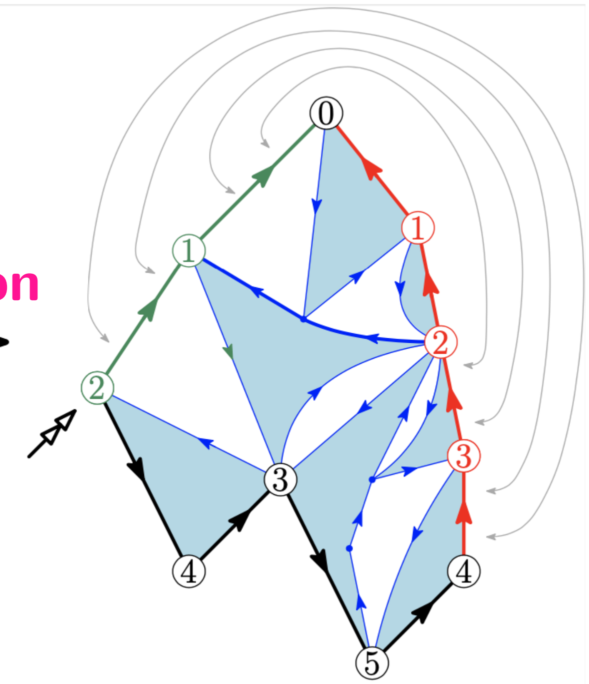

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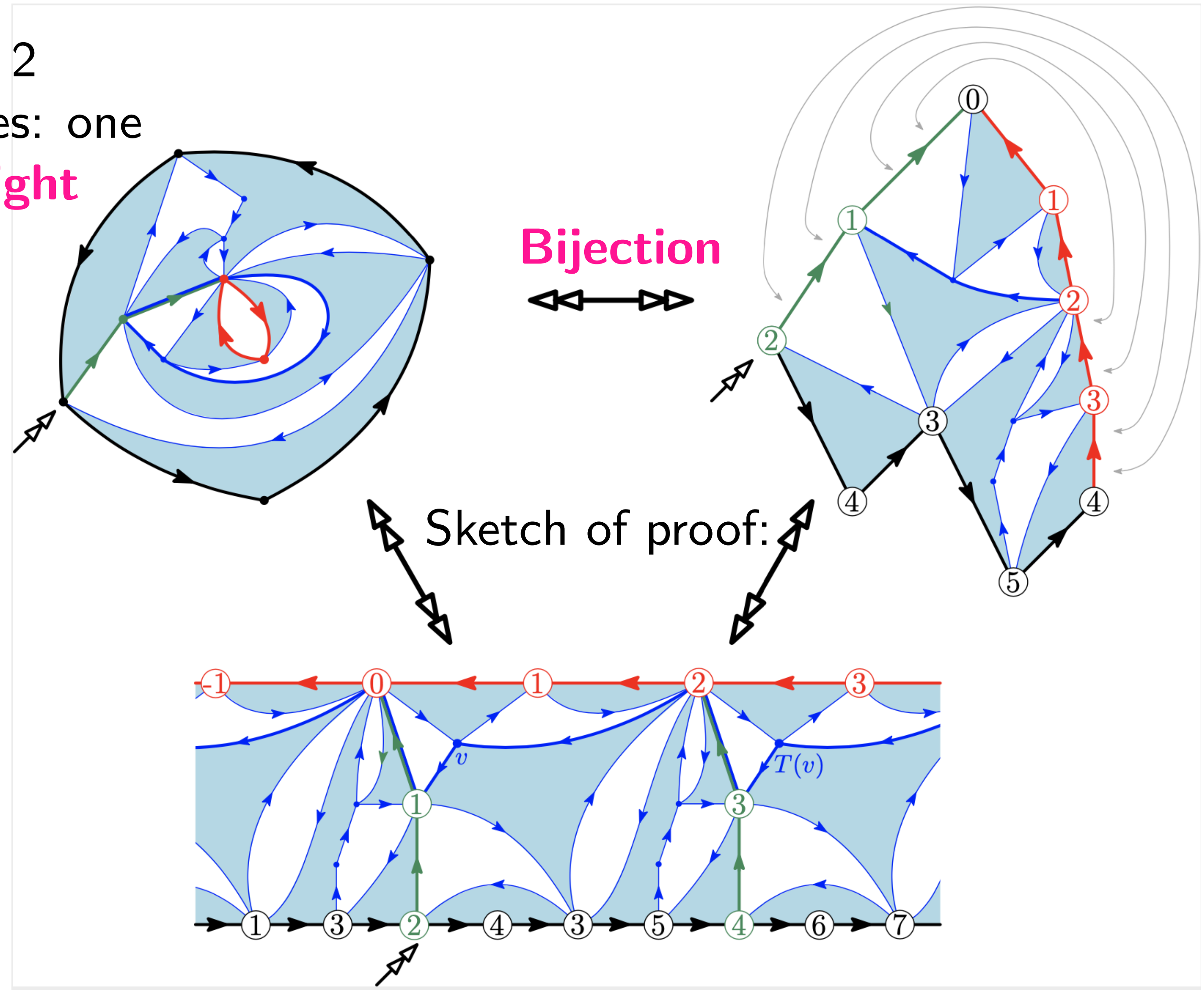

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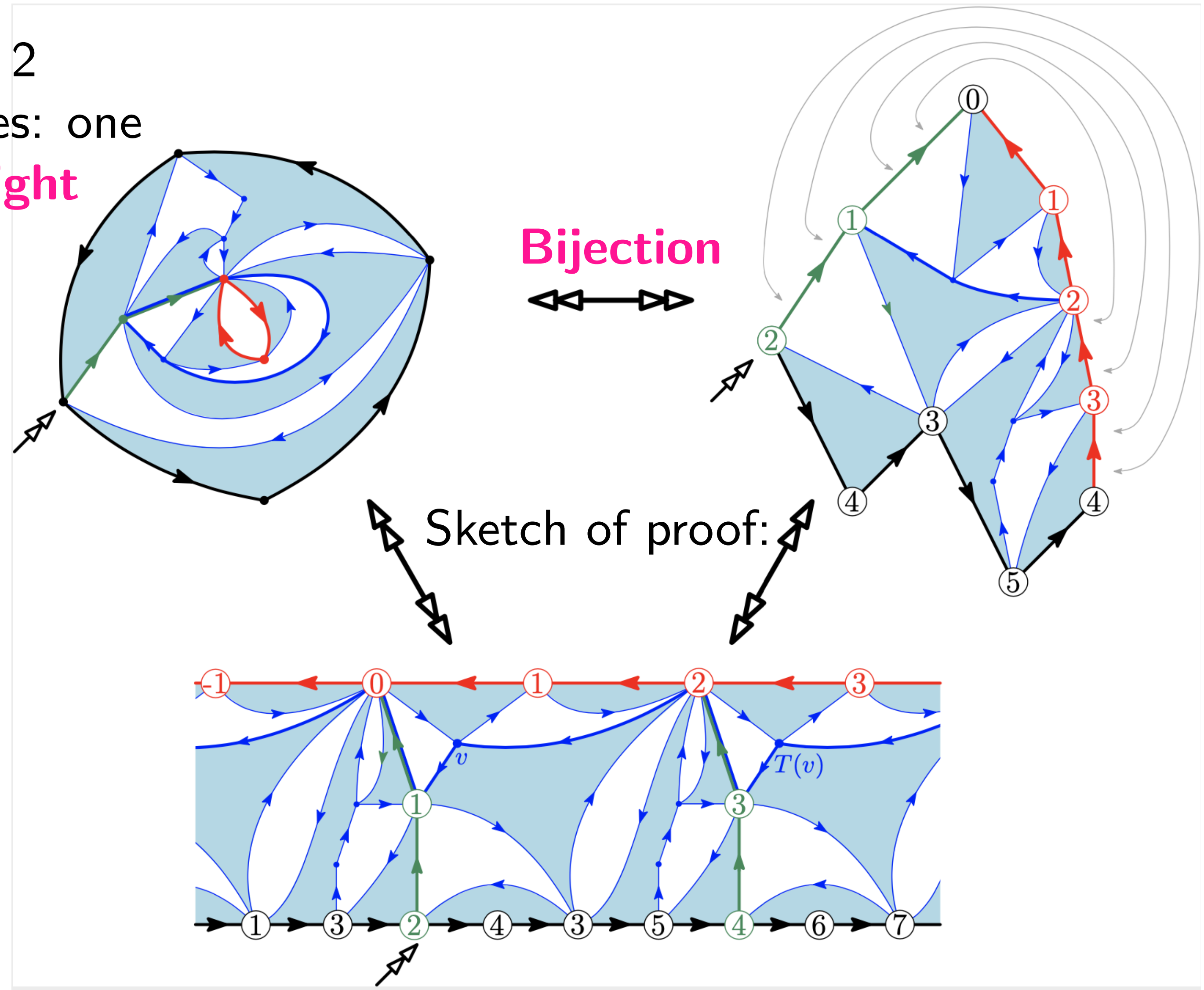

Slice with increment > 0.

Remark: Similar result for slices of type B and **trumpets** with a **tight face**.

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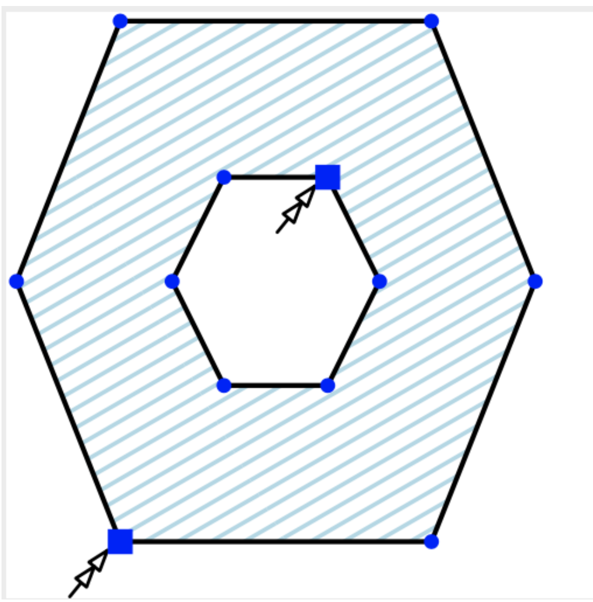
Two monochromatic boundaries: general case

An hypermap with two monochromatic boundaries can be decomposed along the "inner-most" shortest separating cycle: we get an ordered pair trumpet/cornet.



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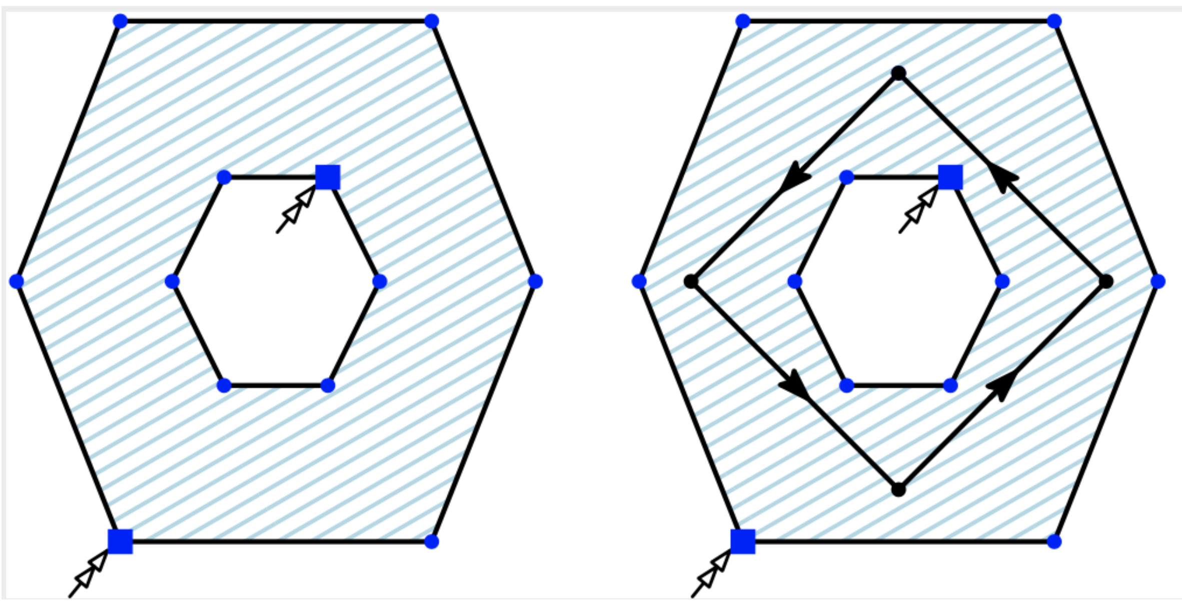

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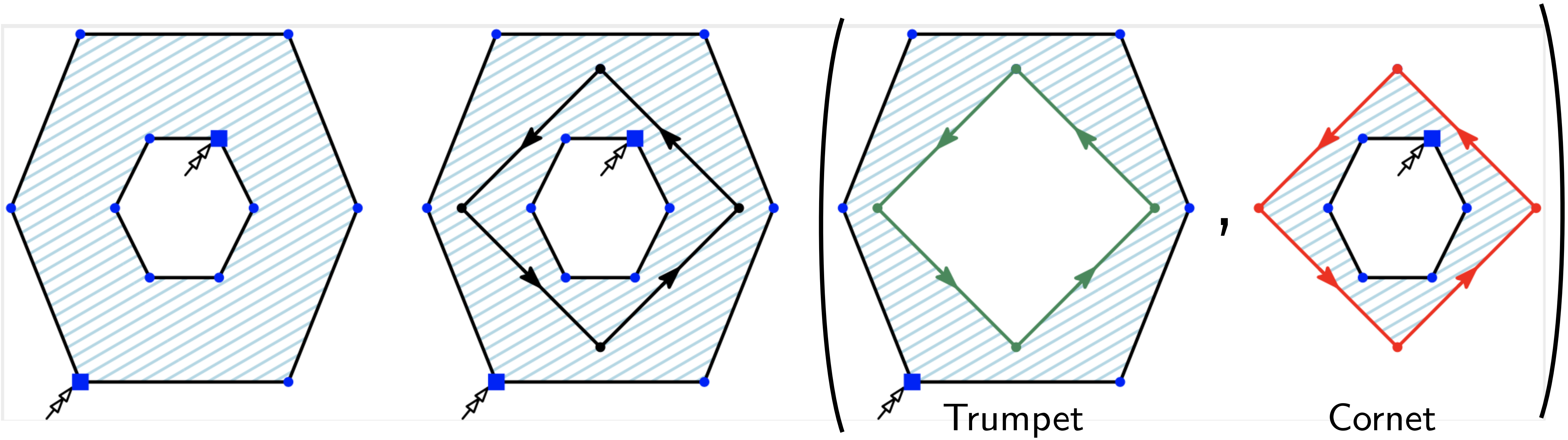

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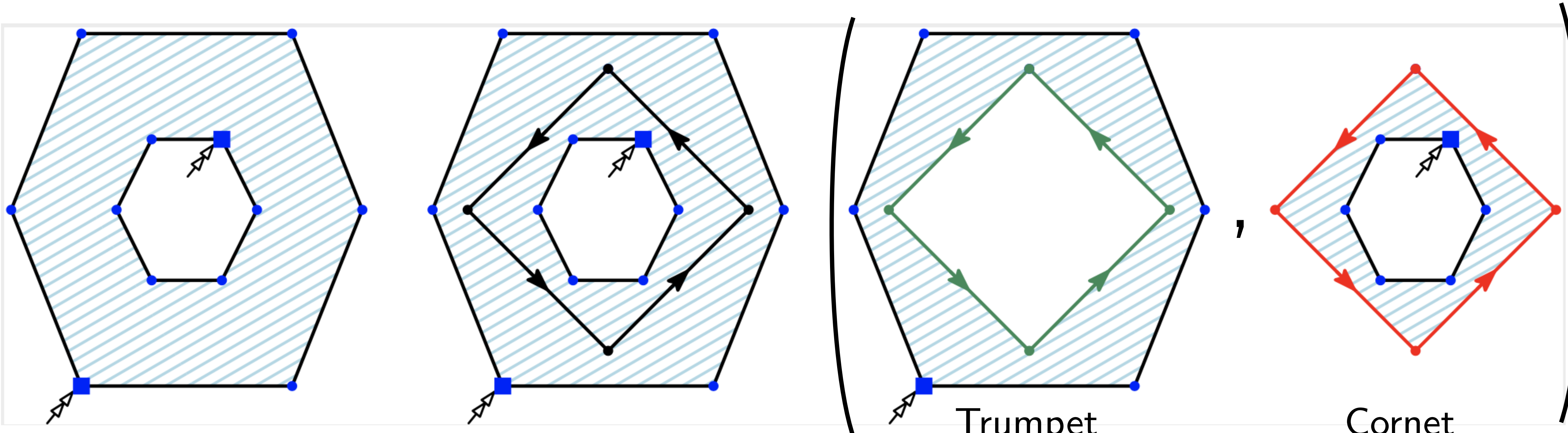

The generating series of hypermaps with two monochromatic boundaries are given by:

$$F^{\circ\circ}_{p,q} = \sum_{h\geq 1} h \left([z^h]x(z)^p\right)\left([z^{-h}]x(z)^q\right), \qquad F^{\circ\bullet}_{p,q} = \sum_{h\geq 1} h \left([z^h]x(z)^p\right)\left([z^{-h}]y(z)^q\right),$$

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Rooted maps via cylinders

$F_p^\circ, F_p^\bullet$:= generating series of hypermaps with a monochromatic white (resp. black) boundary of degree p. We established that:

$$\frac{d}{dt} F_p^\circ = [z^0] x(z)^p, \qquad \text{resp. } \frac{d}{dt} F_p^\bullet = [z^0] y(z)^p.$$

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admit the following rational parametrization in terms of $x(z)$ and $y(z)$:

$$Y(x(z)) = y(z) \quad \text{and} \quad X(y(z)) = x(z),$$

with $Y(x) := W^\circ(x) + \sum_{d\geq 1} t_d^\circ x^{d-1}$ and $X(y) := W^\bullet(y) + \sum_{d\geq 1} t_d^\bullet y^{d-1}$

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So let's do something else than integration !
i.e. let us try to give a combinatorial sense of Eynard's expressions...

Rooted maps via cylinders

Proposition: $$F_p^{\circ} = \frac{1}{p+1}\left(F_{p+1,1}^{\circ\bullet} - \sum_{k\geq 2} t_k^{\circ} F_{p+1,k-1}^{\circ\circ}\right)$$

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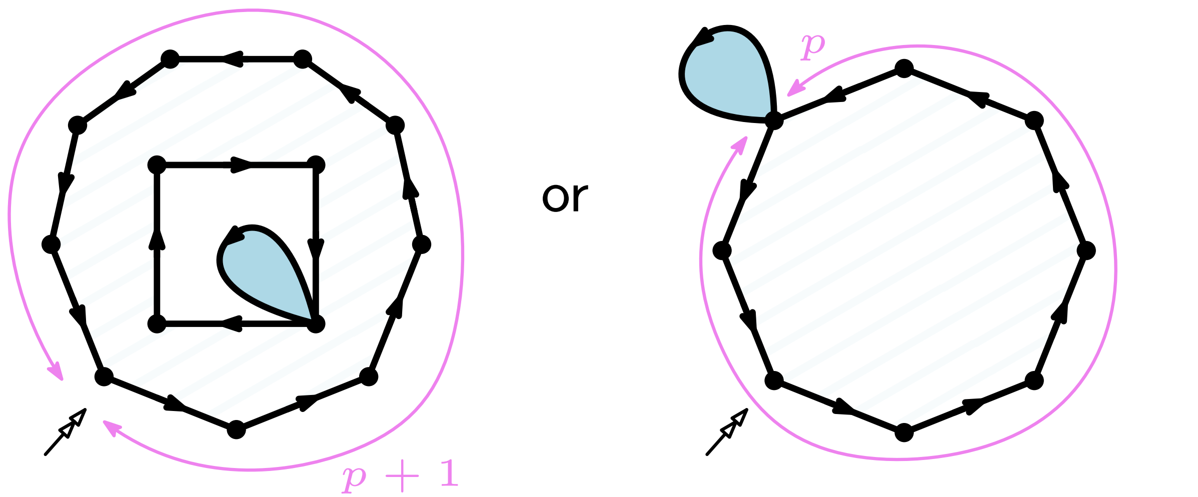

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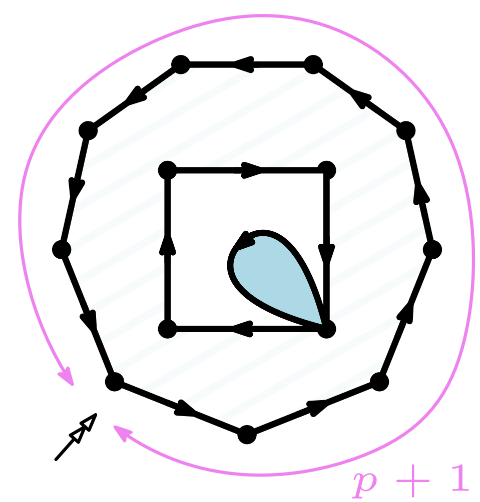

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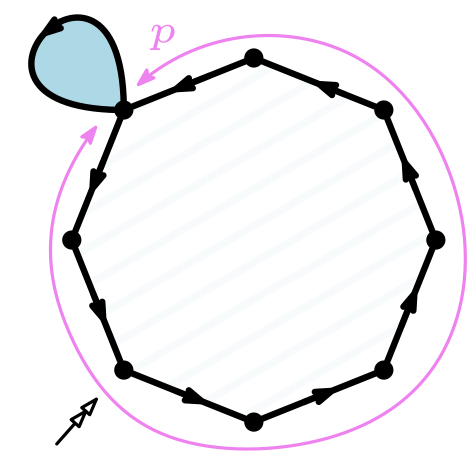

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$\Rightarrow \quad (p+1)F_p^{\circ}$

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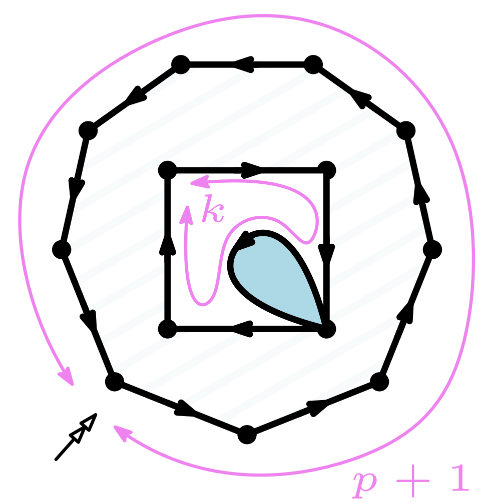

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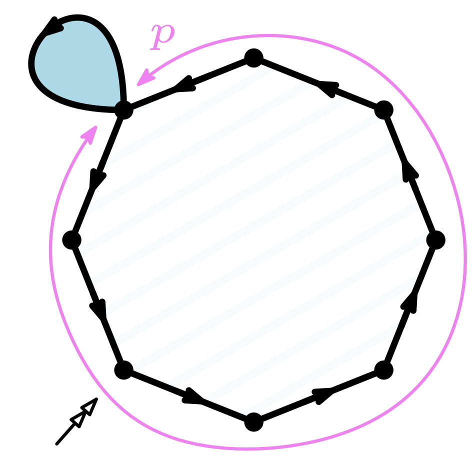

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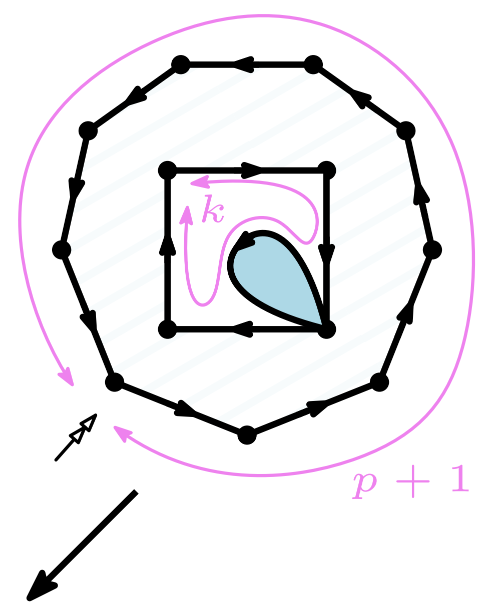

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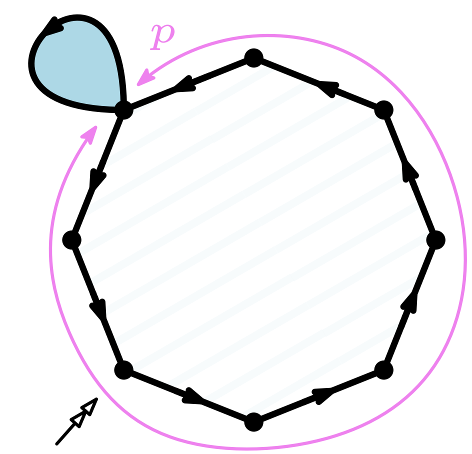

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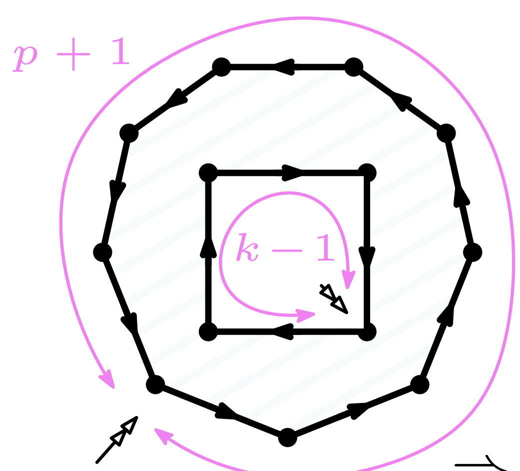

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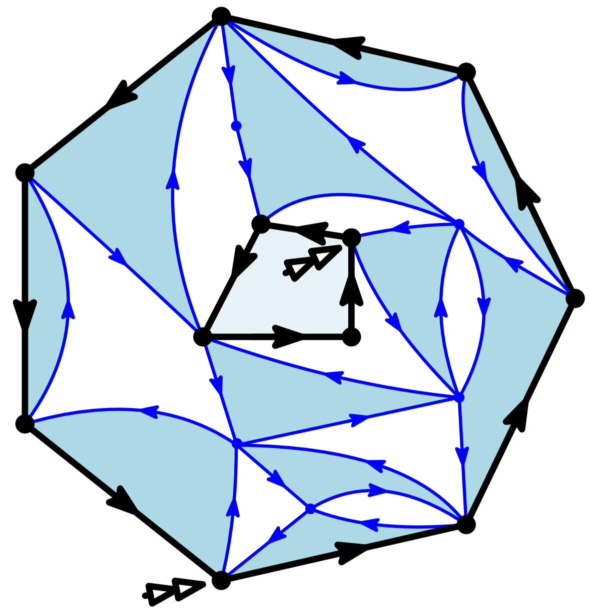
Hypermaps with boundaries

A map **with boundaries** is a map where some faces are marked (and rooted).
Other faces are called **inner faces**.



- Hypermap **with monochromatic boundaries**:

  All faces (inner and boundaries) are colored.
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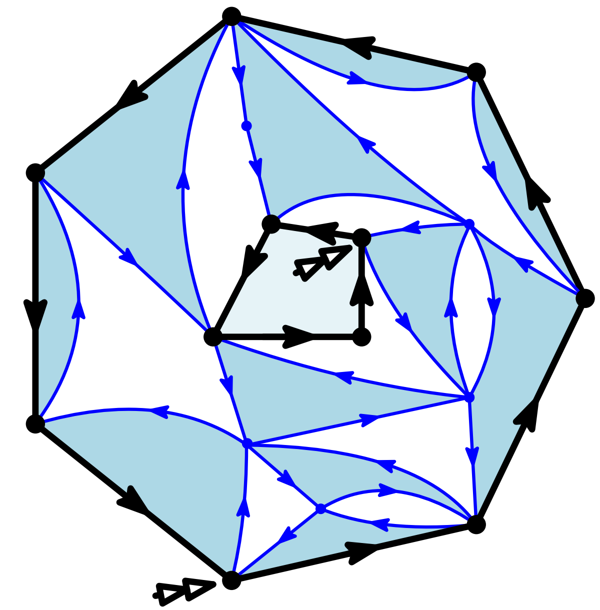
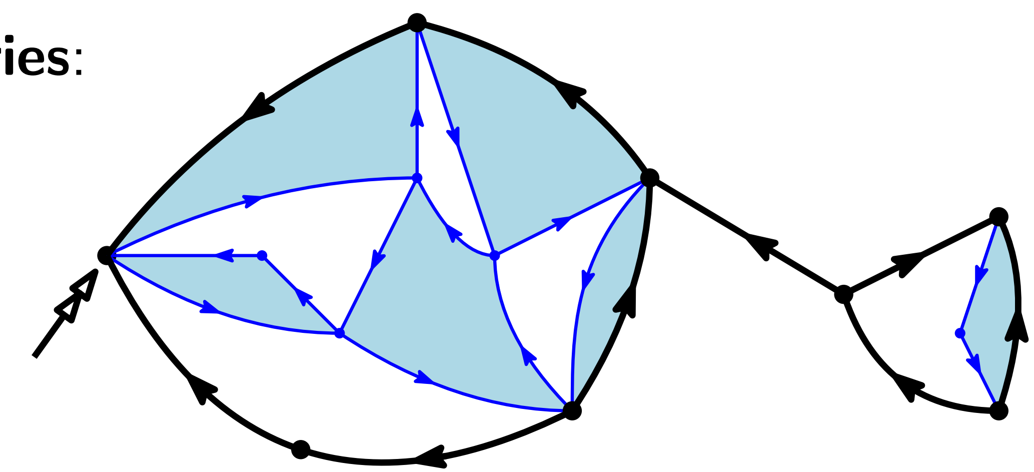


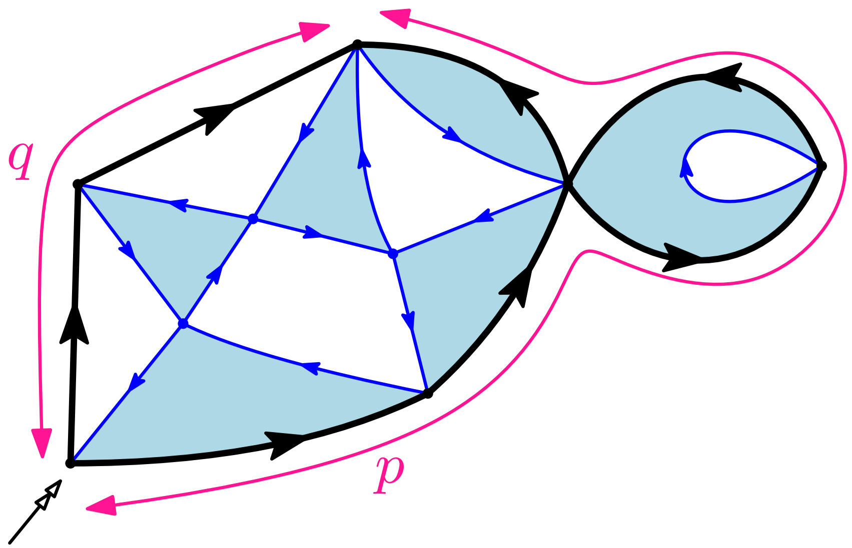
- Hypermap **with non-monochromatic boundaries**:

  Only the contour of the inner faces are required to be oriented.



One more result

Generating series of hypermaps with a **Dobrushin boundary condition**:



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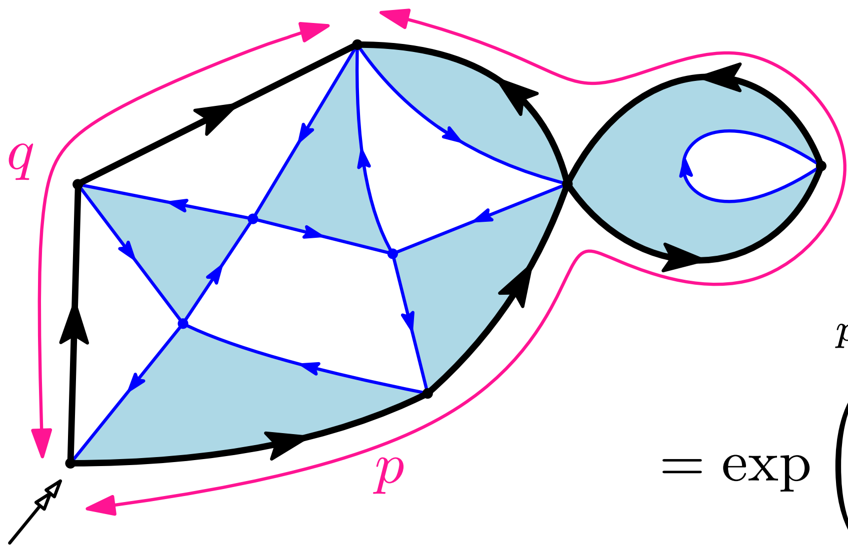

$$\sum_{p,q\geq 0} \frac{F^{◑}_{p,q}}{x^{p+1}y^{q+1}} = \exp\left(\sum_{h\in\mathbb{Z}} h\left([z^h]\ln\left(1-\frac{x(z)}{x}\right)\right)\left([z^{-h}]\ln\left(1-\frac{y(z)}{y}\right)\right)\right)-1$$

Again, the proof relies on some “trick” to interpret Dobrushin boundary condition as some special families of cylinders.

Conclusion

We gave **bijective derivation** of enumeration results for hypermaps with one or two boundaries.

$\rightarrow$ This new proof of known enumerative results, allows us to encode some (oriented) metric properties.

$\rightarrow$ This derivation was applied to constellations with some additional statistics in [Bonzom, Chapuy, Charbonnier, Garcia-Failde 24], to prove topological recursion for colored constellations.

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