

Hands-on Session: Nuclear Shell Model

Shell-model problem for a two-valence nucleons system

Nadezda Smirnova

Joliot-Curie School" "Nuclear Theory to Shape Experiment"

17 Septembre 2026

Objective: Set-up and solve the shell-model problem for a two-nucleon system

- 1 The Angular Momentum Toolbox
- 2 Two-Particle Basis & The Pauli Principle
- 3 Delta interaction and single-shell spectrum
- 4 Setting the Hamiltonian matrix and diagonalization
- 5 Matrix Elements of Transition Operators

1. The Angular Momentum Toolbox

1. Coupling Rules & Clebsch-Gordan Coefficients

When coupling two angular momenta $\vec{j}_1$ and $\vec{j}_2$ to a total $\vec{J} = \vec{j}_1 + \vec{j}_2$:

- **Triangular Condition:** $|j_1 - j_2| \leq J \leq j_1 + j_2$
- **Projection Conservation:** $M = m_1 + m_2$

The transformation from the uncoupled basis $|j_1 m_1 j_2 m_2\rangle$ to the coupled basis $|(j_1 j_2) JM\rangle$ is mediated by the Clebsch-Gordan (CG) coefficients:

$$|(j_1 j_2) JM\rangle = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | JM) |j_1 m_1\rangle |j_2 m_2\rangle$$

Important Property: Symmetry under change of the coupling order

Exchanging the order of coupling introduces a phase shift:

$$|(j_2 j_1) JM\rangle = (-1)^{j_1 + j_2 - J} |(j_1 j_2) JM\rangle$$

Wigner 3j Symbols

To handle multiple couplings elegantly, instead of Clebsch-Gordan coefficients we can use more symmetric objects: **Wigner 3j Symbol**:

$$(j_1 m_1 j_2 m_2 | JM) = (-1)^{j_1 - j_2 + M} \sqrt{2J + 1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}$$

where $m_1 + m_2 - M = 0$.

Highly symmetric under column permutations!

Attention: these are not matrices, but real numbers!

Wigner 6j Symbols

How to couple three angular momenta $\vec{j}_1$, $\vec{j}_2$, and $\vec{j}_3$ to a total $\vec{J}$?

Basis 1: Intermediate J_{12}

First couple 1 and 2, then add 3:

$$\vec{j}_1 + \vec{j}_2 = \vec{J}_{12} \quad \rightarrow \quad \vec{J}_{12} + \vec{j}_3 = \vec{J}$$

Basis 2: Intermediate J_{23}

First couple 2 and 3, then add 1:

$$\vec{j}_2 + \vec{j}_3 = \vec{J}_{23} \quad \rightarrow \quad \vec{j}_1 + \vec{J}_{23} = \vec{J}$$

Re-coupling three angular momenta, e.g., changing from $|(j_1 j_2) J_{12} j_3; JM\rangle$ to $|j_1 (j_2 j_3) J_{23}; JM\rangle$ is mediated by the 6j symbol:

$$|j_1 (j_2 j_3) J_{23}; JM\rangle = \sum_{J_{12}} \kappa \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} |(j_1 j_2) J_{12} j_3; JM\rangle$$

where $\kappa = (-1)^{j_1+j_2+j_3+J} \sqrt{(2J_{12}+1)(2J_{23}+1)}$.

Wigner 9j Symbols

How to couple four angular momenta, for example, $\vec{l}_1, \vec{s}_1, \vec{l}_2, \vec{s}_2$ to a total $\vec{J}$?

Used for re-coupling four angular momenta, vital for changing frameworks from LS -coupling to jj -coupling in the two-nucleon wave function:

$$|(l_1 s_1)j_1, (l_2 s_2)j_2; JM\rangle = \sum_{L,S} \kappa \begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix} |(l_1 l_2)L, (s_1 s_2)S; JM\rangle$$

where for nucleons $s_1 = s_2 = 1/2$ and therefore $S = 0, 1$, and

$$\kappa = \sqrt{(2j_1 + 1)(2j_2 + 1)(2L + 1)(2S + 1)}$$

Spherical Tensors

A spherical tensor $T_{\lambda\mu}$ of rank λ is a set of $(2\lambda + 1)$ operators, $\mu = -\lambda, -\lambda + 1, \dots, \lambda - 1, \lambda$ which obey the following commutation relations with the angular momentum operators $J_{\pm} = J_x \pm iJ_y$ and J_0 :

$$\begin{aligned}[J_{\pm}, T_{\lambda\mu}] &= \sqrt{\lambda(\lambda + 1) - \mu(\mu \pm 1)} T_{\lambda, \mu \pm 1} \\ [J_0, T_{\lambda\mu}] &= \mu T_{\lambda\mu}\end{aligned}$$

These commutation relations can be compared with the action of the angular momentum operator on the states of the angular momentum $J = \lambda$:

$$\begin{aligned}J_{\pm}|\lambda\mu\rangle &= \sqrt{\lambda(\lambda + 1) - \mu(\mu \pm 1)}|\lambda, \mu \pm 1\rangle \\ J_0|\lambda\mu\rangle &= \mu|\lambda\mu\rangle\end{aligned}$$

Examples of Spherical Tensors

Rank	Parity-even	Parity-odd
$\lambda = 0$	Scalar	Pseudoscalar ($\vec{p} \cdot \vec{L}$)
$\lambda = 1$	Polar Vector ($\vec{r}$) $r_{\pm 1} = \mp \frac{1}{\sqrt{2}}(x \pm iy)$ $r_0 = z$	Axial Vector ($\vec{L}$) $L_{\pm}, L_0$
$\lambda \in N$	$Y_{\lambda\mu}(\hat{r})$ λ even	of parity $(-1)^\lambda$ λ odd

Irreducible tensorial operators and the Wigner-Eckart Theorem

The absolute cornerstone for calculating observables. It separates the **geometry** (magnetic projections) from the **physics** (nuclear structure dynamics). For a spherical tensor operator $T_{\lambda\mu}$ of rank λ :

$$\langle j_f m_f | T_{\lambda\mu} | j_i m_i \rangle = (-1)^{j_f - m_f} \begin{pmatrix} j_f & \lambda & j_i \\ -m_f & \mu & m_i \end{pmatrix} \langle j_f || T_\lambda || j_i \rangle$$

What is the importance for physics?

- The matrix element is **zero** unless $|j_f - j_i| \leq \lambda \leq j_f + j_i$ and $m_f = m_i + \mu$.
- To know the transition rate between states, you only need to compute the **Reduced Matrix Element** $\langle j_f || T_\lambda || j_i \rangle$ **once**, regardless of the magnetic sublevels m .

Examples: reduced Matrix Element of the $\vec{J}$ operator and of the spherical harmonic

Let's calculate the matrix elements of $\hat{J}_0$ operator and of the spherical harmonic $Y_{\lambda\mu}$ and then apply the Wigner-Eckart theorem. We can identify the following reduced matrix elements.

Reduced matrix element of the angular momentum operator

The reduced matrix element of $\vec{J}$ within the states of the same j ($j_f = j_i = j$) evaluates to a simple geometric factor:

$$\langle j || \vec{J} || j \rangle = \sqrt{j(j+1)(2j+1)}$$

Examples: $\langle 1/2 || \vec{\sigma} || 1/2 \rangle = 2\langle 1/2 || \vec{s} || 1/2 \rangle = \sqrt{6}$

Reduced matrix element of the spherical harmonics

For states without spin consideration, the spatial integration over three spherical harmonics yields:

$$\langle l_f || Y_\lambda || l_i \rangle = (-1)^{l_f} \sqrt{\frac{(2l_f + 1)(2\lambda + 1)(2l_i + 1)}{4\pi}} \begin{pmatrix} l_f & \lambda & l_i \\ 0 & 0 & 0 \end{pmatrix}$$

2. BASIS CONSTRUCTION and THE PAULI PRINCIPLE

Two-particle wave function: M -scheme basis

Convention for single-particle wave functions: $\langle \vec{r}, \vec{\sigma} | n l s j m \rangle \equiv \langle \vec{r}, \vec{\sigma} | j m \rangle = \phi_{n l s j m}(\vec{r}) = R_{n l}(r) [Y_l \otimes \chi_s]_m^{(j)}$.

For a harmonic oscillator radial wave function:

$$R_{n l}(r) = \mathcal{N}_{n l} r^l \exp\left(-\frac{r^2}{2b^2}\right) L_n^{l+1/2}\left(\frac{r^2}{b^2}\right),$$

$b = \sqrt{\frac{\hbar}{\mu\omega}}$ is the harmonic oscillator length parameter, $L_n^{l+1/2}\left(\frac{r^2}{b^2}\right)$ are generalized Laguerre polynomials,

the normalization factor $\mathcal{N}_{n l}$ is from $\int_0^\infty R_{n l}^2(r) r^2 dr = 1$.

Slater determinants

$$\Phi_{\alpha_1, \alpha_2}(1, 2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_{\alpha_1}(\vec{r}_1) & \phi_{\alpha_1}(\vec{r}_2) \\ \phi_{\alpha_2}(\vec{r}_1) & \phi_{\alpha_2}(\vec{r}_2) \end{vmatrix}$$

where $\alpha_i = (n_i, l_i, s_i, j_i, m_i)$ is a set of spherical quantum numbers ($s_i = 1/2$) and
Our notation of a quantum state: $|\alpha_1, \alpha_2\rangle$

Constructing the Two-Body Basis: Identical Nucleons

Let's build a coupled state $|(j_a j_b) JM\rangle$ for two identical nucleons (e.g., two neutrons) in the model space comprised by two orbitals j_a and j_b .

To respect the **Pauli Exclusion Principle**, the two-body state must be completely antisymmetric under particle exchange ($1 \leftrightarrow 2$):

$$|j_a j_b; JM\rangle_{AS} = \mathcal{N} \left(|j_a(1) j_b(2); JM\rangle - |j_a(2) j_b(1); JM\rangle \right)$$

Using the CG exchange phase property from Part 1, the second term expands to:

$$|j_a(2) j_b(1); JM\rangle = (-1)^{j_a + j_b - J} |j_b(1) j_a(2); JM\rangle$$

Normalization factor for $j_a \neq j_b$: $\mathcal{N} = 1/\sqrt{2}$

Constructing the Two-Body Basis: Identical Nucleons

What happens if two identical nucleons (e.g., two neutrons) in the **same orbital** $j_a = j_b = j$.

The two-body state, antisymmetric under particle exchange ($1 \leftrightarrow 2$):

$$|jj; JM\rangle_{\text{AS}} = \mathcal{N} \left(|j(1)j(2); JM\rangle - |j(2)j(1); JM\rangle \right)$$

Using the Clebsch-Gordan exchange phase property from Part 1, the second term expands to:

$$|j(2)j(1); JM\rangle = (-1)^{j+j-J} |j(1)j(2); JM\rangle = (-1)^{2j-J} |j(1)j(2); JM\rangle$$

Phase simplification:

Since j is a half-integer for nucleons (e.g., $5/2, 7/2$), $2j$ is always an **odd integer**. What happens to the state $|jj; JM\rangle_{\text{AS}}$ if J is chosen to be an odd integer?

The Even- J Constraint

Substituting the phase into the antisymmetric expression:

$$|jj; JM\rangle_{\text{AS}} = \mathcal{N} (1 + (-1)^J) |j(1)j(2); JM\rangle$$

We conclude that:

- If J is **even**: $(-1)^{\text{even}} = +1$. $\implies (1 + 1) = 2 \neq 0 \rightarrow$ **State is allowed!**, $\mathcal{N} = 1/2$.
- If J is **odd**: $(-1)^{\text{odd}} = -1$. $\implies (1 - 1) = 0 \rightarrow$ **State vanishes!**

Conclusion

For a configuration of two identical nucleons in the same orbit j^2 , the Pauli principle restricts the allowed total angular momentum states strictly to **even values**:

$$J = 0, 2, 4, \dots, (2j - 1)$$

Including Isospin

If the two nucleons are not necessarily identical (each can be either neutron or proton), but occupy the same orbital j :

$$|jj; JM, T\rangle_{AS} = \mathcal{N} (1 - (-1)^{J+T}) |j(1)j(2); JM, T\rangle$$

where T can take values 0 or 1.

- If $(J + T)$ is **odd**: $\implies (1 - (-1)^{\text{odd}}) = 2 \neq 0 \rightarrow$ **State is allowed!**
- If $(J + T)$ is **even**: $\implies (1 - (-1)^{\text{even}}) = 0 \rightarrow$ **State vanishes!**

Conclusion

For a configuration of a proton and a neutron the same orbit j^2 , the Pauli principle restricts $(J + T)$ strictly to **odd values**:

$$T = 1 : J = 0, 2, 4, \dots, 2j - 1$$

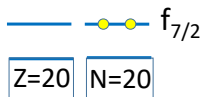
$$T = 0 : J = 1, 3, 5, \dots, 2j$$

Exercise 1: Basis construction

2 neutrons in $0f_{7/2}$ orbital

$J^\pi, T=1$

?

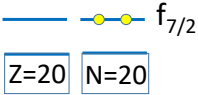


- Write down the basis in J -coupled form.
- Write down the basis in M -scheme (number the Slater determinants).
- How many different J -states exist in this model space?

Solution to exercise 1

2 neutrons in $0f_{7/2}$ orbital

- Basis in J -coupled scheme:

		$J^\pi, T=1 \quad (2J+1)$	
$(f_{7/2})^2$		0^+	1
		2^+	5
		4^+	9
		6^+	13

$$|(0f_{7/2})^2; J, T=1\rangle \Rightarrow J^\pi = 0^+, 2^+, 4^+, 6^+$$

For each J , there are $(2J+1)$ substates with $M = -J, \dots, J$,

that is we have $1 + 5 + 9 + 13 = 28$ states with different $|JM\rangle$.

Solution to exercise 1 (continued)

Basis construction

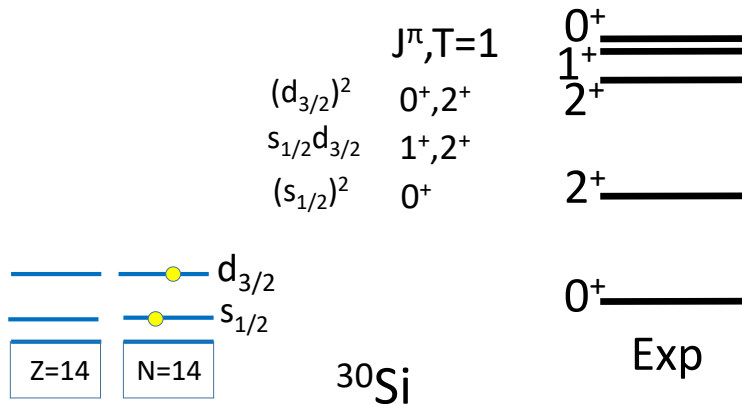
- Basis in M -scheme (antisymmetrized and normalized products of s.p. states - Slater determinants): $|m_a m_b; M\rangle$, where $m_a, m_b = -7/2, -5/2, \dots, 5/2, 7/2$, $M = m_a + m_b$. **Attention: J is not a good quantum number!**

$$\begin{array}{llll}
 | \frac{7}{2}, \frac{5}{2}; 6 \rangle & & & \\
 | \frac{7}{2}, \frac{3}{2}; 5 \rangle & & & \\
 | \frac{7}{2}, \frac{1}{2}; 4 \rangle & | \frac{5}{2}, \frac{3}{2}; 4 \rangle & & \\
 | \frac{7}{2}, -\frac{1}{2}; 3 \rangle & | \frac{5}{2}, \frac{1}{2}; 3 \rangle & & \\
 | \frac{7}{2}, -\frac{3}{2}; 2 \rangle & | \frac{5}{2}, -\frac{1}{2}; 2 \rangle & | \frac{3}{2}, \frac{1}{2}; 2 \rangle & \\
 | \frac{7}{2}, -\frac{5}{2}; 1 \rangle & | \frac{5}{2}, -\frac{3}{2}; 1 \rangle & | \frac{3}{2}, -\frac{1}{2}; 1 \rangle & \\
 | \frac{7}{2}, -\frac{7}{2}; 0 \rangle & | \frac{5}{2}, -\frac{5}{2}; 0 \rangle & | \frac{3}{2}, -\frac{3}{2}; 0 \rangle & | \frac{1}{2}, -\frac{1}{2}; 0 \rangle \\
 | -\frac{7}{2}, \frac{5}{2}; -1 \rangle & | -\frac{5}{2}, \frac{3}{2}; -1 \rangle & | -\frac{3}{2}, \frac{1}{2}; -1 \rangle & \\
 | -\frac{7}{2}, \frac{3}{2}; -2 \rangle & | -\frac{5}{2}, \frac{1}{2}; -2 \rangle & | -\frac{3}{2}, -\frac{1}{2}; -2 \rangle & \\
 | -\frac{7}{2}, \frac{1}{2}; -3 \rangle & | -\frac{5}{2}, -\frac{1}{2}; -3 \rangle & & \\
 | -\frac{7}{2}, -\frac{1}{2}; -4 \rangle & | -\frac{5}{2}, -\frac{3}{2}; -4 \rangle & & \\
 | -\frac{7}{2}, -\frac{3}{2}; -5 \rangle & & & \\
 | -\frac{7}{2}, -\frac{5}{2}; -6 \rangle & & &
 \end{array}$$

- There are 4 different J -states: $J^\pi = 0^+, 2^+, 4^+, 6^+$ (for each J , $M = -J, \dots, J$).

What type of the basis (J -coupled or M -scheme) do you prefer?

Example: $^{30}_{14}\text{Si}_{16}$ as $^{28}_{14}\text{Si}_{14}$ plus 2 neutrons



Three-particle wave function: M -scheme basis

Slater determinants

$$\Phi_{\alpha}(1, 2, 3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \phi_{\alpha_1}(\vec{r}_1) & \phi_{\alpha_1}(\vec{r}_2) & \phi_{\alpha_1}(\vec{r}_3) \\ \phi_{\alpha_2}(\vec{r}_1) & \phi_{\alpha_2}(\vec{r}_2) & \phi_{\alpha_2}(\vec{r}_3) \\ \phi_{\alpha_3}(\vec{r}_1) & \phi_{\alpha_3}(\vec{r}_2) & \phi_{\alpha_3}(\vec{r}_3) \end{vmatrix}$$

where $\alpha_i = (n_i, l_i, j_i, m_i)$ and α stores a set of single-particle configurations $\{\alpha_1, \alpha_2, \alpha_3\}$

$$M = \sum_{i=1}^3 m_i$$

Our notation of a quantum state: $|\alpha_1, \alpha_2, \alpha_3\rangle$

Homework: Three identical nucleons in $d_{5/2}$ orbital

Basis construction

Basis in M -scheme: $|m_a m_b m_c; M\rangle$,

where $m_a, m_b, m_c = -5/2, -3/2, -1/2, 1/2, 3/2, 5/2$:

$$| \frac{5}{2}, \frac{3}{2}, \frac{1}{2}; 9/2 \rangle$$

...

...

...

...

...

What J^π, T values are possible ?

Three-particle wave function: antisymmetrized J -coupled states.

Let's construct a J -coupled state of three nucleons in three orbitals j_a, j_b, j_c antisymmetric under any two-nucleon exchange ($i \leftrightarrow k$) and where two of the nucleons are coupled to the intermediate angular momentum J' :

$$\begin{aligned} |j_a j_b j_c(J'); JM\rangle &= \frac{1}{3!} \sum_{P(\text{permutations})} (-1)^P \hat{P} |(j_a(1)j_b(2))J'j_c(3); JM\rangle \\ &= \frac{1}{6} (|(j_a(1)j_b(2))J'j_c(3); JM\rangle - |(j_a(2)j_b(1))J'j_c(3); JM\rangle \\ &\quad - |(j_a(1)j_b(3))J'j_c(2); JM\rangle + |(j_a(2)j_b(3))J'j_c(1); JM\rangle \\ &\quad + |(j_a(3)j_b(1))J'j_c(2); JM\rangle - |(j_a(3)j_b(2))J'j_c(1); JM\rangle) \end{aligned}$$

Three-particles in the same j -state.

Let's construct a J -coupled state of three identical nucleons in a single- j orbital ($j_a = j_b = j_c = j$)

$$|j^3(J'); JM\rangle = \frac{1}{\mathcal{N}} (|j^2(1,2)[J']j(3); JM\rangle - |j^2(1,3)[J']j(2); JM\rangle + |j^2(2,3)[J']j(1); JM\rangle)$$

Using $6j$ -coefficients

$$|j^2(2,3)[J']j(1); JM\rangle = \sum_{J''=\text{even}} \sqrt{(2J'+1)(2J''+1)} \left\{ \begin{matrix} j & j & J'' \\ j & j & J' \end{matrix} \right\} |j^2(1,2)[J'']j(3); JM\rangle$$

$$|j^2(1,3)[J']j(2); JM\rangle = - \sum_{J''=\text{even}} \sqrt{(2J'+1)(2J''+1)} \left\{ \begin{matrix} j & j & J'' \\ j & j & J' \end{matrix} \right\} |j^2(1,2)[J'']j(3); JM\rangle$$

Three-particle wave function: J -coupled states.

J -coupled state $j_a = j_b = j_c = j$

$$|j^3(J'); JM\rangle = \frac{1}{\mathcal{N}} \sum_{J''=\text{even}} \left[\delta_{J'J''} + 2\sqrt{(2J'+1)(2J''+1)} \left\{ \begin{matrix} j & j & J'' \\ j & j & J' \end{matrix} \right\} \right] |j^2(1,2)[J'']j(3); JM\rangle$$

with the normalization constant from

$$\mathcal{N}^2 = 3 + 6(2J'+1) \left\{ \begin{matrix} j & j & J'' \\ j & j & J' \end{matrix} \right\}.$$

One-nucleon coefficients of fractional parentage (c.f.p.'s):

$$|j^3(J'); JM\rangle = \sum_{J''=\text{even}} \underbrace{\langle j^2[J'']j | j^3 J' J \rangle}_{\text{c.f.p.}} |j^2(1,2)[J'']j(3); JM\rangle$$

Solution of exercise 2: Three identical nucleons in $d_{5/2}$ orbital

Basis in J -coupled scheme: $|(d_{5/2}^3; J, T = 3/2)\rangle$

To get antisymmetrized states, we use the formula for c.f.p.'s from the previous slide:

$$\begin{aligned} |d_{5/2}^3; J^\pi = 3/2^+\rangle &= -\frac{\sqrt{5}}{\sqrt{7}} |d_{5/2}^2 [J''=2] d_{5/2}; J^\pi = 3/2^+\rangle \\ &+ \frac{\sqrt{2}}{\sqrt{7}} |d_{5/2}^2 [J''=4] d_{5/2}; J^\pi = 3/2^+\rangle \\ |d_{5/2}^3; J^\pi = 5/2^+\rangle &= -\frac{\sqrt{2}}{3} |d_{5/2}^2 [J''=0] d_{5/2}; J^\pi = 5/2^+\rangle \\ &+ \frac{\sqrt{5}}{3\sqrt{2}} |d_{5/2}^2 [J''=2] d_{5/2}; J^\pi = 5/2^+\rangle \\ &+ \frac{1}{\sqrt{2}} |d_{5/2}^2 [J''=4] d_{5/2}; J^\pi = 5/2^+\rangle \\ |d_{5/2}^3; J^\pi = 9/2^+\rangle &= \frac{\sqrt{3}}{\sqrt{14}} |d_{5/2}^2 [J''=2] d_{5/2}; J^\pi = 9/2^+\rangle \\ &- \frac{\sqrt{11}}{\sqrt{14}} |d_{5/2}^2 [J''=4] d_{5/2}; J^\pi = 9/2^+\rangle \end{aligned}$$

Remark: For $j > 7/2$, we need J' to distinguish different 3-nucleon states coupled to the same J .

3. DELTA INTERACTION AND SINGLE-SHELL SPECTRUM

Setting up the Hamiltonian for a 2-Nucleon System

Let's construct in the J -coupled basis the Hamiltonian matrix $H = H_0 + V$ for two nucleons. The total energy is the sum of Single-Particle Energies (SPE) and the Two-Body Matrix Elements (TBME).

General Matrix Element Expression

In a given J^π subspace, the matrix elements connecting an initial two-body configuration $|j_a j_b\rangle_J$ to a final configuration $|j_c j_d\rangle_J$ are given by:

$$\langle j_c j_d | H | j_a j_b \rangle_J = \underbrace{(\epsilon_a + \epsilon_b) \delta_{ac} \delta_{bd}}_{\text{Eigenstates of } H_0 \text{ (SPEs)}} + \underbrace{\langle j_c j_d | V | j_a j_b \rangle_J}_{\text{Residual Interaction (TBME)}, \text{ Antisymmetrized}}$$

Single-shell problem: $j_a = j_b = j_c = j_d = j$

$$E_J = \langle j j | H | j j \rangle_J = 2\epsilon_j + \langle j j | V | j j \rangle_{J, \text{AS}}$$

Practical approaches to effective interaction

- **Schematic** interaction (parametrized interaction between two nucleons in a nuclear medium)
- **Empirical** interaction (Fit of TBME's to energy levels of nuclei to be described within the chosen model space)
- **Microscopic** interaction (derived from a bare NN-force) $\Rightarrow$ *ab-initio* approach

Techniques for evaluation of TBMEs of schematic interactions

I. Technique of transformation to relative and center-of-mass coordinates

$$\begin{aligned} H &= \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 r_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 r_2^2 \\ &= \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 r^2 + \frac{P^2}{2M} + \frac{1}{2}M\omega^2 R^2, \end{aligned}$$

where

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2), \vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2), \vec{P} = \vec{p}_1 + \vec{p}_2, \mu = \frac{m}{2}, M = 2m$$

Talmi-Moshinsky transformation from particle-particle to relative and center-of-mass basis:

$$|n_1 l_1(\vec{r}_1), n_2 l_2(\vec{r}_2); LM_L\rangle = \sum_{nl, N\Lambda} \langle nl, N\Lambda; L | n_1 l_1, n_2 l_2; L \rangle |nl(\vec{r}), N\Lambda(\vec{R}); LM_L\rangle$$

with $2n_1 + l_1 + 2n_2 + l_2 = 2n + l + 2N + \Lambda$

Techniques for evaluation of TBMEs of schematic interactions

II. Multipole decomposition

$$V(|\vec{r}_1 - \vec{r}_2|) = \sum_{k=0}^{\infty} v_k(r_1, r_2) P_k(\cos \theta_{12}),$$

where

$$P_k(\cos \theta_{12}) = \sum_{q=-k}^k \frac{4\pi}{2k+1} Y_{kq}^*(\Omega_1) Y_{kq}(\Omega_2)$$

Algorithm:

To compute the Two-Body Matrix Element (TBME) $\langle j_a j_b | V | j_c j_d \rangle_{J,AS}$ (direct and exchange terms), we must decouple the spatial and angular-momentum parts:

- 1 Separate the single-particle wavefunctions into radial and angular parts: $\psi_{nljm}(\vec{r}) = R_{nl}(r)[Y_l \otimes \chi_{1/2}]_{jm}^j$;
- 2 Compute two-dimensional integrals of $v_k(r_1, r_2)$;
- 3 Use angular-momentum algebra to compute the matrix elements of $Y_{kq}^*(\Omega_1) Y_{kq}(\Omega_2)$ within J coupled states.

The 3D Delta Function Interaction

To simulate the extremely short range of the residual strong force between valence nucleons, we use a **3D Delta function interaction**:

$$V(\vec{r}_1 - \vec{r}_2) = -V_0 \delta(\vec{r}_1 - \vec{r}_2) = -V_0 \frac{\delta(r_1 - r_2)}{r_1 r_2} \sum_{q=-k}^k Y_{kq}^*(\Omega_1) Y_{kq}(\Omega_2)$$

Where $V_0 > 0$ represents the attractive strength.

Analytical Result of the 3D Delta Interaction

For diagonal matrix elements ($j_a \neq j_b$) between antisymmetrized J -coupled states, we get:

$$\langle j_a j_b | V_\delta | j_a j_b \rangle_J = -V_0 I_R \frac{(2j_a + 1)(2j_b + 1)}{4\pi} \begin{pmatrix} j_a & j_b & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2$$

Where:

- I_R is the radial overlap integral: $I_R = \int_0^\infty R_{n_a l_a}^2(r) R_{n_b l_b}^2(r) r^2 dr$
- Parity selection rule: $l_a + l_b + J$ must be **even**.

Example: Two-Neutron Spectrum in ^{18}O

Let's calculate the explicit energy levels of the two valence neutrons in ^{18}O outside the doubly-magic ^{16}O core, restricted to the single orbit $(0d_{5/2})^2$.

- **Residual Interaction:** Pure 3D Delta force with $V_0 = 400 \text{ MeV} \cdot \text{fm}^3$.
- **Core Geometry:** For ^{18}O , the radial overlap integral for the $0d$ wavefunctions yields approximately $I_R = \int_0^\infty R_{0d}^4(r) r^2 dr \approx 0.039 \text{ fm}^{-3}$.
- **Single-Particle Energy:** $\epsilon_{d5/2} = -4.14 \text{ MeV}$ (from the ^{17}O and ^{16}O binding energies).

Energies of Two Identical Nucleon states

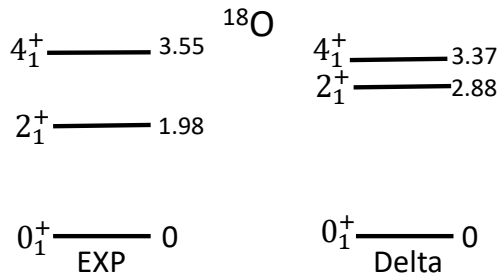
In case of $j_a = j_b = j$ for even J :

$$E(J) = 2\epsilon_{d5/2} - V_0 I_R \frac{(2j+1)^2}{8\pi} \begin{pmatrix} 5/2 & 5/2 & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2$$

Numerical Results and Energy Levels

Plugging in $V_0 \cdot I_R \approx 15.2$ MeV and the analytic values of the 3j symbols:

- $E(0^+) = -8.28 - 3.73$ MeV = **-12.01** MeV
(Ground State)
- $E(2^+) = -8.28 - 0.85$ MeV = **-9.13** MeV
(2.88 MeV excitation energy)
- $E(4^+) = -8.28 - 0.35$ MeV = **-8.63** MeV
(3.37 MeV excitation energy)



Heavy Nuclei Comparison: ^{210}Pb and ^{210}Po

Let's move from light systems to heavy systems near the doubly-magic ^{208}Pb core ($A = 210$). We apply 3D Delta force ($V_0 = 450 \text{ MeV} \cdot \text{fm}^3$).

^{210}Pb : 2 valence neutrons

- **Orbit:** Pure $(1g_{9/2})^2$ configuration.
- **Radial Integral:** $I_R \approx 0.0058 \text{ fm}^{-3}$.
- **Analytical $J = 0$ Shift:**

$$\begin{aligned}\Delta E(0^+) &= -V_0 I_R \frac{(2j+1)^2}{8\pi \times 10} \\ &\approx -1.0 \text{ MeV}\end{aligned}$$

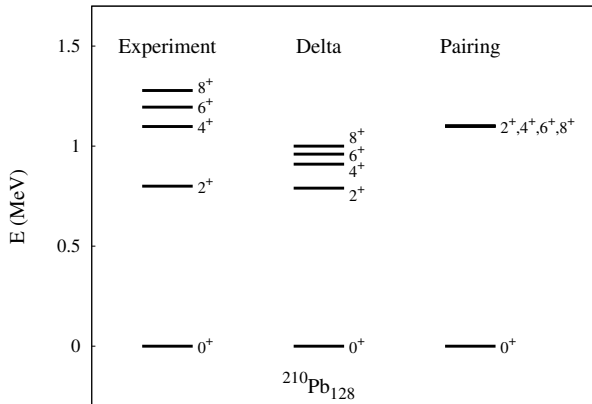
^{210}Po : 2 valence protons

- **Orbit:** Pure $(0h_{9/2})^2$ configuration.
- **Radial Integral:** $I_R \approx 0.0067 \text{ fm}^{-3}$.
- **Analytical $J = 0$ Shift:**

$$\begin{aligned}\Delta E(0^+) &= -V_0 I_R \frac{(2j+1)^2}{8\pi \times 10} \\ &\approx -1.2 \text{ MeV}\end{aligned}$$

Example: ^{210}Pb in $(\nu 1g_{9/2})^2$

$$\langle j^2 | V_{\text{pairing}} | j^2 \rangle_J = -\frac{1}{2} G(2j+1) \delta_{J0}$$



Setting the Hamiltonian matrix and diagonalization

Solution of the eigenvalue problem $H = H_0 + V$

- Construct a basis in the valence space: $|\Phi_k\rangle$
- Expand unknown eigenstate in terms of basis states

$$|\Psi_p\rangle = \sum_{k=1}^n a_{kp} |\Phi_k\rangle$$

- Multiplying $H|\Psi_p\rangle = E_p|\Psi_p\rangle$ by $\langle\Phi_l|$, we get the matrix equation to obtain $E_p, \{a_{kp}\}$:

$$\sum_{k=1}^n H_{lk} a_{kp} = E_p a_{lp}, \quad \text{with} \quad H_{lk} = \langle\Phi_l|\hat{H}|\Phi_k\rangle = E_k^{(0)}\delta_{lk} + V_{lk}$$

$$\begin{pmatrix} H_{11} & H_{12} & \dots & \dots & H_{1n} \\ H_{21} & H_{22} & \dots & \dots & H_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ H_{n1} & H_{n2} & \dots & \dots & H_{nn} \end{pmatrix} \longrightarrow \begin{array}{c} \text{=====} \\ \text{=====} \\ \text{=====} \\ \text{=====} \\ \text{=====} \end{array}$$

Exercise 3: The spectrum of ^{18}O in sd -shell



Z=8

N=8

$J^\pi, T = ?$

$E, \Psi = ?$

$$\langle j_c j_d | \hat{H} | j_a j_b \rangle_J = (\epsilon_a + \epsilon_b) \delta_{ac} \delta_{bd} + \langle j_c j_d | V | j_a j_b \rangle_{J, AS}$$

Single-particle energies:

$$\epsilon(d_{5/2}) = E_B(^{17}\text{O}_9) - E_B(^{16}\text{O}_8) = -4.14 \text{ MeV}$$

$$\epsilon(s_{1/2}) = \epsilon(d_{5/2}) + E_{\text{ex}}(^{17}\text{O}; 1/2_1^+) = -3.28 \text{ MeV}$$

$$\epsilon(d_{3/2}) = \epsilon(d_{5/2}) + E_{\text{ex}}(^{17}\text{O}; 3/2_1^+) = 0.93 \text{ MeV}$$

2-nucleon basis:

$$(d_{5/2})^2 \quad J^\pi = 0^+, 2^+, 4^+$$

$$(s_{1/2})^2 \quad J^\pi = 0^+$$

$$(d_{3/2})^2 \quad J^\pi = 0^+, 2^+,$$

$$d_{5/2} d_{1/2} \quad J^\pi = 2^+, 3^+$$

$$d_{5/2} d_{3/2} \quad J^\pi = 1^+, 2^+, 3^+, 4^+$$

$$d_{3/2} s_{1/2} \quad J^\pi = 1^+, 2^+$$

Setting up the Hamiltonian Matrix in each J^π subspace in the sd -shell

The Hamiltonian Matrix for $J^\pi = 0^+$

The allowed 0^+ states are $|(d_{5/2})^2\rangle_0$, $|(s_{1/2})^2\rangle_0$ and $|(d_{3/2})^2\rangle_0$, the matrix looks like:

	$ (d_{5/2})^2\rangle_0$	$ (s_{1/2})^2\rangle_0$	$ (d_{3/2})^2\rangle_0$
$\langle(d_{5/2})^2 $	$2\epsilon_{d5/2} + \langle d_{5/2}^2 V d_{5/2}^2 \rangle_0$	$\langle d_{5/2}^2 V s_{1/2}^2 \rangle_0$	$\langle d_{5/2}^2 V d_{3/2}^2 \rangle_0$
$\langle(s_{1/2})^2 $	$\langle s_{1/2}^2 V d_{5/2}^2 \rangle_0$	$2\epsilon_{s1/2} + \langle s_{1/2}^2 V s_{1/2}^2 \rangle_0$	$\langle s_{1/2}^2 V d_{3/2}^2 \rangle_0$
$\langle(d_{3/2})^2 $	$\langle d_{3/2}^2 V d_{5/2}^2 \rangle_0$	$\langle d_{3/2}^2 V s_{1/2}^2 \rangle_0$	$2\epsilon_{d3/2} + \langle d_{3/2}^2 V d_{3/2}^2 \rangle_0$

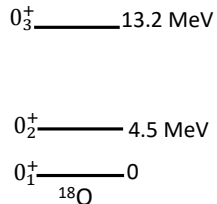
The off-diagonal delta terms drive the **configuration mixing**, shifting the lowest eigenvalue to form the ground state.

Practical shell-model for ^{18}O in sd -shell

Eigenvalues (g.s. and two excited 0^+ states):

Taking the TBMEs from USDB “Universal” SD
(Brown, Richter, PRC74, 034315 (2006)):

$$\begin{array}{ll} E(0_1^+) = -12.617 \text{ MeV} & E_{\text{ex}}(0_1^+) = 0 \text{ MeV} \\ E(0_2^+) = -8.111 \text{ MeV} & \Rightarrow E_{\text{ex}}(0_2^+) = 4.506 \text{ MeV} \\ E(0_3^+) = 0.600 \text{ MeV} & E_{\text{ex}}(0_3^+) = 13.217 \text{ MeV} \end{array}$$



Eigenstates:

	$ (\mathbf{d}_{5/2})^2\rangle_0$	$ (\mathbf{s}_{1/2})^2\rangle_0$	$ (\mathbf{d}_{3/2})^2\rangle_0$
0_1^+	79%	14%	7%
0_2^+	14%	85%	1%
0_3^+	1%	7%	92%

Setting up the Hamiltonian Matrix each J^π subspace in the sd -shell

The Hamiltonian Matrix $J^\pi = 1^+$

The allowed 1^+ states are $|d_{5/2}d_{3/2}\rangle_1$ and $|d_{3/2}s_{1/2}\rangle_1$, the matrix looks like:

	$ d_{5/2}d_{3/2}\rangle_1$	$ d_{3/2}s_{1/2}\rangle_1$
$\langle d_{5/2}d_{3/2} $	$\epsilon_{d5/2} + \epsilon_{d3/2} + \langle d_{5/2}d_{3/2} V d_{5/2}d_{3/2}\rangle_1$	$\langle d_{5/2}d_{3/2} V d_{3/2}s_{1/2}\rangle_1$
$\langle d_{3/2}s_{1/2} $	$\langle d_{3/2}s_{1/2} V d_{5/2}d_{3/2}\rangle_1$	$\epsilon_{d3/2} + \epsilon_{s1/2} + \langle d_{3/2}s_{1/2} V d_{3/2}s_{1/2}\rangle_1$

Energies of 1^+ states in ^{18}O

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} a_{1p} \\ a_{2p} \end{pmatrix} = E_p \begin{pmatrix} a_{1p} \\ a_{2p} \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$E_{1,2} = \frac{1}{2} \left\{ H_{11} + H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2} \right\}$$

$$|1_1^+\rangle = a_{11}|d_{5/2}d_{3/2}\rangle_1 + a_{21}|d_{3/2}s_{1/2}\rangle_1$$

$$|1_2^+\rangle = a_{12}|d_{5/2}d_{3/2}\rangle_1 + a_{22}|d_{3/2}s_{1/2}\rangle_1$$

$$\sum_k a_{kp}^2 = 1$$

Energies of 1^+ states in ^{18}O

Diagonal and non-diagonal TBMEs

$$\begin{aligned}
 H_{11} &= \varepsilon_{d5/2} + \varepsilon_{d3/2} + \underbrace{\langle d_{5/2}d_{3/2} | V | d_{5/2}d_{3/2} \rangle_1}_{1.030 \text{ MeV}} \\
 H_{12} &= \underbrace{\langle d_{5/2}d_{3/2} | V | s_{1/2}d_{3/2} \rangle_1}_{0.187 \text{ MeV}} \\
 H_{21} &= \underbrace{\langle d_{5/2}d_{3/2} | V | s_{1/2}d_{3/2} \rangle_1}_{0.187 \text{ MeV}} \\
 H_{22} &= \varepsilon_{s1/2} + \varepsilon_{d3/2} + \underbrace{\langle s_{1/2}d_{3/2} | V | s_{1/2}d_{3/2} \rangle_1}_{0.606 \text{ MeV}}
 \end{aligned}$$

Eigenvalues (g.s. and two excited 0^+ states):

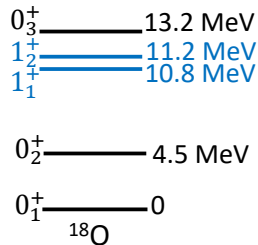
$$E(1_1^+) = -2.164 \text{ MeV}$$

$$E(1_2^+) = -1.725 \text{ MeV}$$

↓

$$E_{\text{ex}}(1_1^+) = 10.786 \text{ MeV}$$

$$E_{\text{ex}}(1_2^+) = 11.225 \text{ MeV}$$



Shell-model codes

M-scheme codes

- KShell (U. Tokyo, N. Shimizu et al) — MPI+OpenMP
Public version: <https://github.com/GaffaSnobb/kshell>
arXiv:1310.5431
- BIGSTICK (U. St Diego, C. Johnson et al) — MPI
Public version: <https://github.com/cwjdsu/BigstickPublick>
- ANTOINE (Strasbourg/Madrid, E. Caurier, **F. Nowacki** et al)
- NuShellX@MSU (Oxford/MSU, W. Rae, **B. A. Brown**)
- MSHELL (U. Tokyo, T. Mizusaki)
- OXBASH (Oxford/MSU)
- ...

Features:

- Matrix dimension: $\sim 10^{10}$ (for most performant)
- Lanczos diagonalization algorithm
- Calculation of the matrix elements on-the-fly

5. Matrix elements of transition operators

Calculation of observables

General scheme:

- Basis construction: $|\Phi_k\rangle$
- Expand the wave function: $|\Psi_p\rangle = \sum_k a_{kp} |\Phi_k\rangle$
- Solution of the Schrödinger equation by Hamiltonian matrix diagonalization: $\{H_{lk}\} \Rightarrow E_p, \{a_{kp}\}$
- Calculation of matrix elements of the operators

Transition probability:

$$T_{if} \propto |\langle \Psi_f | \hat{O} | \Psi_i \rangle|^2$$

Partial lifetime:

$$\tau_{if} = \frac{1}{T_{if}}$$

- Electromagnetic transitions
- Beta decay
- Nucleon transfer operators

Electromagnetic operators

Electric 2^λ -pole operators:

$$O(E\lambda) = \sum_{k=1}^A e(k) r^\lambda(k) Y_{\lambda\mu}(\hat{r}(k))$$

Magnetic 2^λ -pole operators:

$$O(M\lambda) = \sum_{k=1}^A \mu_N \left(g_s(k) \vec{s}(k) + \frac{2g_l(k)}{\lambda+1} \vec{l}(k) \right) \cdot \nabla(k) r_k^\lambda Y_{\lambda\mu}(\Omega_k)$$

$$\hat{O}(M1) = \sum_{k=1}^A \mu_N (g_s(k) \vec{s}(k) + g_l(k) \vec{l}(k))$$

$g_s = 5.586$, $g_l = 1$ for a proton

$g_s = -3.826$, $g_l = 0$ for a neutron

Reduced transition probabilities:

$$\begin{aligned} B(\lambda; J_i \rightarrow J_f) &= \frac{1}{2J_i + 1} \sum_{M_i, \mu, M_f} |\langle \alpha_f; J_f M_f | \hat{O}_{\lambda\mu} | \alpha_i; J_i M_i \rangle|^2 \\ &= \frac{1}{2J_i + 1} \sum_{M_i, \mu, M_f} \begin{pmatrix} J_f & \lambda & J_i \\ -M_f & \mu & M_i \end{pmatrix}^2 |\langle \alpha_f; J_f || \hat{O}_\lambda || \alpha_i; J_i \rangle|^2 \\ &= \frac{1}{2J_i + 1} |\langle \alpha_f; J_f || \hat{O}_\lambda || \alpha_i; J_i \rangle|^2. \end{aligned}$$

Selection rules:

- Triangle rule:

$$\Delta(J_i, J_f, \lambda)$$

- Parity conservation:

$$\begin{aligned} P_i P_f (-1)^\lambda &= 1 \quad (\text{electric}) \\ P_i P_f (-1)^{\lambda+1} &= 1 \quad (\text{magnetic}) \end{aligned}$$

Electric Quadrupole (E2) Operator

The E2 operator evaluates structural shapes and spatial deformations. The single-particle operator is:

$$\hat{O}(E2) = e^{\text{eff}} r^2 Y_{2\mu}(\hat{r})$$

Applying the Wigner-Eckart theorem, its reduced single-particle matrix element splits into a radial integral and a spherical harmonic angular matrix element:

$$\langle n_f l_f s_f j_f || \hat{O}(E2) || n_i l_i s_i j_i \rangle = e^{\text{eff}} \int_0^\infty R_{n_f l_f}(r) r^4 R_{n_i l_i}(r) dr \times \langle l_f s_f; j_f || Y_2 || l_i s_i; j_i \rangle$$

Using re-coupling rules to isolate the spin component, the angular part evaluates to:

$$\langle l_f s_f; j_f || Y_2 || l_i s_i; j_i \rangle = (-1)^{j_i+1/2} \frac{\hat{j}_i \hat{j}_f \hat{2}}{\sqrt{4\pi}} \begin{pmatrix} j_f & 2 & j_i \\ 1/2 & 0 & -1/2 \end{pmatrix} \times \text{Parity Rule}$$

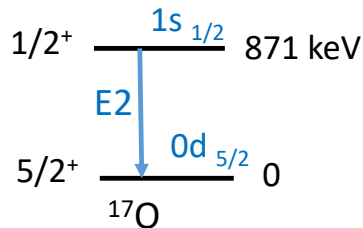
Selection Rules: $|j_f - j_i| \leq 2$, and $(l_f + l_i)$ must be **even** (no parity change).

Exercise 4

Electromagnetic transitions in ^{17}O

Calculate the $B(E2; 1/2_1^+ \rightarrow 5/2_{g.s.}^+)$ in ^{17}O .

Compare your result to experimental value $B_{\text{exp}}(E2) = 6.3 \text{ e}^2 \cdot \text{fm}^4$.



Electromagnetic transitions in ^{17}O

In our case, there is one valence particle, the initial state is $1s_{1/2}$, the final state is $0d_{5/2}$:

$$B(E2; 1/2^+ \rightarrow 5/2^+) = \frac{1}{2} |\langle 0d_{5/2} || O(E2) || 1s_{1/2} \rangle|^2 =$$
$$\frac{1}{2} e^2 \underbrace{\left(\int R_{0d_{5/2}}^*(r) r^2 R_{1s_{1/2}}(r) dr \right)^2}_{\langle r^2 \rangle^2} |\langle d_{5/2} || Y_2(\theta, \phi) || s_{1/2} \rangle|^2.$$

Evaluation of the radial part:

$$\int_0^\infty R_{1s_{1/2}}(r) r^2 R_{0d_{5/2}}(r) r^2 dr \approx -10 \text{ fm}^2.$$

for $\hbar\omega = 14 \text{ MeV}$ (near $A = 18$).

Solution to exercise 4

Electromagnetic transitions in ^{17}O (continued)

$$B(E2; 1/2^+ \rightarrow 5/2^+) = (e^{\text{eff}})^2 \frac{1}{4\pi} \langle r^2 \rangle^2 6 \times 5 \begin{pmatrix} 5/2 & 2 & 1/2 \\ 1/2 & 0 & -1/2 \end{pmatrix}^2 \approx 24 (e^{\text{eff}})^2 \cdot \text{fm}^4.$$

Experimental value $B_{\text{exp}}(E2; 1/2^+ \rightarrow 5/2^+) = 6.3 e^2 \cdot \text{fm}^4$.

This means that the neutron should have an effective charge: $e^{\text{eff}}(n) \approx 0.5 e$, because we work in a severely restricted model space (one valence neutron!).

“Standard” effective $E2$ and $M1$ operators

$$\begin{aligned} e^{\text{eff}}(p) &\approx 1.5 e & e^{\text{eff}}(n) &\approx 0.5 e \\ g_s^{\text{eff}}(p) &\approx 0.7 \underbrace{g_s(p)}_{5.586} & g_l^{\text{eff}}(p) &= g_l(p) = 1 \\ g_s^{\text{eff}}(n) &\approx 0.7 \underbrace{g_s(n)}_{-3.826} & g_l^{\text{eff}}(n) &= g_l(n) = 0 \end{aligned}$$

Two-nucleon system: ^{18}F

Magnetic moment expression:

$$\mu = \sqrt{\frac{4\pi}{3}} \langle J, M = J | O(M1) | J, M = J \rangle = \sqrt{\frac{4\pi}{3}} \begin{pmatrix} J & 1 & J \\ -J & 0 & J \end{pmatrix} \langle J || O(M1) || J \rangle$$

Let's calculate the magnetic moment of 5^+ state in ^{18}F : $|pd_{5/2}nd_{5/2}; 5^+, T = 0\rangle$

Decoupling of the matrix element:

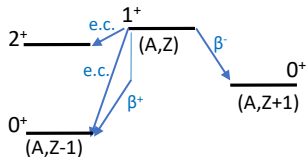
$$\langle pj_a nj_b; J || O_p(M1) + O_n(M1) || pj_a nj_b; J \rangle = (2J + 1) \times \\ (-1)^{j_a + j_b + J + 1} \left[\begin{Bmatrix} J & J & 1 \\ j_a & j_a & j_b \end{Bmatrix} \langle pj_a || O_p(M1) || pj_a \rangle + \begin{Bmatrix} J & J & 1 \\ j_b & j_b & j_a \end{Bmatrix} \langle nj_b || O_n(M1) || nj_b \rangle \right]$$

Result:

Theory: $\mu(5_1^+) = 2.96 \mu_N$

Experiment: $\mu(5_1^+) = +2.86(3) \mu_N$

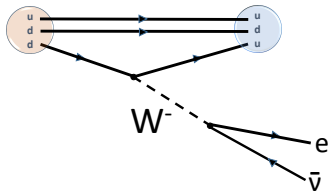
Nuclear beta decay



$${}_Z X_N \rightarrow {}_{Z+1} X_{N-1} + e^- + \bar{\nu}$$

$${}_Z X_N \rightarrow {}_{Z-1} X_{N+1} + e^+ + \nu$$

$${}_Z X_N + e^- \rightarrow {}_{Z-1} X_{N+1} + \nu$$



At low energies, the charge-changing semi-leptonic effective weak-interaction Lagrangian reads:

$$\mathcal{L}_\beta = -\frac{G_F}{\sqrt{2}} V_{ud} \bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_{\nu_e} \bar{\psi}_u \gamma^\mu (1 - \gamma_5) \psi_d + h.c.$$

Nuclear beta decay

Approximations

- Nucleons are non-relativistic point-like particles
- leptonic wave functions are constant in the nucleus interior (allowed transitions)

The decay rate:

$$T(\beta^\pm) = \frac{m^5 c^3}{2\pi^3 \hbar^7} f(Z, W_0) [G_V^2 B(F) + G_A^2 B(GT)]$$

The phase-space integral:

$$f(Z, W_0) = \int_1^{W_0} F(Z, W) S(Z, W) (W - W_0)^2 p W dW$$

Fermi operator

$$\hat{O}(F) = \sum_{k=1}^A \hat{t}_\pm(k)$$

Gamow-Teller operator:

$$\hat{O}(GT) = \sum_{k=1}^A \hat{\sigma}_m(k) \hat{t}_\pm(k)$$

ft -values

$$f(Z, W_0)t_{1/2}(\beta^\pm) = f(Z, W_0) \frac{\ln(2)}{T(\beta^\pm)} = \frac{K}{G_V^2 B(F) + G_A^2 B(GT)}$$

where $K/G_V^2 = 6163$ sec, $G_A/G_V \approx 1.26$.

Fermi operator

$$\hat{O}(F) = \sum_{k=1}^A \hat{t}_\pm(k)$$

Gamow-Teller operator:

$$\hat{O}(GT) = \sum_{k=1}^A \hat{\sigma}_m(k) \hat{t}_\pm(k)$$

Nuclear beta decay

Reduced Fermi transition probability:

$$\begin{aligned} B(F) &= \frac{1}{2J_i + 1} \sum_{M_i M_f} \langle J_f M_f T_f M_{Tf} | \sum_{k=1}^A \hat{t}_+(k) | J_i M_i T_i M_{Ti} \rangle^2 \\ &= [T_i(T_i + 1) - M_{Ti} M_{Tf}] \delta_{J_i J_f} \delta_{T_i T_f} \end{aligned}$$

Selection rules: $J_i = J_f$, $P_i = P_f$, $T_i = T_f$.

Reduced Gamow-Teller transition probability:

$$\begin{aligned} B(GT) &= \frac{1}{2J_i + 1} \sum_{M_i M_f m} \langle J_f M_f T_f M_{Tf} | \sum_{k=1}^A \hat{\sigma}_m(k) t_+(k) | J_i M_i T_i M_{Ti} \rangle^2 \\ &= \frac{1}{2(2J_i + 1)} \begin{pmatrix} T_f & 1 & T_i \\ -M_{Tf} & 1 & M_{Ti} \end{pmatrix}^2 \underbrace{\langle J_f T_f || \sum_{k=1}^A \hat{\sigma}(k) \hat{\tau}(k) || J_i T_i \rangle^2}_{\text{doubly reduced matrix element}} \end{aligned}$$

Selection rules: $J_f = J_i, J_i \pm 1$, $P_f = P_i$, $T_f = T_i, T_i \pm 1$ (no $J_i = 0 \rightarrow J_f = 0$ transition).

Beta decay of the neutron: $n \rightarrow p + e^- + \bar{\nu}_e$

Neutron: $l = 0, s = 1/2, j = 1/2, t = 1/2, m_t = 1/2$

Proton: $l = 0, s = 1/2, j = 1/2, t = 1/2, m_t = -1/2$

Fermi and Gamow-Teller reduced probabilities

$$B(F) = t(t+1) - m_{ti}m_{tf} = \frac{3}{4} + \frac{1}{4} = 1$$

$$B(GT) = \left(\frac{1}{2}\right)^2 \begin{pmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}^2 \langle \frac{1}{2} || \hat{\sigma} || \frac{1}{2} \rangle^2 \langle \frac{1}{2} || \hat{\tau} || \frac{1}{2} \rangle^2 = \frac{1}{4} \times \frac{1}{3} \times 6 \times 6 = 3$$

Beta for single-particle configurations

Initial state: $l_i, s_i = 1/2, j_i, t_i = 1/2, m_t = 1/2$

Final state : $l_f, s_f = 1/2, j_f, t_f = 1/2, m_t = -1/2$

Fermi and Gamow-Teller reduced probabilities

$$B(F) = t(t+1) - m_{ti}m_{tf} = \frac{3}{4} + \frac{1}{4} = 1$$

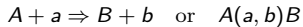
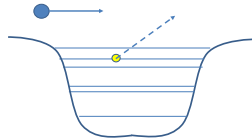
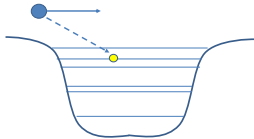
$$B(GT) = 6(2j_f + 1) \left\{ \begin{matrix} 1/2 & 1/2 & 1 \\ j_i & j_f & l_f \end{matrix} \right\} \delta_{l_i l_f}.$$

Where we have exploited:

$$\langle n_f l_f j_f, t = \frac{1}{2} || \hat{\sigma} \hat{\tau} || n_i l_i j_i, t = \frac{1}{2} \rangle = (-1)^{l_f + j_f + 3/2} 6 \sqrt{(2j_f + 1)(2j_i + 1)} \left\{ \begin{matrix} 1/2 & 1/2 & 1 \\ j_i & j_f & l_f \end{matrix} \right\} \delta_{l_i l_f}.$$

$$(\tau_{\pm} = \sqrt{2}t_{\pm}).$$

One-nucleon transfer reactions



Proton stripping	Proton pick-up	Neutron stripping	Neutron pick-up
(d,n)	(n,d)	(d,p)	(p,d)
(³ He,d)	(d, ³ He)	(t,d)	(d,t)
(α,t)	(t, α)	(α, ³ He)	(³ He,α)

Direct reactions (with minimum of rearrangement of the nucleons)

$$\sigma_I(\theta) \propto S \sigma_I^{\text{DWBA}}(\theta)$$

S is the spectroscopic factor

Spectroscopic factors for pick-up reactions (particle removal: $A \rightarrow A - 1$)

Reaction cross-section

$$\begin{aligned}\sigma_l^{p.u.} &\sim \frac{1}{2J_i + 1} \sum_{M_i M_f m} |\langle (A-1) \alpha_f J_f M_f | \tilde{a}_{jm} | A \alpha_i J_i M_i \rangle|^2 \\ &= \frac{1}{2J_i + 1} \sum_{M_i M_f m} \left(\begin{matrix} J_f & j & J_i \\ -M_f & m & M_i \end{matrix} \right)^2 |\langle (A-1) \alpha_f J_f || \tilde{a}_j || A \alpha_i J_i \rangle|^2 \\ &= \frac{1}{2J_i + 1} |\langle (A-1) \alpha_f J_f || \tilde{a}_j || A \alpha_i J_i \rangle|^2 \equiv S \\ \sigma_l^{p.u.} &= S \sigma_l^{s.p.}\end{aligned}$$

$S \equiv S(nlj)$ is the spectroscopic factor

Selection rules

- $|J_i - j| \leq J_f \leq J_i + j$, where $j = l \pm 1/2$
- $\Pi_i \Pi_f = (-1)^l$

Spectroscopic factors for stripping reactions (particle addition)

Reaction cross-section

$$\begin{aligned}\sigma_l^{str} &\sim \frac{1}{2J_i + 1} \sum_{M_i M_f m} |\langle (A+1) \alpha_f J_f M_f | a_{jm}^+ | A \alpha_i J_i M_i \rangle|^2 \\ &= \frac{1}{2J_i + 1} \left(\begin{matrix} J_f & j & J_i \\ -M_f & m & M_i \end{matrix} \right)^2 |\langle (A+1) \alpha_f J_f || a_j^+ || A \alpha_i J_i \rangle|^2 \\ &= \frac{1}{2J_i + 1} |\langle (A+1) \alpha_f J_f || \tilde{a}_j || A \alpha_i J_i \rangle|^2 \equiv \frac{2J_f + 1}{2J_i + 1} S \\ \sigma_l^{str} &= \frac{2J_f + 1}{2J_i + 1} S \sigma_l^{s.p.}\end{aligned}$$

$S \equiv S(nlj)$ is the spectroscopic factor

Selection rules

- $|J_i - j| \leq J_f \leq J_i + j$, where $j = l \pm 1/2$
- $\Pi_i \Pi_f = (-1)^l$

Spectroscopic factors

Spectroscopic factors are related to reduced overlap integrals:

$$S = \frac{|\langle (A-1) \alpha' J' || \tilde{a}_j || A \alpha J \rangle|^2}{2J+1} = \frac{|\langle A \alpha J || a_j^+ || (A-1) \alpha' J' \rangle|^2}{2J+1}$$

All n nucleons are on the same j shell

The n -nucleon wave functions can be expressed via c.f.p.'s:

$$|j^n; JM\rangle = \sum_{J'} \langle j^{n-1} [J'] j | j^n J \rangle |j^{n-1} [J'] j(n); JM\rangle$$

The overlap integral is given by c.f.p.:

$$S = n \langle j^{n-1} [J_i] j | j^n J_f \rangle^2$$

Example: Spectroscopic factors for $^{51}\text{V}(\text{d}, ^3\text{He})^{50}\text{Ti}$

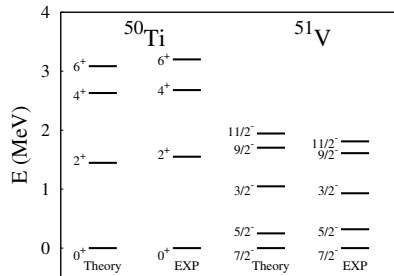
The spectra of the initial and final nuclei from FPD6 interaction (Richter et al, NPA523, 325 (1991)):

$^{50}_{22}\text{Ti}_{28}$: $pf_{7/2}^2$ configurations $\rightarrow J^\pi = 0^+, 2^+, 4^+, 6^+$

$$E(J) = 2\varepsilon_{f7/2} + \langle f_{7/2}^2; J | V | f_{7/2}^2; J \rangle$$

$^{51}_{23}\text{V}_{28}$: $pf_{7/2}^3$ configurations $\rightarrow$
 $J^\pi = 3/2^-, 5/2^-, 7/2^-, 9/2^-, 11/2^-, 15/2^-$

$$E(J) = 3\varepsilon_{f7/2} + 3 \sum_{J''} \langle j^2 [J''] j J | \{ j^3; J \}^2 \langle f_{7/2}^2; J'' | V | f_{7/2}^2; J'' \rangle$$



Spectroscopic factors for $^{51}\text{V}(\text{d}, ^3\text{He})^{50}\text{Ti}$

Model space: $f_{7/2}$ orbital

$$\sigma_I^{p.u.}(\theta) = S \sigma_I^{s.p.}(\theta)$$

$$S = 3 \langle f^2 [J''] f; 7/2 | \} f^3 7/2 \rangle^2$$

Spectroscopic factors for a transfer from $7/2_{gs}^-$ of ^{51}V to the states of ^{50}Ti in comparison with experiment at 52 MeV from Hinterberger et al, Z. Phys. 202, 236 (1968).

$S_{th}(0^+) = 3 \times \frac{1}{4} = 0.75$	$S_{exp}(0^+) = 0.73$
$S_{th}(2^+) = 3 \times \left(\frac{\sqrt{5}}{6}\right)^2 \approx 0.42$	$S_{exp}(2^+) = 0.39$
$S_{th}(4^+) = 3 \times \left(\frac{1}{2}\right)^2 = 0.75$	$S_{exp}(4^+) = 0.64$
$S_{th}(6^+) = 3 \times \left(\frac{\sqrt{13}}{6}\right)^2 \approx 1.08$	$S_{exp}(6^+) = 1.05$

Shell model theory

- P.J. Brussaard, P.W.M. Glaudemans, *Shell-Model Applications in Nuclear Spectroscopy*, (North-Holland, Amsterdam, 1977).
- K. Heyde, *The Nuclear Shell Model* (Springer-Verlag, Heidelberg, 2004)
- J. Suhonen, *From Nucleons to Nuclei* (Springer-Verlag, Heidelberg, 2007)
- I. Talmi, *Simple Models of Complex Nuclei, The Shell Model and the Interacting Boson Model*, (Harwood Academic Publishers, New York, 1993)
- E. Caurier et al, Rev. Mod. Phys. 77 (2005) 427.

No-Core Shell Model

B.R. Barrett, P. Navratil, J.P. Vary, Prog. Part. Nucl. Phys. 69 (2013) 235