# Hands-on Session: Nuclear Shell Model

## Shell-model problem for a two-valence nucleons system

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# Roadmap for our 75-Minute Session

**Objective:** Set-up and solve the shell-model problem for a two-nucleon system

1. The Angular Momentum Toolbox
2. Two-Particle Basis & The Pauli Principle
3. Delta interaction and single-shell spectrum
4. Setting the Hamiltonian matrix and diagonalization
5. Matrix Elements of Transition Operators

# 1. The Angular Momentum Toolbox

# 1. Coupling Rules & Clebsch-Gordan Coefficients

When coupling two angular momenta $\vec{j_1}$ and $\vec{j_2}$ to a total $\vec{J} = \vec{j_1} + \vec{j_2}$:

- **Triangular Condition:** $|j_1 - j_2| \leq J \leq j_1 + j_2$
- **Projection Conservation:** $M = m_1 + m_2$

The transformation from the uncoupled basis $|j_1 m_1 j_2 m_2\rangle$ to the coupled basis $|(j_1 j_2) JM\rangle$ is mediated by the Clebsch-Gordan (CG) coefficients:

$$|(j_1 j_2) JM\rangle = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | JM) |j_1 m_1\rangle |j_2 m_2\rangle$$

## Important Property: Symmetry under change of the coupling order

Exchanging the order of coupling introduces a phase shift:

$$|(j_2 j_1) JM\rangle = (-1)^{j_1 + j_2 - J} |(j_1 j_2) JM\rangle$$

# Wigner 3j Symbols

To handle multiple couplings elegantly, instead of Clebsch-Gordan coefficients we can use more symmetric objects: **Wigner 3j Symbol:**

$$(j_1 m_1 j_2 m_2 | JM) = (-1)^{j_1 - j_2 + M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}$$

where $m_1 + m_2 - M = 0$.

Highly symmetric under column permutations!

Attention: these are not matrices, but real numbers!

# Wigner 6j Symbols

How to couple three angular momenta $\vec{j}_1$, $\vec{j}_2$, and $\vec{j}_3$ to a total $\vec{J}$?

**Basis 1: Intermediate $J_{12}$**

First couple 1 and 2, then add 3:

$$\vec{j}_1 + \vec{j}_2 = \vec{\mathbf{J}}_{12} \quad \rightarrow \quad \vec{\mathbf{J}}_{12} + \vec{j}_3 = \vec{J}$$

**Basis 2: Intermediate $J_{23}$**

First couple 2 and 3, then add 1:

$$\vec{j}_2 + \vec{j}_3 = \vec{\mathbf{J}}_{23} \quad \rightarrow \quad \vec{j}_1 + \vec{\mathbf{J}}_{23} = \vec{J}$$

Re-coupling three angular momenta, e.g., changing from $|(j_1 j_2) J_{12}\, j_3; JM\rangle$ to $|j_1\, (j_2 j_3) J_{23}; JM\rangle$ is mediated by the 6j symbol:

$$|j_1\, (j_2 j_3) J_{23}; JM\rangle = \sum_{\mathbf{J_{12}}} \kappa \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} |(j_1 j_2) J_{12}\, j_3; JM\rangle$$

where $\kappa = (-1)^{j_1 + j_2 + j_3 + J} \sqrt{(2J_{12}+1)(2J_{23}+1)}$.

# Wigner 9j Symbols

How to couple four angular momenta, for example, $\vec{l}_1$, $\vec{s}_1$, $\vec{l}_2$, $\vec{s}_2$ to a total $\vec{J}$?

Used for re-coupling four angular momenta, vital for changing frameworks from $LS$-coupling to $jj$-coupling in the two-nucleon wave function:

$$|(l_1 s_1) j_1, (l_2 s_2) j_2; JM\rangle = \sum_{L,S} \kappa \begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix} |(l_1 l_2) L, (s_1 s_2) S; JM\rangle$$

where for nucleons $s_1 = s_2 = 1/2$ and therefore $S = 0, 1$, and

$\kappa = \sqrt{(2j_1 + 1)(2j_2 + 1)(2L + 1)(2S + 1)}$

# Spherical Tensors

A spherical tensor $T_{\lambda\mu}$ of rank $\lambda$ is a set of $(2\lambda+1)$ operators, $\mu = -\lambda, -\lambda+1, \ldots, \lambda-1, \lambda$ which obey the following commutation relations with the angular momentum operators $J_\pm = J_x \pm iJ_y$ and $J_0$ :

$$[J_\pm, T_{\lambda\mu}] = \sqrt{\lambda(\lambda+1) - \mu(\mu\pm1)}\, T_{\lambda,\mu\pm1}$$
$$[J_0, T_{\lambda\mu}] = \mu T_{\lambda\mu}$$

These commutation relations can be compared with the action of the angular momentum operator on the states of the angular momentum $J = \lambda$ :

$$J_\pm|\lambda\mu\rangle = \sqrt{\lambda(\lambda+1) - \mu(\mu\pm1)}|\lambda, \mu\pm1\rangle$$
$$J_0|\lambda\mu\rangle = \mu|\lambda\mu\rangle$$

# Examples of Spherical Tensors

| Rank | Parity-even | Parity-odd |
|---|---|---|
| $\lambda = 0$ | Scalar | Pseudoscalar $(\vec{p} \cdot \vec{L})$ |
| $\lambda = 1$ | Polar Vector $(\vec{r})$ | Axial Vector $(\vec{L})$ |
| | $r_{\pm 1} = \mp \frac{1}{\sqrt{2}}(x \pm iy)$ | $L_{\pm}, L_0$ |
| | $r_0 = z$ | |
| $\lambda \in N$ | $Y_{\lambda\mu}(\hat{r})$ | of parity $(-1)^{\lambda}$ |
| | $\lambda$ even | $\lambda$ odd |

# Irreducible tensorial operators and the Wigner-Eckart Theorem

The absolute cornerstone for calculating observables. It separates the **geometry** (magnetic projections) from the **physics** (nuclear structure dynamics). For a spherical tensor operator $T_{\lambda\mu}$ of rank $\lambda$:

$$\langle j_f m_f | T_{\lambda\mu} | j_i m_i \rangle = (-1)^{j_f - m_f} \begin{pmatrix} j_f & \lambda & j_i \\ -m_f & \mu & m_i \end{pmatrix} \langle j_f || T_\lambda || j_i \rangle$$

## What is the importance for physics?

- The matrix element is **zero** unless $|j_f - j_i| \leq \lambda \leq j_f + j_i$ and $m_f = m_i + \mu$.
- To know the transition rate between states, you only need to compute the **Reduced Matrix Element** $\langle j_f || T_\lambda || j_i \rangle$ **once**, regardless of the magnetic sublevels $m$.

# Examples: reduced Matrix Element of the $\vec{J}$ operator and of the spherical harmonic

Let's calculate the matrix elements of $\hat{J}_0$ operator and of the spherical harmonic $Y_{\lambda\mu}$ and then apply the Wigner-Eckart theorem. We can identify the following reduced matrix elements.

## Reduced matrix element of the angular momentum operator

The reduced matrix element of $\vec{J}$ within the states of the same $j$ ($j_f = j_i = j$) evaluates to a simple geometric factor:

$$\langle j||\vec{J}||j\rangle = \sqrt{j(j+1)(2j+1)}$$

Examples: $\langle 1/2||\vec{\sigma}||1/2\rangle = 2\langle 1/2||\vec{s}||1/2\rangle = \sqrt{6}$

## Reduced matrix element of the spherical harmonics

For states without spin consideration, the spatial integration over three spherical harmonics yields:

$$\langle l_f||Y_\lambda||l_i\rangle = (-1)^{l_f}\sqrt{\frac{(2l_f+1)(2\lambda+1)(2l_i+1)}{4\pi}}\begin{pmatrix} l_f & \lambda & l_i \\ 0 & 0 & 0 \end{pmatrix}$$

# 2. BASIS CONSTRUCTION and THE PAULI PRINCIPLE

# Two-particle wave function: $M$-scheme basis

Convention for single-particle wave functions: $\langle \vec{r}, \vec{\sigma} | nlsjm \rangle \equiv \langle \vec{r}, \vec{\sigma} | jm \rangle = \phi_{nlsjm}(\vec{r}) = R_{nl}(r) \left[ Y_l \otimes \chi_s \right]^{(j)}_m$.
For a harmonic oscillator radial wave function:

$$R_{nl}(r) = \mathcal{N}_{nl}\, r^l \, \exp\left(-\frac{r^2}{2b^2}\right) L_n^{l+1/2}\left(\frac{r^2}{b^2}\right) ,$$

$b = \sqrt{\frac{\hbar}{\mu\omega}}$ is the harmonic oscillator length parameter, $L_n^{l+1/2}\left(\frac{r^2}{b^2}\right)$ are generalized Laguerre polynomials, the normalization factor $\mathcal{N}_{nl}$ is from $\int_0^\infty R_{nl}^2(r) r^2 dr = 1$.

## Slater determinants

$$\Phi_{\alpha_1,\alpha_2}(1,2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_{\alpha_1}(\vec{r}_1) & \phi_{\alpha_1}(\vec{r}_2) \\ \phi_{\alpha_2}(\vec{r}_1) & \phi_{\alpha_2}(\vec{r}_2) \end{vmatrix}$$

where $\alpha_i = (n_i, l_i, s_i, j_i, m_i)$ is a set of sperical quantum numbers ($s_i = 1/2$) and
Our notation of a quantum state: $|\alpha_1, \alpha_2\rangle$

# Constructing the Two-Body Basis: Identical Nucleons

Let's build a coupled state $|(j_a j_b)JM\rangle$ for two identical nucleons (e.g., two neutrons) in the model space comprised by two orbitals $j_a$ and $j_b$.

To respect the **Pauli Exclusion Principle**, the two-body state must be completely antisymmetric under particle exchange ($1 \leftrightarrow 2$):

$$|j_a j_b; JM\rangle_{\text{AS}} = \mathcal{N}\Big(|j_a(1)j_b(2); JM\rangle - |j_a(2)j_b(1); JM\rangle\Big)$$

Using the CG exchange phase property from Part 1, the second term expands to:

$$|j_a(2)j_b(1); JM\rangle = (-1)^{j_a+j_b-J}|j_b(1)j_a(2); JM\rangle$$

Normalization factor for $j_a \neq j_b$: $\mathcal{N} = 1/\sqrt{2}$

# Constructing the Two-Body Basis: Identical Nucleons

What happens if two identical nucleons (e.g., two neutrons) in the **same orbital** $j_a = j_b = j$.

The two-body state, antisymmetric under particle exchange ($1 \leftrightarrow 2$):

$$|jj; JM\rangle_{\text{AS}} = \mathcal{N}\Big(|j(1)j(2); JM\rangle - |j(2)j(1); JM\rangle\Big)$$

Using the Clebsch-Gordan exchange phase property from Part 1, the second term expands to:

$$|j(2)j(1); JM\rangle = (-1)^{j+j-J}|j(1)j(2); JM\rangle = (-1)^{2j-J}|j(1)j(2); JM\rangle$$

**Phase simplification:**

Since $j$ is a half-integer for nucleons (e.g., $5/2, 7/2$), $2j$ is always an **odd integer**. What happens to the state $|jj; JM\rangle_{\text{AS}}$ if $J$ is chosen to be an odd integer?

# The Even-$J$ Constraint

Substituting the phase into the antisymmetric expression:

$$|jj; JM\rangle_{\mathrm{AS}} = \mathcal{N}\left(1 + (-1)^J\right)|j(1)j(2); JM\rangle$$

We conclude that:

- If $J$ is **even**: $(-1)^{\mathrm{even}} = +1. \implies (1+1) = 2 \neq 0 \quad \rightarrow$ **State is allowed!**, $\mathcal{N} = 1/2$.
- If $J$ is **odd**: $(-1)^{\mathrm{odd}} = -1. \implies (1-1) = 0 \quad \rightarrow$ **State vanishes!**

## Conclusion

For a configuration of two identical nucleons in the same orbit $j^2$, the Pauli principle restricts the allowed total angular momentum states strictly to **even values**:

$$J = 0, 2, 4, \ldots, (2j-1)$$

# Including Isospin

If the two nucleons are not necessarily identical (each can be either neutron or proton), but occupy the same orbital $j$:

$$|jj; JM, T\rangle_{\text{AS}} = \mathcal{N}\left(1 - (-1)^{J+T}\right)|j(1)j(2); JM, T\rangle$$

where $T$ can take values 0 or 1.

- If $(J+T)$ is **odd**: $\implies (1-(-1)^{\text{odd}}) = 2 \neq 0 \quad \rightarrow$ **State is allowed!**
- If $(J+T)$ is **even**: $\implies (1-(-1)^{\text{even}}) = 0 \quad \rightarrow$ **State vanishes!**

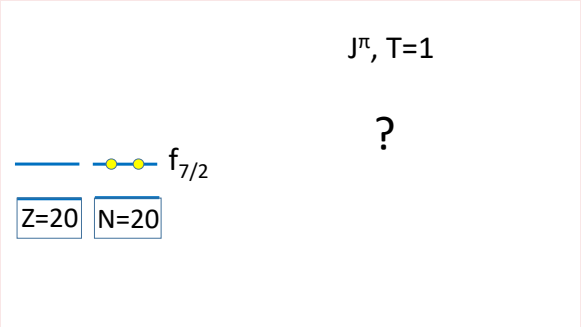
## Conclusion

For a configuration of a proton and a neutron the same orbit $j^2$, the Pauli principle restricts $(J+T)$ strictly to **odd values**:

$$T = 1 : J = 0, 2, 4, \ldots, 2j-1$$
$$T = 0 : J = 1, 3, 5, \ldots, 2j$$

# Exercise 1: Basis construction

## 2 neutrons in $0f_{7/2}$ orbital

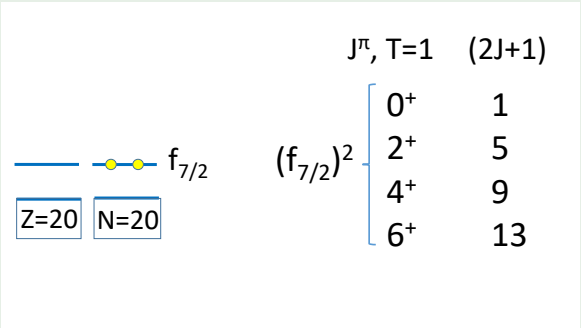



- Write down the basis in $J$-coupled form.
- Write down the basis in $M$-scheme (number the Slater determinants).
- How many different $J$-states exist in this model space?

# Solution to exercise 1

## 2 neutrons in $0f_{7/2}$ orbital

- Basis in $J$-coupled scheme:




$$|(0f_{7/2})^2;\ J,\ T=1\rangle \quad \Rightarrow \quad J^\pi = 0^+, 2^+, 4^+, 6^+$$

For each $J$, there are $(2J+1)$ substates with $M = -J, \ldots, J$,

that is we have $1 + 5 + 9 + 13 = 28$ states with different $|JM\rangle$.

# Solution to exercise 1 (continued)

## Basis construction

- Basis in $M$-scheme (antisymmetrized and normalized products of s.p. states - Slater determinants): $|m_a m_b; M\rangle$, where $m_a, m_b = -7/2, -5/2, \ldots, 5/2, 7/2$, $M = m_a + m_b$. Attention: $J$ is not a good quantum number!

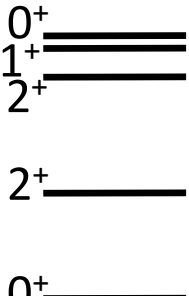
$$
\begin{array}{llll}
|\tfrac{7}{2},\ \tfrac{5}{2};\ 6\rangle & & & \\
|\tfrac{7}{2},\ \tfrac{3}{2};\ 5\rangle & & & \\
|\tfrac{7}{2},\ \tfrac{1}{2};\ 4\rangle & |\tfrac{5}{2},\ \tfrac{3}{2};\ 4\rangle & & \\
|\tfrac{7}{2},-\tfrac{1}{2};\ 3\rangle & |\tfrac{5}{2},\ \tfrac{1}{2};\ 3\rangle & & \\
|\tfrac{7}{2},-\tfrac{3}{2};\ 2\rangle & |\tfrac{5}{2},-\tfrac{1}{2};\ 2\rangle & |\tfrac{3}{2},\ \tfrac{1}{2};\ 2\rangle & \\
|\tfrac{7}{2},-\tfrac{5}{2};\ 1\rangle & |\tfrac{5}{2},-\tfrac{3}{2};\ 1\rangle & |\tfrac{3}{2},-\tfrac{1}{2};\ 1\rangle & \\
|\tfrac{7}{2},-\tfrac{7}{2};\ 0\rangle & |\tfrac{5}{2},-\tfrac{5}{2};\ 0\rangle & |\tfrac{3}{2},-\tfrac{3}{2};0\rangle & |\tfrac{1}{2},-\tfrac{1}{2};0\rangle \\
|-\tfrac{7}{2},\tfrac{5}{2};-1\rangle & |-\tfrac{5}{2},\ \tfrac{3}{2};-1\rangle & |-\tfrac{3}{2},\tfrac{1}{2};-1\rangle & \\
|-\tfrac{7}{2},\tfrac{3}{2};-2\rangle & |-\tfrac{5}{2},\ \tfrac{1}{2};-2\rangle & |-\tfrac{3}{2},-\tfrac{1}{2};-2\rangle & \\
|-\tfrac{7}{2},\tfrac{1}{2};-3\rangle & |-\tfrac{5}{2},\ \tfrac{1}{2};-3\rangle & & \\
|-\tfrac{7}{2},\tfrac{1}{2};-4\rangle & |-\tfrac{5}{2},\ \tfrac{3}{2};-4\rangle & & \\
|-\tfrac{7}{2},\tfrac{3}{2};-5\rangle & & & \\
|-\tfrac{7}{2},\tfrac{5}{2};-6\rangle & & &
\end{array}
$$

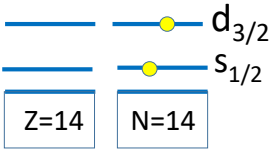
- There are 4 different $J$-states: $J^\pi = 0^+, 2^+, 4^+, 6^+$ (for each $J$, $M = -J, \ldots, J$).

What type of the basis ($J$-coupled or $M$-scheme) do you prefer?

# Example: $^{30}_{14}Si_{16}$ as $^{28}_{14}Si_{14}$ plus 2 neutrons

| | $J^\pi$,T=1 |
|---|---|
| $(d_{3/2})^2$ | $0^+,2^+$ |
| $s_{1/2}d_{3/2}$ | $1^+,2^+$ |
| $(s_{1/2})^2$ | $0^+$ |




$^{30}Si$

# Three-particle wave function: $M$-scheme basis

## Slater determinants

$$\Phi_\alpha(1,2,3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \phi_{\alpha_1}(\vec{r}_1) & \phi_{\alpha_1}(\vec{r}_2) & \phi_{\alpha_1}(\vec{r}_3) \\ \phi_{\alpha_2}(\vec{r}_1) & \phi_{\alpha_2}(\vec{r}_2) & \phi_{\alpha_2}(\vec{r}_3) \\ \phi_{\alpha_A}(\vec{r}_1) & \phi_{\alpha_A}(\vec{r}_2) & \phi_{\alpha_3}(\vec{r}_3) \end{vmatrix}$$

where $\alpha_i = (n_i, l_i, j_i, m_i)$ and $\alpha$ stores a set of single-particle configurations $\{\alpha_1, \alpha_2, \alpha_3\}$

$$M = \sum_{i=1}^{3} m_i$$

Our notation of a quantum state: $|\alpha_1, \alpha_2, \alpha_3\rangle$

# Homework: Three identical nucleons in $d_{5/2}$ orbital

## Basis construction

Basis in $M$-scheme: $|m_a m_b m_c; M\rangle$,

where $m_a, m_b, m_c = -5/2, -3/2, -1/2, 1/2, 3/2, 5/2$:

$$|\ \tfrac{5}{2},\ \tfrac{3}{2},\ \tfrac{1}{2};\ 9/2\rangle$$
$$\dots$$
$$\dots$$
$$\dots$$
$$\dots$$
$$\dots$$

What $J^\pi$, $T$ values are possible ?

# Three-particle wave function: antisymmetrized $J$-coupled states.

Let's construct a $J$-coupled state of three nucleons in three orbitals $j_a, j_b, j_c$ antisymmetric under any two-nucleon exchange ($i \leftrightarrow k$) and where two of the nucleons are coupled to the intermediate angular momentum $J'$:

$$
\begin{aligned}
|j_a j_b j_c(J'); JM\rangle &= \frac{1}{3!} \sum_{P(permutations)} (-1)^P \hat{P} |(j_a(1) j_b(2)) J' j_c(3); JM\rangle \\
&= \frac{1}{6} \left( |(j_a(1) j_b(2)) J' j_c(3); JM\rangle - |(j_a(2) j_b(1)) J' j_c(3); JM\rangle \right. \\
&- |(j_a(1) j_b(3)) J' j_c(2); JM\rangle + |(j_a(2) j_b(3)) J' j_c(1); JM\rangle \\
&+ \left. |(j_a(3) j_b(1)) J' j_c(2); JM\rangle - (j_a(3) j_b(2)) J' j_c(1); JM\rangle \right)
\end{aligned}
$$

# Three-particles in the same $j$-state.

Let's construct a $J$-coupled state of three identical nucleons in a single-$j$ orbital ($j_a = j_b = j_c = j$)

$$|j^3\,(J');\,JM\rangle = \frac{1}{\mathcal{N}}\left(|j^2(1,2)[J']j(3);\,JM\rangle - |j^2(1,3)[J']\,j(2);\,JM\rangle + |j^2(2,3)[J']\,j(1);\,JM\rangle\right)$$

## Using $6j$-coefficients

$$|j^2(2,3)\,[J']\,j(1);\,JM\rangle = \sum_{J''=even}\sqrt{(2J'+1)(2J''+1)}\left\{\begin{array}{ccc} j & j & J'' \\ J & j & J' \end{array}\right\}|j^2(1,2)\,[J'']\,j(3);\,JM\rangle$$

$$|j^2(1,3)\,[J']\,j(2);\,JM\rangle = -\sum_{J''=even}\sqrt{(2J'+1)(2J''+1)}\left\{\begin{array}{ccc} j & j & J'' \\ J & j & J' \end{array}\right\}|j^2(1,2)\,[J'']\,j(3);\,JM\rangle$$

# Three-particle wave function: $J$-coupled states.

## $J$-coupled state $j_a = j_b = j_c = j$

$$|j^3\,(J');\,JM\rangle = \frac{1}{\mathcal{N}}\sum_{J''=even}\left[\delta_{J'J''} + 2\sqrt{(2J'+1)(2J''+1)}\left\{\begin{array}{ccc} j & j & J'' \\ J & j & J' \end{array}\right\}\right] |j^2(1,2)\,[J'']\,j(3);\,JM\rangle$$

with the normalization constant from

$$\mathcal{N}^2 = 3 + 6(2J'+1)\left\{\begin{array}{ccc} j & j & J' \\ J & j & J' \end{array}\right\}.$$

## One-nucleon coefficients of fractional parentage (c.f.p.'s):

$$|j^3\,(J');JM\rangle = \sum_{J''=even} \underbrace{\langle j^2\,[J'']\,j\,|\}\,j^3\,J'\,J\rangle}_{\text{c.f.p.}} |j^2(1,2)\,[J'']\,j(3);\,JM\rangle$$

# Solution of exercise 2: Three identical nucleons in $d_{5/2}$ orbital

## Basis in $J$-coupled scheme: $|(d^3_{5/2}; J, T = 3/2\rangle$

To get antisymmetrized states, we use the formula for c.f.p.'s from the previous slide:

$$
\begin{aligned}
|d^3_{5/2}; J^\pi = 3/2^+\rangle &= -\frac{\sqrt{5}}{\sqrt{7}}|d^2_{5/2}\,[J''{=}2]\; d_{5/2}; J^\pi = 3/2^+\rangle \\
&+ \frac{\sqrt{2}}{\sqrt{7}}|d^2_{5/2}\,[J''{=}4]\; d_{5/2}; J^\pi = 3/2^+\rangle \\
|d^3_{5/2}; J^\pi = 5/2^+\rangle &= -\frac{\sqrt{2}}{3}|d^2_{5/2}\,[J''{=}0]\; d_{5/2}; J^\pi = 5/2^+\rangle \\
&+ \frac{\sqrt{5}}{3\sqrt{2}}|d^2_{5/2}\,[J''{=}2]\; d_{5/2}; J^\pi = 5/2^+\rangle \\
&+ \frac{1}{\sqrt{2}}|d^2_{5/2}\,[J''{=}4]\; d_{5/2}; J^\pi = 5/2^+\rangle \\
|d^3_{5/2}; J^\pi = 9/2^+\rangle &= \frac{\sqrt{3}}{\sqrt{14}}|d^2_{5/2}\,[J''{=}2]\; d_{5/2}; J^\pi = 9/2^+\rangle \\
&- \frac{\sqrt{11}}{\sqrt{14}}|d^2_{5/2}\,[J''{=}4]\; d_{5/2}; J^\pi = 9/2^+\rangle
\end{aligned}
$$

Remark: For $j > 7/2$, we need $J'$ to distinguish different 3-nucleon states coupled to the same $J$.

# 3. DELTA INTERACTION AND SINGLE-SHELL SPECTRUM

# Setting up the Hamiltonian for a 2-Nucleon System

Let's construct in the $J$-coupled basis the Hamiltonian matrix $H = H_0 + V$ for two nucleons. The total energy is the sum of Single-Particle Energies (SPE) and the Two-Body Matrix Elements (TBME).

## General Matrix Element Expression

In a given $J^\pi$ subspace, the matrix elements connecting an initial two-body configuration $|j_a j_b\rangle_J$ to a final configuration $|j_c j_d\rangle_J$ are given by:

$$\langle j_c j_d | H | j_a j_b \rangle_J = \underbrace{(\epsilon_a + \epsilon_b)\,\delta_{ac}\delta_{bd}}_{\text{Eigenstates of } H_0 \text{ (SPEs)}} + \underbrace{\langle j_c j_d | V | j_a j_b \rangle_{J,\,\text{Antisymmetrized}}}_{\text{Residual Interaction (TBME)}}$$

## Single-shell problem: $j_a = j_b = j_c = j_d = j$

$$E_J = \langle j\,j\,|H|j\,j\,\rangle_J = 2\epsilon_j + \langle j\,j\,|V|j\,j\,\rangle_{J,\,\text{AS}}$$

# Residual (Effective) Interaction

## Practical approaches to effective interaction

- **Schematic** interaction (parametrized interaction between two nucleons in a nuclear medium)
- **Empirical** interaction (Fit of TBME's to energy levels of nuclei to be described within the chosen model space)
- **Microscopic** interaction (derived from a bare NN-force) $\Rightarrow$ *ab-initio* approach

# Technique of transformation to relative and center-of-mass coordinates (h.o. basis)

## I. Two-particle harmonic-oscillator Hamiltonian

$$\begin{aligned} H &= \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 r_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 r_1^2 \\ &= \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 r^2 + \frac{P^2}{2M} + \frac{1}{2}M\omega^2 R^2\,, \end{aligned}$$

where

$$\vec{r} = \vec{r_1} - \vec{r_2},\ \vec{R} = \frac{1}{2}\left(\vec{r_1} + \vec{r_2}\right),\ \vec{p} = \frac{1}{2}\left(\vec{p_1} - \vec{p_2}\right),\ \vec{P} = \vec{p_1} + \vec{p_2},\ \mu = \frac{m}{2},\ M = 2m$$

## Talmi-Moshinsky transformation from particle-particle to relative and center-of-mass basis:

$$|n_1 l_1(\vec{r_1}), n_2 l_2(\vec{r_2});\ LM_L\rangle = \sum_{nl,N\Lambda} \langle\, nl,\ N\Lambda;\ L|\, n_1 l_1, n_2 l_2;\ L\rangle\, |nl(\vec{r}),\ N\Lambda(\vec{R});\ LM_L\rangle$$

with $2n_1 + l_1 + 2n_2 + l_2 = 2n + l + 2N + \Lambda$

## II. Multipole decomposition

$$V(|\vec{r_1} - \vec{r_2}|) = \sum_{k=0}^{\infty} v_k(r_1, r_2) P_k(\cos\theta_{12}),$$

where

$$P_k(\cos\theta_{12}) = \sum_{q=-k}^{k} \frac{4\pi}{2k+1} Y^*_{kq}(\Omega_1) Y_{kq}(\Omega_2)$$

## Algorithme:

To compute the Two-Body Matrix Element (TBME) $\langle j_a j_b | V | j_c j_d \rangle_J$ (direct and exchange terms), we must decouple the spatial and angular-momentum parts:

1. Separate the single-particle wavefunctions into radial and angular parts: $\psi_{nljm}(\vec{r}) = R_{nl}(r)[Y_l \otimes \chi_{1/2}]^j_m$;
2. Compute two-dimensional integrals of $v_k(r_1, r_2)$;
3. Use angular-momentum algebra to compute the matrix elements of $Y^*_{kq}(\Omega_1) Y_{kq}(\Omega_2)$ within $J$ coupled states.

# The 3D Delta Function Interaction

To simulate the extremely short range of the residual strong force between valence nucleons, we use a **3D Delta function interaction**:

$$V(\vec{r}_1 - \vec{r}_2) = -V_0\,\delta(\vec{r}_1 - \vec{r}_2) = -V_0\,\frac{\delta(r_1 - r_2)}{r_1 r_2}\sum_{q=-k}^{k} Y^*_{kq}(\Omega_1)\,Y_{kq}(\Omega_2)$$

Where $V_0 > 0$ represents the attractive strength.

# Analytical Result of the 3D Delta Interaction

For diagonal matrix elements ($j_a \neq j_b$) between antisymmetrized $J$-coupled states, we get:

$$\langle j_a j_b | V_\delta | j_a j_b \rangle_J = -V_0 \, I_R \, \frac{(2j_a+1)(2j_b+1)}{4\pi} \begin{pmatrix} j_a & j_b & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2$$

Where:

- $I_R$ is the radial overlap integral: $I_R = \int_0^\infty R^2_{n_a l_a}(r) R^2_{n_b l_b}(r) r^2 dr$
- Parity selection rule: $l_a + l_b + J$ must be **even**.

# Example: Two-Neutron Spectrum in $^{18}$O

Let's calculate the explicit energy levels of the two valence neutrons in $^{18}$O outside the doubly-magic $^{16}$O core, restricted to the single orbit $(0d_{5/2})^2$.

- **Residual Interaction:** Pure 3D Delta force with $V_0 = 400$ MeV $\cdot$ fm$^3$.
- **Core Geometry:** For $^{18}$O, the radial overlap integral for the $0d$ wavefunctions yields approximately $I_R = \int_0^\infty R_{0d}^4(r) r^2 dr \approx 0.038$ fm$^{-3}$.
- **Single-Particle Energy:** $\epsilon_{d5/2} = -4.14$ MeV (from the $^{17}$O and $^{16}$O binding energies).

## Energies of Two Identical Nucleon states

In case of $j_a = j_b = j$ for even $J$:

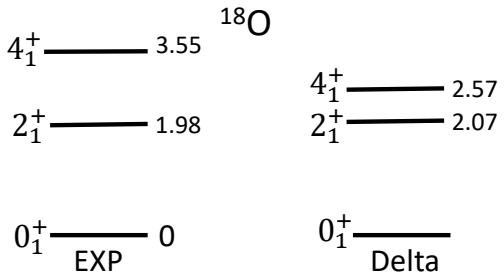
$$E(J) = 2\epsilon_{d5/2} - V_0 I_R \frac{(2j+1)^2}{8\pi} \begin{pmatrix} 5/2 & 5/2 & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2$$

# Energy spectrum

## Numerical Results and Energy Levels

Plugging in $V_0 \cdot I_R \approx 15.2$ MeV and the analytic values of the 3j symbols:

- $E(0^+) = -8.28 - 2.90$ MeV $= \mathbf{-11.18}$ MeV *(Ground State)*
- $E(2^+) = -8.28 - 0.83$ MeV $= \mathbf{-9.11}$ MeV *(2.07 MeV excitation energy)*
- $E(4^+) = -8.28 - 0.33$ MeV $= \mathbf{-8.61}$ MeV *(2.57 MeV excitation energy)*



# Heavy Nuclei Comparison: $^{210}$Pb and $^{210}$Po

Let's move from light systems to heavy systems near the doubly-magic $^{208}$Pb core ($A = 210$). We apply the same 3D Delta force ($V_0 = 400$ MeV $\cdot$ fm$^3$) to observe the impact of a larger nuclear volume.

## $^{210}$Pb: 2 valence neutrons

- **Orbit:** Pure $(0h_{9/2})^2$ configuration.
- **Radial Integral:** $I_R \approx 0.007$ fm$^{-3}$.
- **Analytical $J = 0$ Shift:**

$$\begin{aligned} \Delta E(0^+) &= -V_0 I_R \frac{(2j+1)}{8\pi} \\ &\approx -1.1 \text{ MeV} \end{aligned}$$

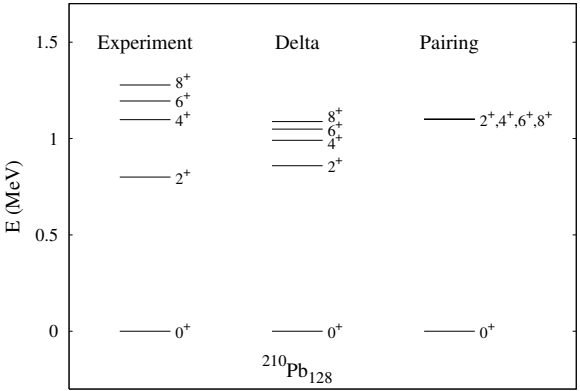
## $^{210}$Po: 2 valence protons

- **Orbit:** Pure $(1g_{9/2})^2$ configuration.
- **Radial Integral:** $I_R \approx 0.0075$ fm$^{-3}$.
- **Analytical $J = 0$ Shift:**

$$\begin{aligned} \Delta E(0^+) &= -V_0 I_R \frac{(2j+1)}{8\pi} \\ &\approx -1.2 \text{ MeV} \end{aligned}$$

# Example: $^{210}$Pb in $(\nu 0h_{9/2})^2$

$$\langle j^2|V_{pairing}|j^2\rangle_J = -\frac{1}{2}G(2j+1)\delta_{J0}$$



# Setting the Hamiltonian matrix and diagonalization

# Solution of the eigenvalue problem $H = H_0 + V$

- Construct a basis in the valence space: $|\Phi_k\rangle$
- Expand unknown eigenstate in terms of basis states

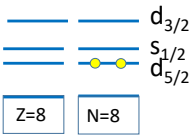
$$|\Psi_p\rangle = \sum_{k=1}^{n} a_{kp}|\Phi_k\rangle$$

- Multiplying $H|\Psi_p\rangle = E_p|\Psi_p\rangle$ by $\langle\Phi_l|$, we get the matrix equation to obtain $E_p, \{a_{kp}\}$:

$$\sum_{k=1}^{n} H_{lk}a_{kp} = E_p a_{lp}, \qquad \text{with} \quad H_{lk} = \langle\Phi_l|\hat{H}|\Phi_k\rangle = E_k^{(0)}\delta_{lk} + V_{lk}$$

$$\begin{pmatrix} H_{11} & H_{12} & \dots & \dots & H_{nn} \\ H_{21} & H_{22} & \dots & \dots & H_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ H_{n1} & H_{n2} & \dots & \dots & H_{nn} \end{pmatrix} \Rightarrow$$

# Exercice 3: The spectrum of $^{18}$O in *sd*-shell




$J^\pi$, T = ?

E,Ψ = ?

$$\langle j_c j_d | \hat{H} | j_a j_b \rangle_J = (\epsilon_a + \epsilon_b)\, \delta_{ac}\delta_{bd} + \langle j_c j_d | V | j_a j_b \rangle_{J,\,\mathrm{AS}}$$

## Single-particle energies:

$$\varepsilon(d_{5/2}) = E_B(^{17}_{8}\mathrm{O}_9) - E_B(^{16}_{8}\mathrm{O}_8) = -4.14\ \mathrm{MeV}$$

$$\varepsilon(s_{1/2}) = \varepsilon(d_{5/2}) + \mathrm{E_{ex}}(^{17}\mathrm{O};\ 1/2^+_1) = -3.28\ \mathrm{MeV}$$

$$\varepsilon(d_{3/2}) = \varepsilon(d_{5/2}) + \mathrm{E_{ex}}(^{17}\mathrm{O};\ 3/2^+_1) = 0.93\ \mathrm{MeV}$$

## 2-nucleon basis:

| | |
|---|---|
| $(d_{5/2})^2$ | $J^\pi = 0^+, 2^+, 4^+$ |
| $(s_{1/2})^2$ | $J^\pi = 0^+$ |
| $(d_{3/2})^2$ | $J^\pi = 0^+, 2^+,$ |
| $d_{5/2}d_{1/2}$ | $J^\pi = 2^+, 3^+$ |
| $d_{5/2}d_{3/2}$ | $J^\pi = 1^+, 2^+, 3^+, 4^+$ |
| $d_{3/2}s_{1/2}$ | $J^\pi = 1^+, 2^+$ |

# Setting up the Hamiltonian Matrix in each $J^\pi$ subspace in the *sd*-shell

## The Hamiltonian Matrix for $J^\pi = 0^+$

The allowed $0^+$ states are $|(d_{5/2})^2\rangle_0$, $|(s_{1/2})^2\rangle_0$ and $|(d_{3/2})^2\rangle_0$, the matrix looks like:

| | $\mathbf{\|(d_{5/2})^2\rangle_0}$ | $\mathbf{\|(s_{1/2})^2\rangle_0}$ | $\mathbf{\|(d_{3/2})^2\rangle_0}$ |
|---|---|---|---|
| $\mathbf{\langle(d_{5/2})^2\|}$ | $2\epsilon_{d5/2} + \langle d^2_{5/2}\|V\|d^2_{5/2}\rangle_0$ | $\langle d^2_{5/2}\|V\|s^2_{1/2}\rangle_0$ | $\langle d^2_{5/2}\|V\|d^2_{3/2}\rangle_0$ |
| $\mathbf{\langle(s_{1/2})^2\|}$ | $\langle s^2_{1/2}\|V\|d^2_{5/2}\rangle_0$ | $2\epsilon_{s1/2} + \langle s^2_{1/2}\|V\|s^2_{1/2}\rangle_0$ | $\langle s^2_{1/2}\|V\|d^2_{3/2}\rangle_0$ |
| $\mathbf{\langle(d_{3/2})^2\|}$ | $\langle d^2_{3/2}\|V\|d^2_{5/2}\rangle_0$ | $\langle d^2_{3/2}\|V\|s^2_{1/2}\rangle_0$ | $2\epsilon_{d3/2} + \langle d^2_{3/2}\|V\|d^2_{3/2}\rangle_0$ |

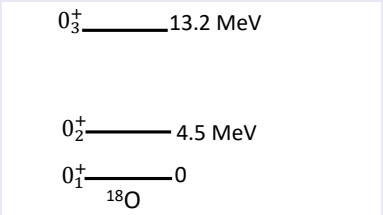
*The off-diagonal delta terms drive the **configuration mixing**, shifting the lowest eigenvalue to form the ground state.*

# Practical shell-model for $^{18}$O in *sd*-shell

## Eigenvalues (g.s. and two excited $0^+$ states):

Taking the TBMEs from USDB "Universal" SD (Brown, Richter, PRC74, 034315 (2006)):

$$E(0_1^+) = -12.617 \text{ MeV} \qquad E_{ex}(0_1^+) = 0 \text{ MeV}$$
$$E(0_2^+) = -8.111 \text{ MeV} \quad \Rightarrow \quad E_{ex}(0_2^+) = 4.506 \text{ MeV}$$
$$E(0_3^+) = 0.600 \text{ MeV} \qquad E_{ex}(0_3^+) = 13.217 \text{ MeV}$$




## Eigenstates:

| | $\|(\mathbf{d}_{5/2})^2\rangle_0$ | $\|(\mathbf{s}_{1/2})^2\rangle_0$ | $\|(\mathbf{d}_{3/2})^2\rangle_0$ |
|---|---|---|---|
| $\mathbf{0}_1^+$ | 79% | 14% | 7% |
| $\mathbf{0}_2^+$ | 14% | 85% | 1% |
| $\mathbf{0}_3^+$ | 1% | 7% | 92% |

# Setting up the Hamiltonian Matrix each $J^\pi$ subspace in the *sd*-shell

## The Hamiltonian Matrix $J^\pi = 1^+$

The allowed $1^+$ states are $|d_{5/2}d_{3/2}\rangle_1$ and $|d_{3/2}s_{1/2}\rangle_1$, the matrix looks like:

| | $\mathbf{\|d_{5/2}d_{3/2}\rangle_1}$ | $\mathbf{\|d_{3/2}s_{1/2}\rangle_1}$ |
|---|---|---|
| $\mathbf{\langle d_{5/2}d_{3/2}\|}$ | $\epsilon_{d5/2} + \epsilon_{d3/2} + \langle d_{5/2}d_{3/2}\|V\|d_{5/2}d_{3/2}\rangle_1$ | $\langle d_{5/2}d_{3/2}\|V\|d_{3/2}s_{1/2}\rangle_1$ |
| $\mathbf{\langle d_{3/2}s_{1/2}\|}$ | $\langle d_{3/2}s_{1/2}\|V\|d_{5/2}d_{3/2}\rangle_1$ | $\epsilon_{d3/2} + \epsilon_{s1/2} + \langle d_{3/2}s_{1/2}\|V\|d_{3/2}s_{1/2}\rangle_1$ |

# Energies of $1^+$ states in $^{18}$O

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} a_{1p} \\ a_{2p} \end{pmatrix} = E_p \begin{pmatrix} a_{1p} \\ a_{2p} \end{pmatrix}$$

## Eigenvalues and eigenvectors:

$$E_{1,2} = \frac{1}{2}\left\{ H_{11} + H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2} \right\}$$

$$|1_1^+\rangle = a_{11}|d_{5/2}d_{3/2}\rangle_1 + a_{21}|d_{3/2}s_{1/2}\rangle_1$$
$$|1_2^+\rangle = a_{12}|d_{5/2}d_{3/2}\rangle_1 + a_{22}|d_{3/2}s_{1/2}\rangle_1$$

$$\sum_k a_{kp}^2 = 1$$

# Energies of $1^+$ states in $^{18}$O

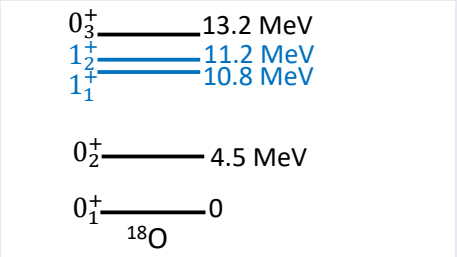
## Diagonal and non-diagonal TBMEs

$$H_{11} = \varepsilon_{d5/2}+\varepsilon_{d3/2}+\underbrace{\langle d_{5/2}d_{3/2}|V|d_{5/2}d_{3/2}\rangle_1}_{1.030\ \text{MeV}} \qquad H_{12} = \underbrace{\langle d_{5/2}d_{3/2}|V|s_{1/2}d_{3/2}\rangle_1}_{0.187\ \text{MeV}}$$

$$H_{21} = \underbrace{\langle d_{5/2}d_{3/2}|V|s_{1/2}d_{3/2}\rangle_1}_{0.187\ \text{MeV}} \qquad H_{22} = \varepsilon_{s1/2}+\varepsilon_{d3/2}+\underbrace{\langle s_{1/2}d_{3/2}|V|s_{1/2}d_{3/2}\rangle_1}_{0.606\ \text{MeV}}$$

## Eigenvalues (g.s. and two excited $0^+$ states):

$$E(1_1^+) = -2.164 \quad \text{MeV}$$
$$E(1_2^+) = -1.725 \quad \text{MeV}$$
$$\Downarrow$$
$$E_{ex}(1_1^+) = 10.786 \quad \text{MeV}$$
$$E_{ex}(1_2^+) = 11.225 \quad \text{MeV}$$



# Shell-model codes

## *M*-scheme codes

- KShell (U. Tokyo, N. Shimizu et al) — MPI+OpenMP
  Public version: `https://github.com/GaffaSnobb/kshell`
  arXiv:1310.5431
- BIGSTICK (U. St Diego, C. Johnson et al) — MPI
  Public version: `https://github.com/cwjsdsu/BigstickPublick`
- ANTOINE (Strasbourg/Madrid, E. Caurier, **F. Nowacki** et al)
- NuShellX@MSU (Oxford/MSU, W. Rae, **B. A. Brown**)
- MSHELL (U. Tokyo, T. Mizusaki)
- OXBASH (Oxford/MSU)
- ...

## Features

- Matrix dimension: $\sim 10^{10}$
- Lanczos diagonalization algorithm
- Calculation of the matrix elements on-the-fly

# 5. Matrix elements of transition operators

# Calculation of observables

## General scheme:

- Basis construction: $|\Phi_k\rangle$
- Expand the wave function: $|\Psi_p\rangle = \sum_k a_{kp}|\Phi_k\rangle$
- Solution of the Shrödinger equation by Hamiltonian matrix diagonalization: $\{H_{lk}\} \Rightarrow E_p, \{a_{kp}\}$
- Calculation of matrix elements of the operators

  Transition probability:

$$T_{if} \propto |\langle\Psi_f|\hat{O}|\Psi_i\rangle|^2$$

  Partial lifetime:

$$\tau_{if} = \frac{1}{T_{if}}$$

  - Electormagnetic transitions
  - Beta decay
  - Nucleon transfer operators

# Electromagnetic operators

## Electric $2^\lambda$-pole operators:

$$O(E\lambda) = \sum_{k=1}^{A} e(k) r^\lambda(k) Y_{\lambda\mu}(\hat{r}(k))$$

## Magnetic $2^\lambda$-pole operators:

$$O(M\lambda) = \sum_{k=1}^{A} \mu_N \left( g_s(k)\vec{s}(k) + \frac{2g_l(k)}{\lambda+1}\vec{l}(k) \right) \cdot \nabla(k) r_k^\lambda Y_{\lambda\mu}(\Omega_k)$$

$$\hat{O}(M1) = \sum_{k=1}^{A} \mu_N (g_s(k)\vec{s}(k) + g_l(k)\vec{l}(k))$$

$g_s = \quad 5.586$, $g_l = 1$ for a proton
$g_s = -3.826$, $g_l = 0$ for a neutron

## Reduced transition probabilities:

$$
\begin{aligned}
B(\lambda; J_i \to J_f) &= \frac{1}{2J_i+1} \sum_{M_i,\mu,M_f} |\langle \alpha_f; J_f M_f | \hat{O}_{\lambda\mu} | \alpha_i; J_i M_i \rangle|^2 \\
&= \frac{1}{2J_i+1} \sum_{M_i,\mu,M_f} \begin{pmatrix} J_f & \lambda & J_i \\ -M_f & \mu & M_i \end{pmatrix}^2 |\langle \alpha_f; J_f || \hat{O}_{\lambda} || \alpha_i; J_i \rangle|^2 \\
&= \frac{1}{2J_i+1} |\langle \alpha_f; J_f || \hat{O}_{\lambda} || \alpha_i; J_i \rangle|^2 \, .
\end{aligned}
$$

## Selection rules:

- Triangle rule:

$$\Delta(J_i, J_f, \lambda)$$

- Parity conservation:

$$
\begin{aligned}
P_i \, P_f \, (-1)^{\lambda} &= 1 \quad \text{(electric)} \\
P_i \, P_f \, (-1)^{\lambda+1} &= 1 \quad \text{(magnetic)}
\end{aligned}
$$

# Electric Quadrupole (E2) Operator

The E2 operator evaluates structural shapes and spatial deformations. The single-particle operator is:

$$\hat{O}(E2) = e^{\text{eff}}\, r^2\, Y_{2\mu}(\hat{r})$$

Applying the Wigner-Eckart theorem, its reduced single-particle matrix element splits into a radial integral and a spherical harmonic angular matrix element:

$$\langle n_f l_f s_f j_f || \hat{O}(E2) || n_i l_i s_i j_i \rangle = e^{\text{eff}} \int_0^\infty R_{n_f l_f}(r) r^4 R_{n_i l_i}(r) dr \times \langle l_f s_f; j_f || Y_2 || l_i s_i; j_i \rangle$$

Using re-coupling rules to isolate the spin component, the angular part evaluates to:

$$\langle l_f s_f; j_f || Y_2 || l_i s_i; j_i \rangle = (-1)^{j_i + 1/2} \frac{\hat{j}_i \hat{j}_f \hat{2}}{\sqrt{4\pi}} \begin{pmatrix} j_f & 2 & j_i \\ 1/2 & 0 & -1/2 \end{pmatrix} \times \text{Parity Rule}$$

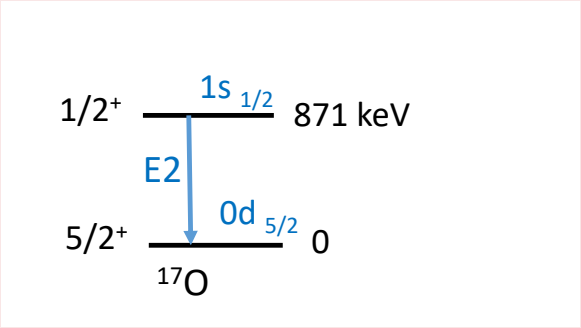
*Selection Rules:* $|j_f - j_i| \leq 2$, and $(l_f + l_i)$ must be **even** (no parity change).

# Exercise 4

## Electromagnetic transitions in $^{17}$O

Calculate the $B(E2; 1/2_1^+ \rightarrow 5/2_{g.s.}^+)$ in $^{17}$O.

Compare your result to experimental value $B_{exp}(E2) = 6.3$ e$^2$.fm$^4$.



# Solution to exercise 6

## Electromagnetic transitions in $^{17}$O

In our case, there is one valence particle, the initial state is $1s_{1/2}$, the final state is $0d_{5/2}$:

$$B(E2; 1/2^+ \to 5/2^+) = \frac{1}{2}|\langle 0d_{5/2}||O(E2)||1s_{1/2}\rangle|^2 =$$

$$\frac{1}{2}e^2 \underbrace{\left(\int R^*_{0d_{5/2}}(r) r^2 R_{1s_{1/2}}(r) dr\right)^2}_{\langle r^2\rangle^2} |\langle d_{5/2}||Y_2(\theta,\phi)||s_{1/2}\rangle|^2 .$$

Evaluation of the radial part:

$$\int_0^\infty R_{1s_{1/2}}(r) r^2 R_{0d_{5/2}}(r) r^2 dr \approx -10\,\mathrm{fm}^2.$$

for $\hbar\omega = 14$ MeV (near $A = 18$).

# Solution to exercise 4

## Electromagnetic transitions in $^{17}$O (continued)

$$B(E2; 1/2^+ \to 5/2^+) = (e^{\text{eff}})^2 \tfrac{1}{4\pi} \langle r^2 \rangle^2 6 \times 5 \begin{pmatrix} 1/2 & 5/2 & 2 \\ 1/2 & -1/2 & 0 \end{pmatrix}^2 \approx 24\,(\text{e}^{\text{eff}})^2.\text{fm}^4.$$

Experimental value $B_{exp}(E2; 1/2^+ \to 5/2^+) = 6.3\ \text{e}^2.\text{fm}^4$.

This means that the neutron should have an effective charge: $e^{\text{eff}}(n) \approx 0.5\,e$, because we work in a severly restricted model space (one valence neutron!).

## "Standard" effective $E2$ and $M1$ operators

$$e^{\text{eff}}(p) \approx 1.5\,e \qquad e^{\text{eff}}(n) \approx 0.5\,e$$
$$g_s^{\text{eff}}(p) \approx 0.7\ \underbrace{g_s(p)}_{5.586} \qquad g_l^{\text{eff}}(p) = g_l(p) = 1$$
$$g_s^{\text{eff}}(n) \approx 0.7\ \underbrace{g_s(n)}_{-3.826} \qquad g_l^{\text{eff}} l(n) = g_l(n) = 0$$

# Two-nucleon system: $^{18}$F

## Magentic moment expression:

$$\mu = \sqrt{\frac{4\pi}{3}}\langle J, M = J|O(M1)|J, M = J\rangle = \sqrt{\frac{4\pi}{3}}\begin{pmatrix} J & 1 & J \\ -J & 0 & J \end{pmatrix}\langle J||O(M1)||J\rangle$$

Let's calculate the magnetic moment of $5^+$ state in $^{18}$F: $|pd_{5/2}nd_{5/2}; 5^+, T = 0\rangle$

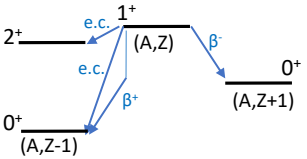
## Decoupling of the matrix element:

$$\langle pj_a\, nj_b; J||O_p(M1) + O_n(M1)||pj_a\, nj_b; J\rangle =$$
$$(-1)^{j_a+j_b+J+1}\left[\begin{Bmatrix} J & J & 1 \\ j_a & j_a & j_b \end{Bmatrix}\langle pj_a||O_p(M1)||pj_a\rangle + \begin{Bmatrix} J & J & 1 \\ j_b & j_b & j_a \end{Bmatrix}\langle nj_b||O_n(M1)||nj_b\rangle\right]$$

## Result:

Theory: $\mu(5^+_1) = 2.96\ \mu_N$
Experiment: $\mu(5^+_1) = +2.86(3)\ \mu_N$

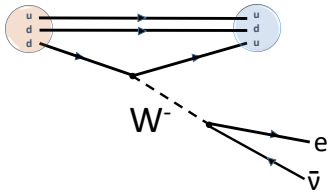
# Nuclear beta decay




$$ {}_{Z}X_{N} \rightarrow {}_{Z+1}X_{N-1} + e^{-} + \overline{\nu} $$

$$ {}_{Z}X_{N} \rightarrow {}_{Z-1}X_{N+1} + e^{+} + \nu $$

$$ {}_{Z}X_{N} + e^{-} \rightarrow {}_{Z-1}X_{N+1} + \nu $$




At low energies, the charge-changing semi-leptonic effective weak-interaction Lagrangian reads:

$$ \mathcal{L}_{\beta} = -\frac{G_F}{\sqrt{2}} V_{ud} \overline{\psi}_{e^-} \gamma_{\mu} (1-\gamma_5) \psi_{\nu_e} \overline{\psi}_{u} \gamma^{\mu} (1-\gamma_5) \psi_{d} + h.c. $$

# Nuclear beta decay

## Approximations

- Nucleons are non-relativistic point-like particles
- leptonic wave functions are constant in the nucleus interior (allowed transitions)

The decay rate:

$$T(\beta^{\pm}) = \frac{m^5 c^3}{2\pi^3 \hbar^7} f(Z, W_0) \left[ G_V^2 B(F) + G_A^2 B(GT) \right]$$

Statistical rate function:

$$f(Z, W_0) = \int_1^{W_0} F(Z, W) S(Z, W) (W - W_0)^2 p W dW$$

### Fermi operator

$$\hat{O}(F) = \sum_{k=1}^{A} \hat{t}_{\pm}(k)$$

### Gamow-Teller operator:

$$\hat{O}(GT) = \sum_{k=1}^{A} \hat{\sigma}_m(k) \hat{t}_{\pm}(k)$$

# Nuclear beta decay

## $ft$-values

$$f(Z, W_0) t_{1/2}(\beta^{\pm}) = f(Z, W_0) \frac{\ln(2)}{T(\beta^{\pm})} = \frac{K}{G_V^2 B(F) + G_A^2 B(GT)}$$

where $K/G_V^2 = 6163$ sec, $G_A/G_V \approx 1.26$.

### Fermi operator

$$\hat{O}(F) = \sum_{k=1}^{A} \hat{t}_{\pm}(k)$$

### Gamow-Teller operator:

$$\hat{O}(GT) = \sum_{k=1}^{A} \hat{\sigma}_m(k) \hat{t}_{\pm}(k)$$

# Nuclear beta decay

## Reduced Fermi transition probability:

$$\begin{aligned} B(F) &= \frac{1}{2J_i+1}\sum_{M_i M_f}\langle J_f M_f\, T_f M_{Tf}|\sum_{k=1}^{A}\hat{t}_+(k)|J_i M_i\, T_i M_{Ti}\rangle^2 \\ &= [T_i(T_i+1)-M_{Ti}M_{Tf}]\,\delta_{J_i J_f}\delta_{T_i T_f} \end{aligned}$$

Selection rules: $J_i = J_f$, $P_i = P_f$, $T_i = T_f$.

## Reduced Gamow-Teller transition probability:

$$\begin{aligned} B(GT) &= \frac{1}{2J_i+1}\sum_{M_i M_f m}\langle J_f M_f\, T_f M_{Tf}|\sum_{k=1}^{A}\hat{\sigma}_m(k)t_+(k)|J_i M_i\, T_i M_{Ti}\rangle^2 \\ &= \frac{1}{2(2J_i+1)}\begin{pmatrix} T_f & 1 & T_i \\ -M_{Tf} & 1 & M_{Ti}\end{pmatrix}^2 \underbrace{\langle J_f\, T_f|||\sum_{k=1}^{A}\hat{\sigma}(k)\hat{\tau}(k)|||J_i\, T_i\rangle^2}_{\textit{doubly reduced matrix element}} \end{aligned}$$

Selection rules: $J_f = J_i, J_i \pm 1$, $P_f = P_i$, $T_f = T_i, T_i \pm 1$ (no $J_i = 0 \to J_f = 0$ transition).

# Beta decay of the neutron: $n \rightarrow p + e^- + \overline{\nu_e}$

Neutron: $l = 0, s = 1/2, j = 1/2, t = 1/2,\ m_t = 1/2$
Proton: $l = 0, s = 1/2, j = 1/2, t = 1/2,\ m_t = -1/2$

## Fermi and Gamow-Teller reduced probabilities

$$B(F) = t(t+1) - m_{ti} m_{tf} = \frac{3}{4} + \frac{1}{4} = 1$$

$$B(GT) = \left(\frac{1}{2}\right)^2 \begin{pmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}^2 \langle \tfrac{1}{2} || \hat{\sigma} || \tfrac{1}{2} \rangle^2 \langle \tfrac{1}{2} || \hat{\tau} || \tfrac{1}{2} \rangle^2 = \tfrac{1}{4} \times \tfrac{1}{3} \times 6 \times 6 = 3$$

# Beta for single-particle configurations

Initial state: $l_i, s_i = 1/2, j_i, t_i = 1/2,\ m_t = 1/2$
Final state : $l_f, s_f = 1/2, j_f, t_f = 1/2,\ m_t = -1/2$

## Fermi and Gamow-Teller reduced probabilities

$$B(F) = t(t+1) - m_{ti} m_{tf} = \frac{3}{4} + \frac{1}{4} = 1$$

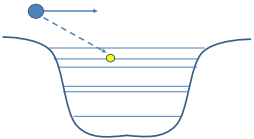
$$B(GT) = 6(2j_f+1) \left\{ \begin{array}{ccc} 1/2 & 1/2 & 1 \\ j_i & j_f & l_f \end{array} \right\} \delta_{l_i l_f} .$$

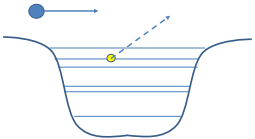
Where we have exploited:

$$\langle n_f l_f j_f, t = \frac{1}{2} ||| \hat{\sigma}\hat{\tau} ||| n_i l_i j_i, t = \frac{1}{2} \rangle = (-1)^{l_f + j_f + 3/2} 6 \sqrt{(2j_f+1)(2j_i+1)} \left\{ \begin{array}{ccc} 1/2 & 1/2 & 1 \\ j_i & j_f & l_f \end{array} \right\} \delta_{l_i l_f} .$$

($\tau_\pm = \sqrt{2} t_\pm$).

# One-nucleon transfer reactions





$$A + a \Rightarrow B + b \quad \text{or} \quad A(a,b)B$$

| Proton stripping | Proton pick-up | Neutron stripping | Neutron pick-up |
|---|---|---|---|
| (d,n) | (n,d) | (d,p) | (p,d) |
| ($^3$He,d) | (d,$^3$He) | (t,d) | (d,t) |
| ($\alpha$,t) | (t, $\alpha$) | ($\alpha$,$^3$He) | ($^3$He,$\alpha$) |

**Direct reactions (with minimum of rearrangement of the nucleons)**

$$\sigma_l(\theta) \propto S\sigma_l^{\mathrm{DWBA}}(\theta)$$

$S$ is the spectroscopic factor

# Spectroscopic factors for pick-up reactions (particle removal: $A \to A-1$)

## Reaction cross-section

$$\begin{aligned}
\sigma_l^{p.u.} &\sim \frac{1}{2J_i+1} \sum_{M_i M_f m} |\langle (A-1)\alpha_f J_f M_f | \tilde{a}_{jm} | A\alpha_i J_i M_i \rangle|^2 \\
&= \frac{1}{2J_i+1} \sum_{M_i M_f m} \begin{pmatrix} J_f & j & J_i \\ -M_f & m & M_i \end{pmatrix}^2 |\langle (A-1)\alpha_f J_f || \tilde{a}_j || A\alpha_i J_i \rangle|^2 \\
&= \frac{1}{2J_i+1} |\langle (A-1)\alpha_f J_f || \tilde{a}_j || A\alpha_i J_i \rangle|^2 \equiv S
\end{aligned}$$

$$\sigma_l^{p.u.} = S \sigma_l^{s.p.}$$

$S \equiv S(nlj)$ is the spectroscopic factor

## Selection rules

- $|J_i - j| \leq J_f \leq J_i + j$, where $j = l \pm 1/2$
- $\Pi_i \Pi_f = (-1)^l$

# Spectroscopic factors for stripping reactions (particle addition)

## Reaction cross-section

$$
\begin{aligned}
\sigma_l^{str} &\sim \frac{1}{2J_i+1} \sum_{M_i M_f m} |\langle (A+1)\,\alpha_f J_f M_f | a^+_{jm} | A\,\alpha_i J_i M_i \rangle|^2 \\
&= \frac{1}{2J_i+1} \begin{pmatrix} J_f & j & J_i \\ -M_f & m & M_i \end{pmatrix}^2 |\langle (A+1)\,\alpha_f J_f || a^+_j || A\,\alpha_i J_i \rangle|^2 \\
&= \frac{1}{2J_i+1} |\langle (A+1)\,\alpha_f J_f || \tilde{a}_j || A\,\alpha_i J_i \rangle|^2 \equiv \frac{2J_f+1}{2J_i+1} S
\end{aligned}
$$

$$
\sigma_l^{str} = \frac{2J_f+1}{2J_i+1}\, S\, \sigma_l^{s.p.}
$$

$S \equiv S(nlj)$ is the spectroscopic factor

## Selection rules

- $|J_i - j| \le J_f \le J_i + j$, where $j = l \pm 1/2$
- $\Pi_i \Pi_f = (-1)^l$

# Spectroscopic factors

**Spectroscopic factors are related to reduced overlap integrals:**

$$S = \frac{|\langle (A-1)\,\alpha' J' || \tilde{a}_j || A\,\alpha J\rangle|^2}{2J+1} = \frac{\left|\langle A\,\alpha J || a_j^+ || (A-1)\,\alpha' J'\rangle\right|^2}{2J+1}$$

**All $n$ nucleons are on the same $j$ shell**

The $n$-nucleon wave functions can be expressed via c.f.p.'s:

$$|j^n; JM\rangle = \sum_{J'} \langle j^{n-1}\,[J']\,j\,|\}\,j^n\,J\rangle |j^{n-1}\,[J']\,j(n);\,JM\rangle$$

The overlap integral is given by c.f.p.:

$$S = n\,\langle j^{n-1}\,[J_i] j |\} j^n\,J_f\rangle^2$$

# Example: Spectroscopic factors for $^{51}$V(d,$^{3}$He)$^{50}$Ti

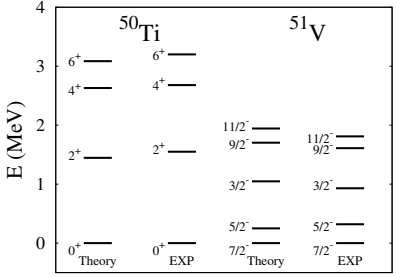
The spectra of the initial and final nuclei:

**$^{50}_{22}$Ti$_{28}$: $pf^2_{7/2}$ configurations $\rightarrow J^\pi = 0^+, 2^+, 4^+, 6^+$**

$$E(J) = 2\varepsilon_{f7/2} + \langle f^2_{7/2}; J|V|f^2_{7/2}; J\rangle$$

**$^{50}_{23}$V$_{28}$: $pf^3_{7/2}$ configurations $\rightarrow$**
**$J^\pi = 3/2^-, 5/2^-, 7/2^-, 9/2^-, 11/2^-, 15/2^-$**

$$E(J) = 3\varepsilon_{f7/2} + 3\sum_{J''}\langle j^2\,[J'']\,j\;J|\}\,j^3\,;J\rangle^2\langle f^2_{7/2}; J''|V|f^2_{7/2}; J''\rangle$$



# Spectroscopic factors for $^{51}$V(d,$^{3}$He)$^{50}$Ti

## Model space: $f_{7/2}$ orbital

$$\sigma_I^{p.u.}(\theta) = S\,\sigma_I^{\mathrm{s.p}}(\theta)$$

$$S = 3\,\langle f^2\,[J'']f; 7/2|\}f^3\,7/2\rangle^2$$

## Sspectroscopic factors for a transfer from $7/2^-_{gs}$ of $^{51}$V to the states of $^{50}$Ti in comparison with experiment at 52 MeV from Hinterberger et al, Z. Phys. 202, 236 (1968).

$$S_{th}(0^+) = 3 \times \frac{1}{4} = 0.75 \qquad S_{exp}(0^+) = 0.73$$

$$S_{th}(2^+) = 3 \times \left(\frac{\sqrt{5}}{6}\right)^2 \approx 0.42 \qquad S_{exp}(2^+) = 0.39$$

$$S_{th}(4^+) = 3 \times \left(\frac{1}{2}\right)^2 = 0.75 \qquad S_{exp}(4^+) = 0.64$$

$$S_{th}(6^+) = 3 \times \left(\frac{\sqrt{13}}{6}\right)^2 \approx 1.08 \qquad S_{exp}(6^+) = 1.05$$

# Some references

## Shell model theory


- P.J. Brussaard, P.W.M. Glaudemans, *Shell-Model Applications in Nuclear Spectroscopy*, (North-Holland, Amsterdam, 1977).
- K. Heyde, *The Nuclear Shell Model* (Springer-Verlag, Heidelberg, 2004)
- J. Suhonen, *From Nucleons to Nuclei* (Springer-Verlag, Heidelberg, 2007)
- I. Talmi, *Simple Models of Complex Nuclei, The Shell Model and the Interacting Boson Model*, (Harwood Academic Publishers, New York, 1993)
- E. Caurier et al, Rev. Mod. Phys. 77 (2005) 427.


## No-Core Shell Model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