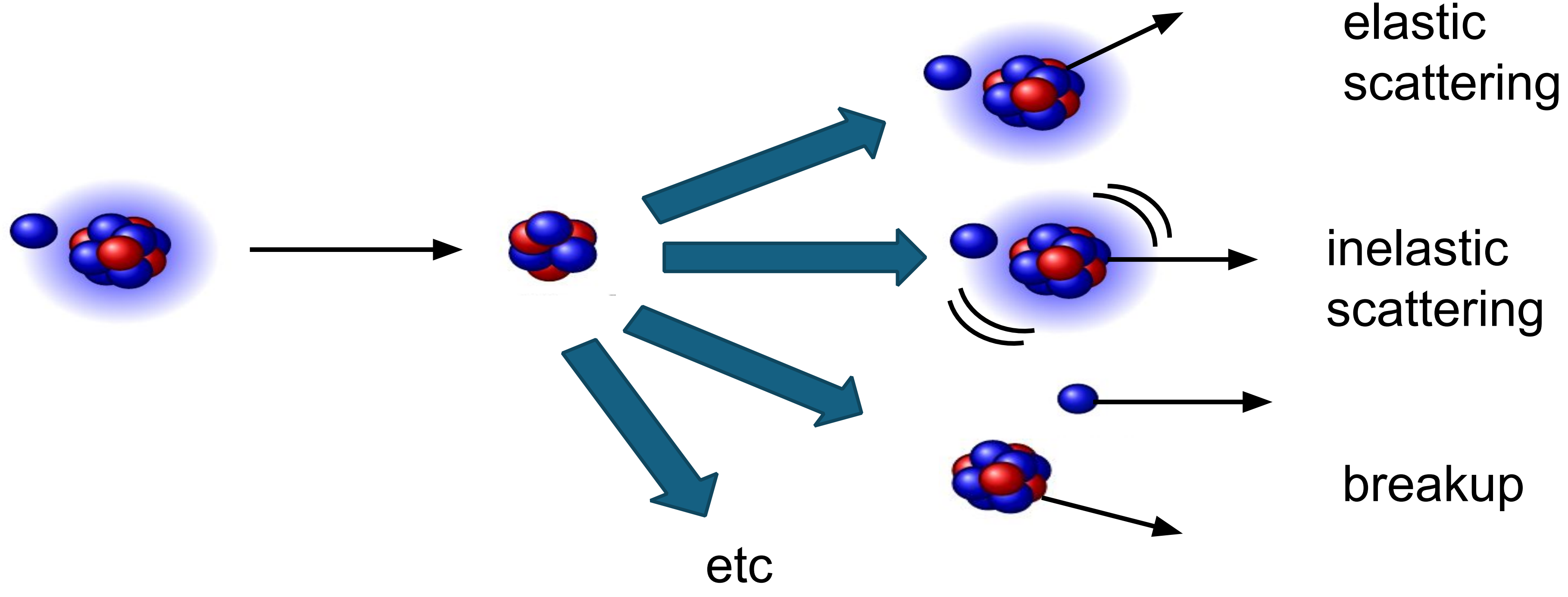
Introduction to the eikonal approximation and its applications

Ecole Joliot Curie

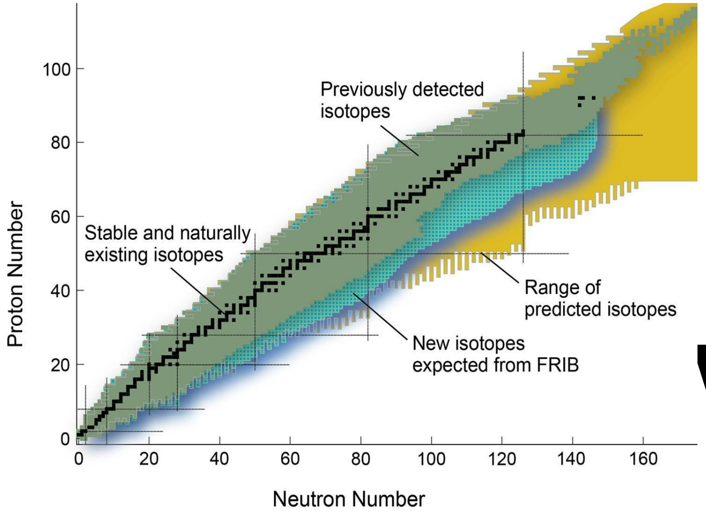
Chloë Hebborn – September 2026

Nuclear Reactions

Nuclear reactions are processes in which two nuclear species collide, allowing them to interact, exchanging matter, energy or momentum

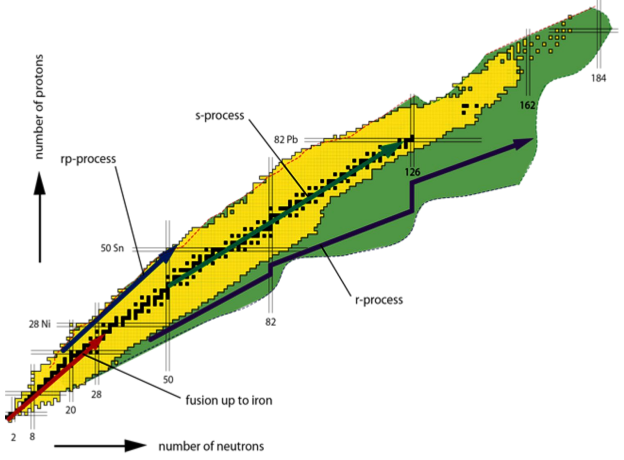


Why do we care?



Studying properties of unstable nuclei

(Source: FRIB)



Nuclear astrophysics

(source GSI)

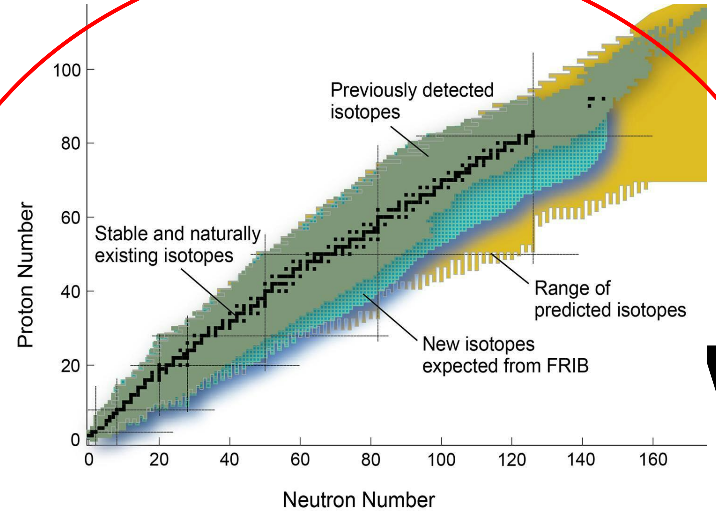
Stockpile stewardship (Source: DOE)

Nuclear energy

Nuclear waste

Nuclear medicine

Why do we care?



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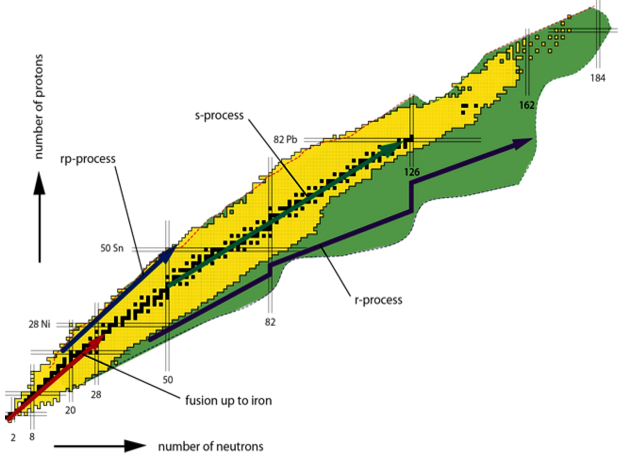

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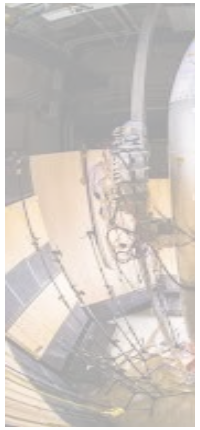
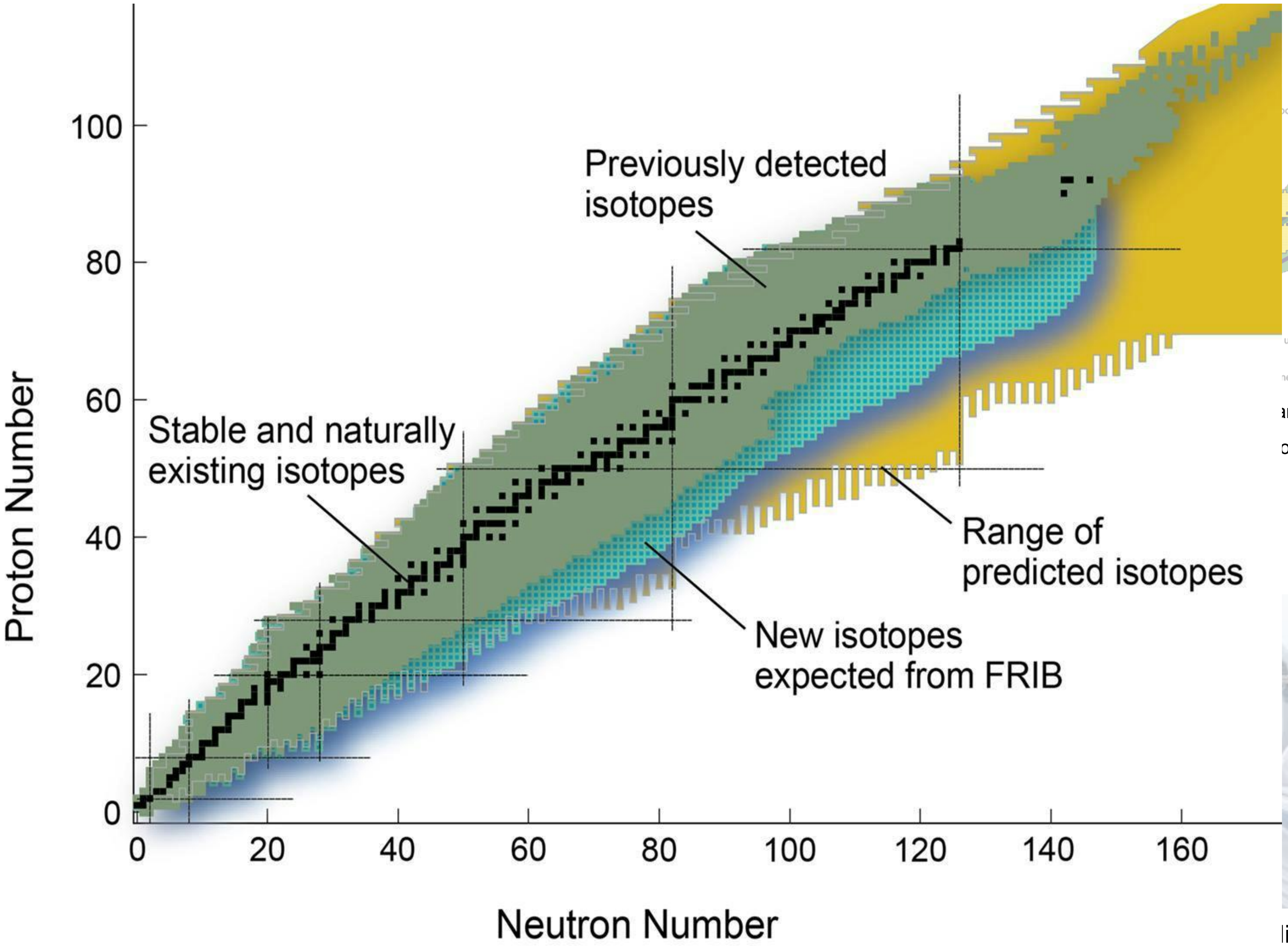


Stockpile stew

D



licine

Studying properties of unstable nuclei

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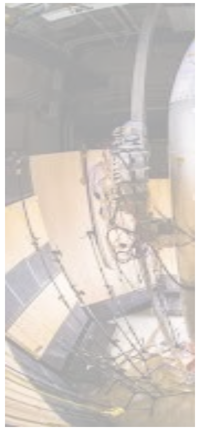

Stockpile stew
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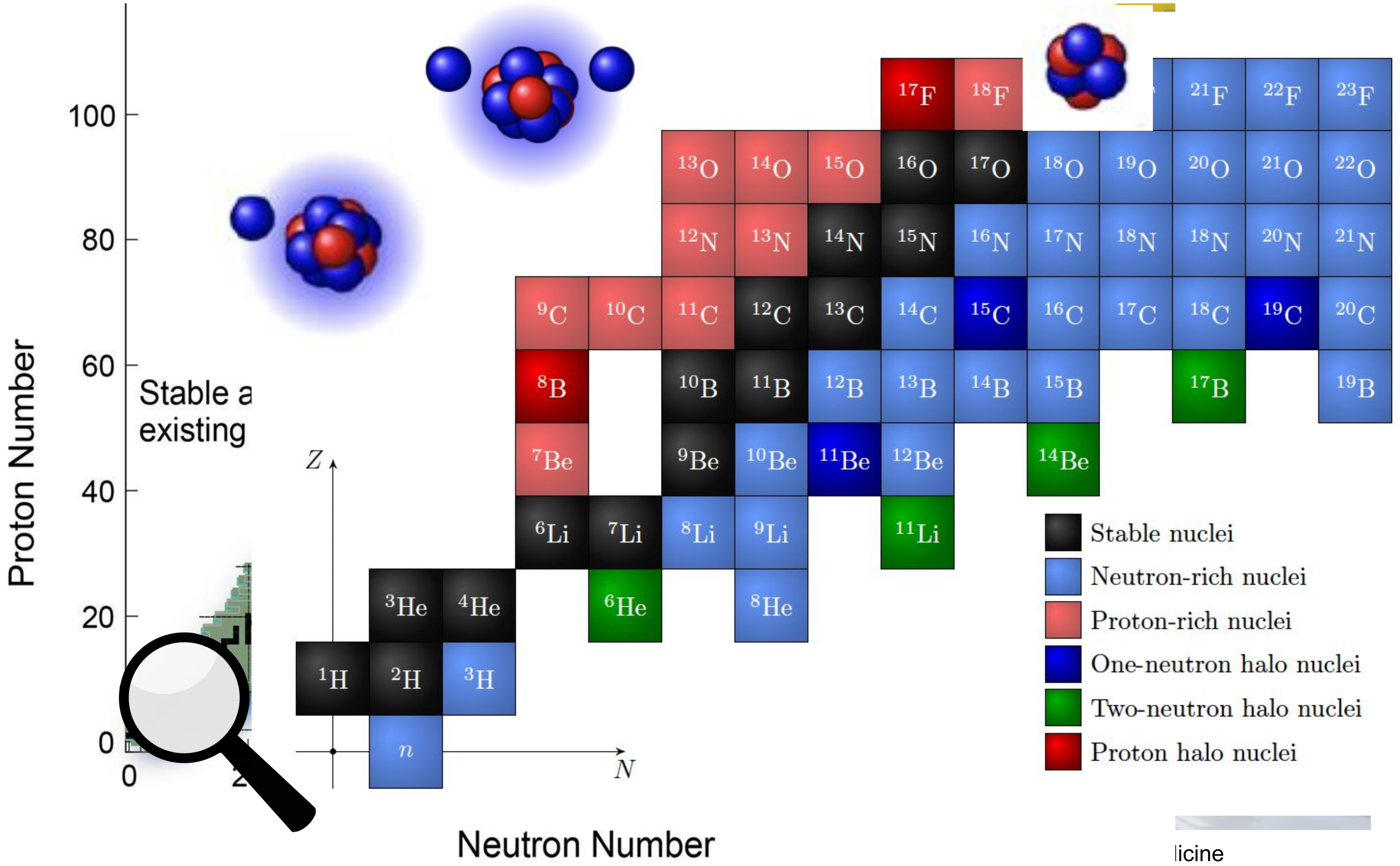

licine

Studying properties of unstable nuclei

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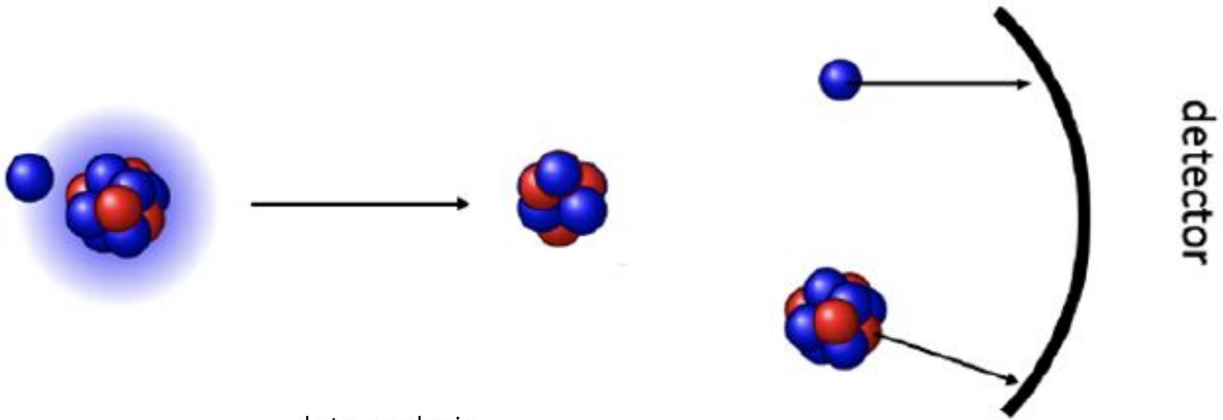
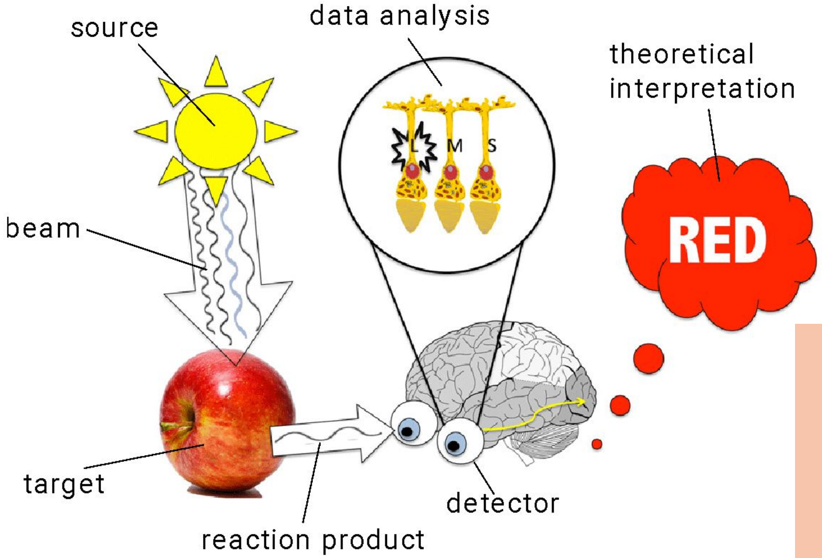
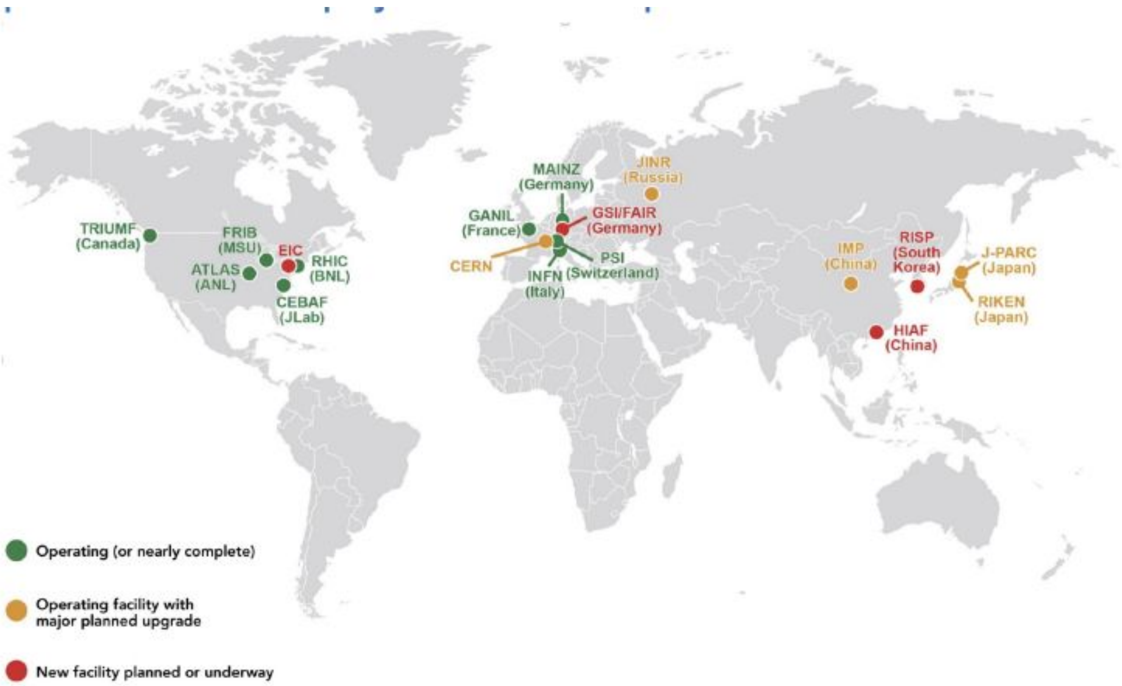
Unstable nuclei are often produced and studied through reactions







Accurate measurements and theory crucial!

During this hour and 1/2, we'll try to reply to few questions..

- When is the eikonal approximation used? For which reactions & at what energies?
- Formal derivation of the eikonal approximation for a two-body problem
- Extension to a few-body ones
- Hands on part with a python notebook

At what energy is the eikonal approximation used?

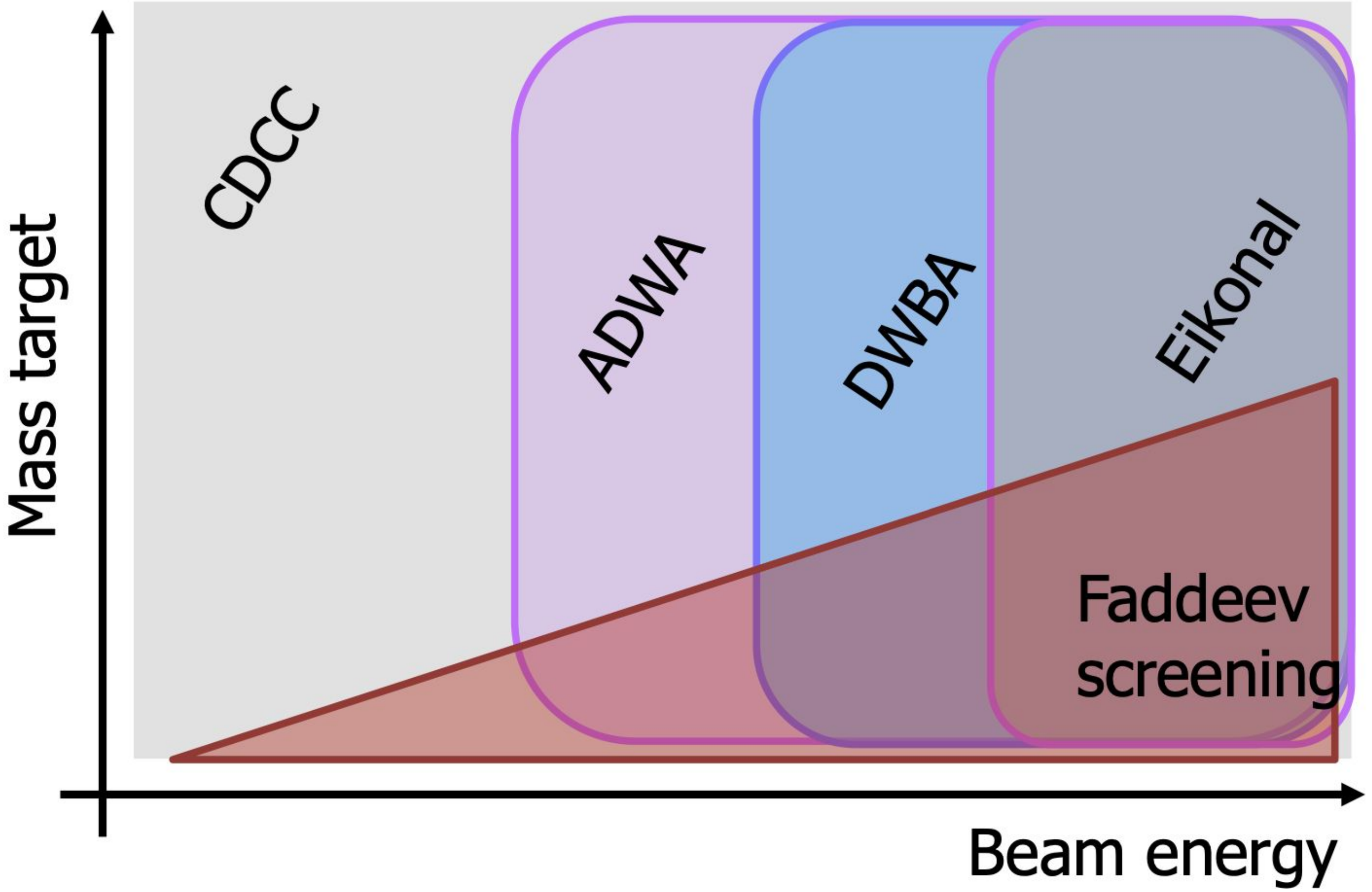
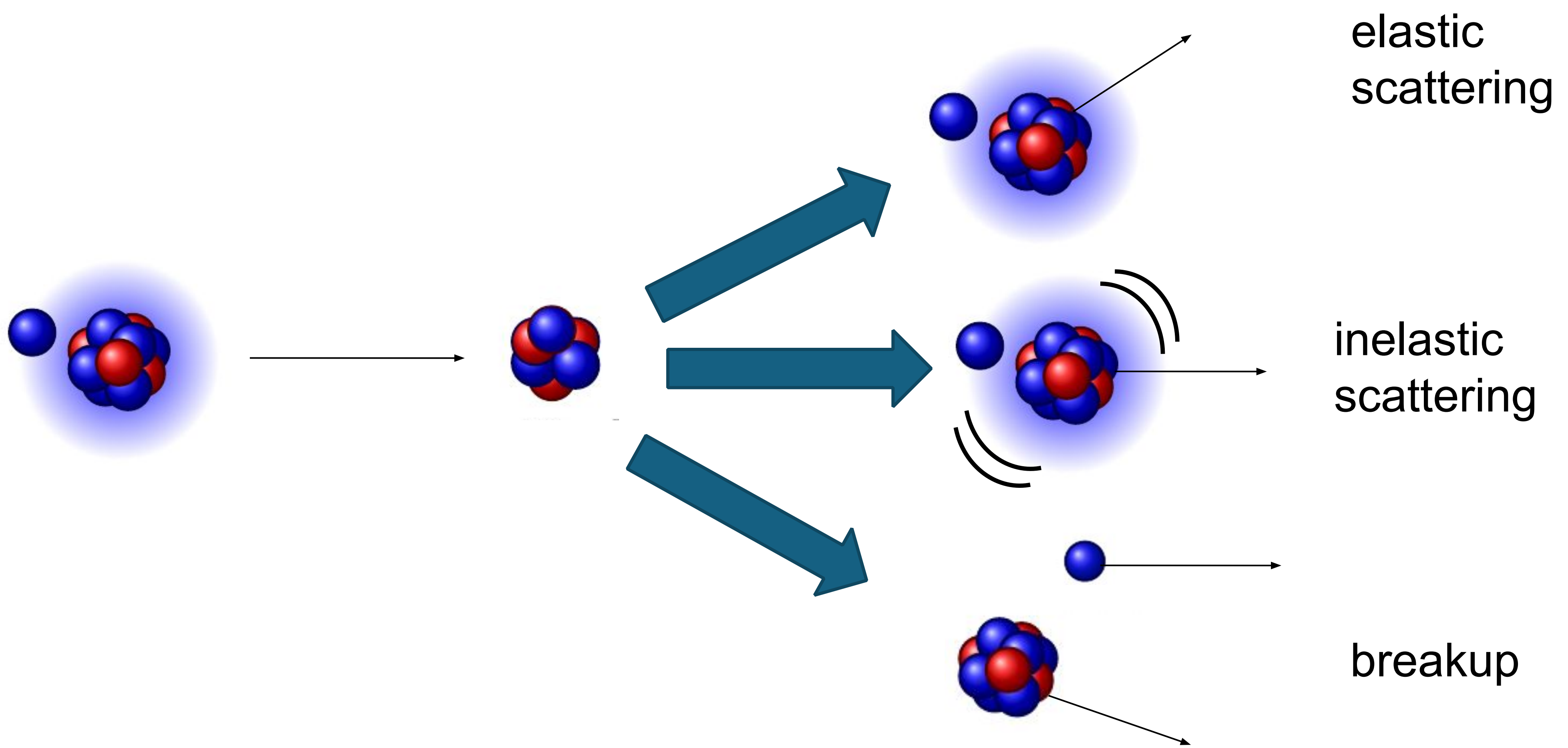


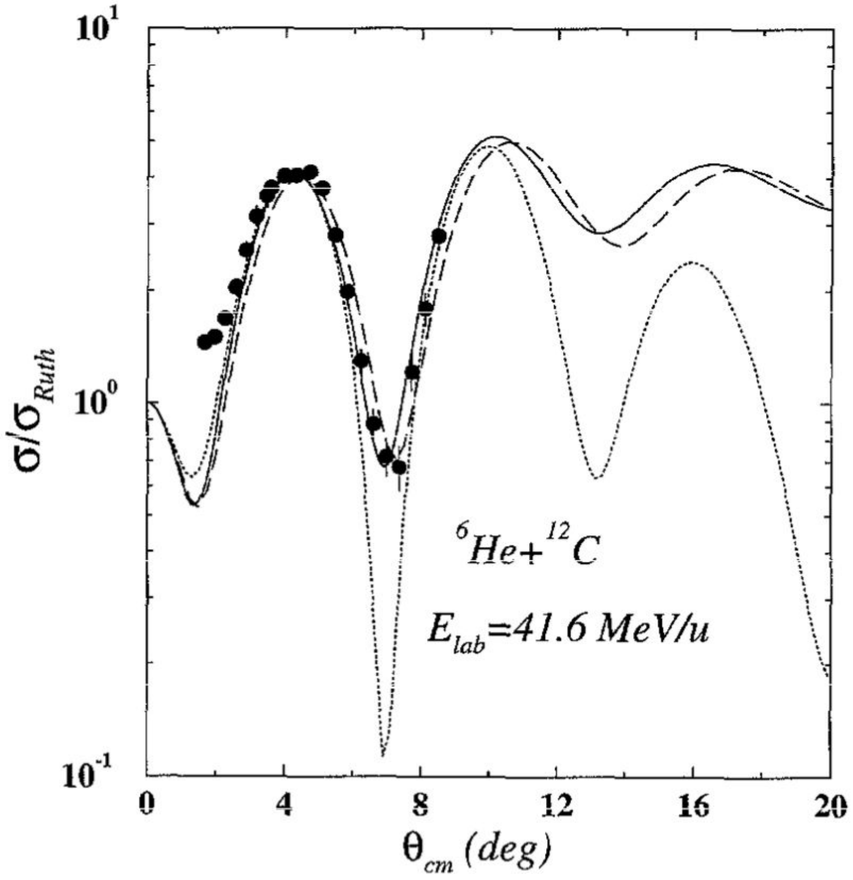
Image courtesy of Filomena Nunes

Eikonal model is cheap computationally, it can be generalized to complex reactions…

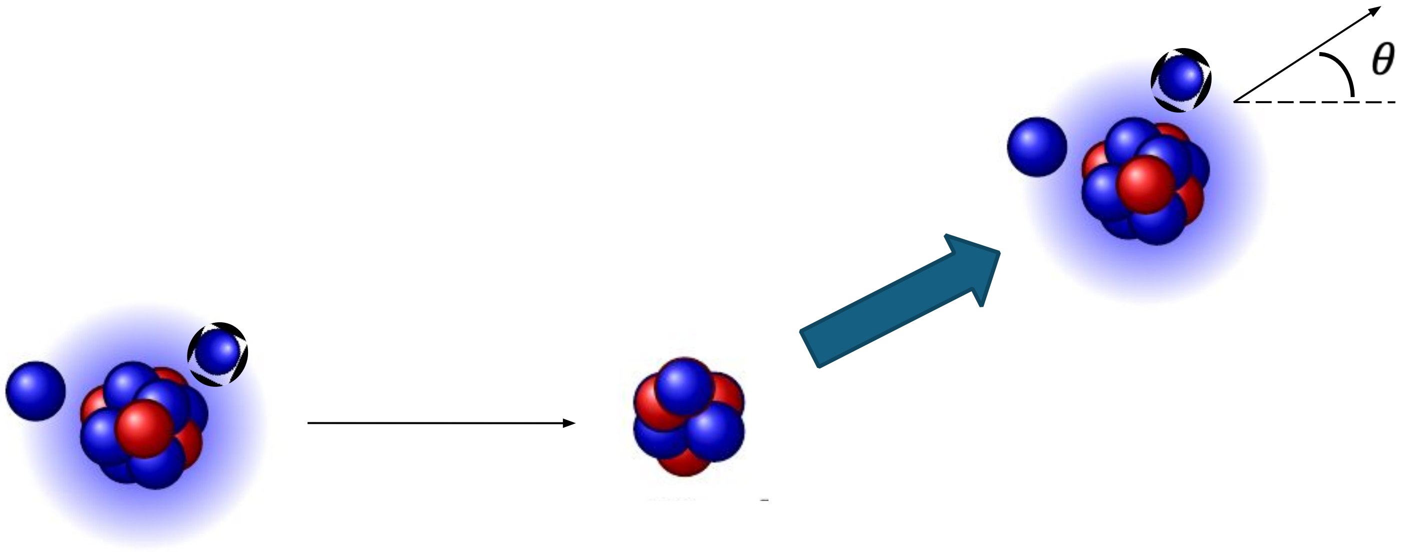
And for which reaction probes ?



Intermediate energy elastic scattering of ^{6}He in a four-body reaction model

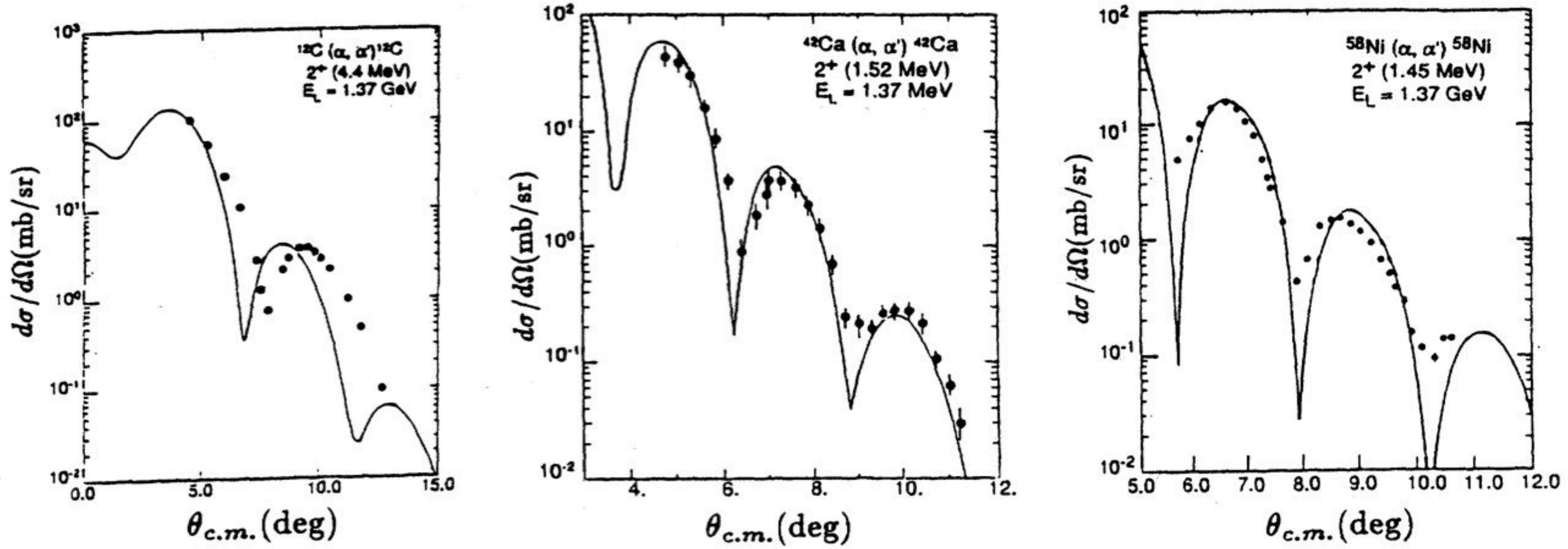


Al Khalili et al. PLB 378 (1996) 45-49



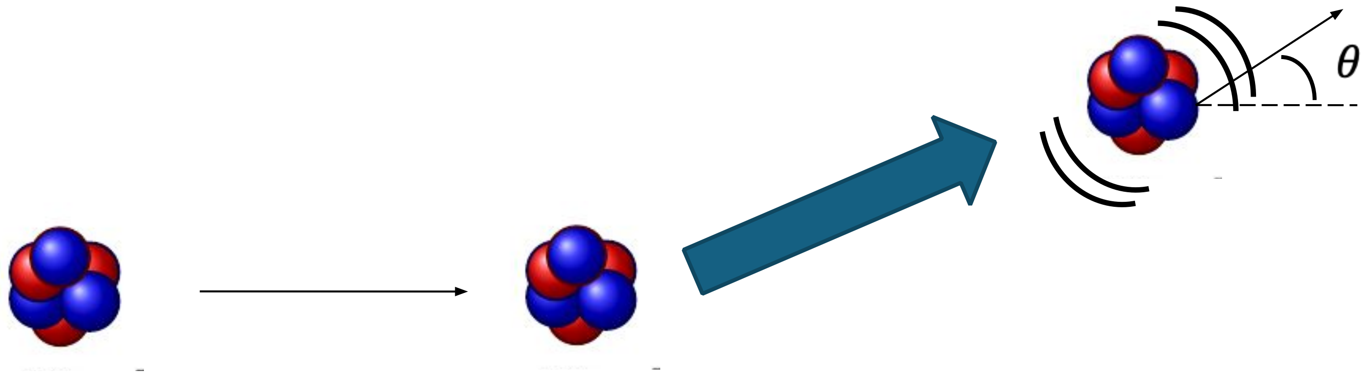
Elastic scattering observables can inform about the cluster structure in nuclei and their rms radii !

High energy inelastic scattering of nuclei

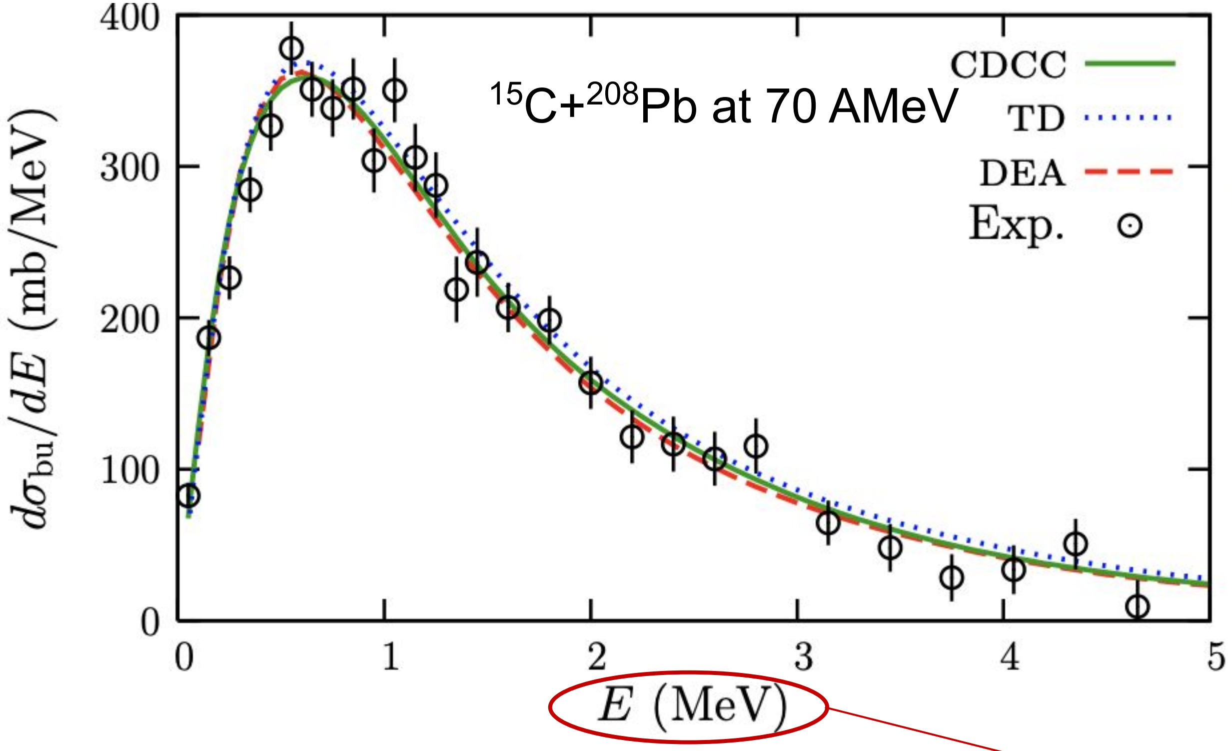


Lenzi et al. PRC 40 6 1989

Inelastic scattering observables can inform about the electromagnetic transition (e.g. B(E2)) in nuclei !

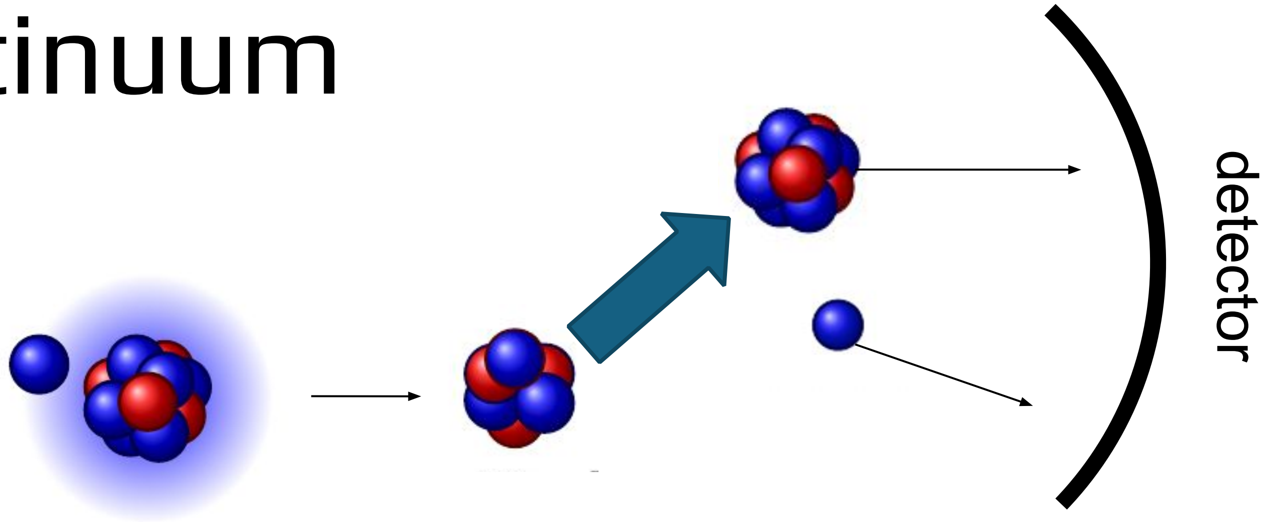


Breakup to study electromagnetic response in the continuum



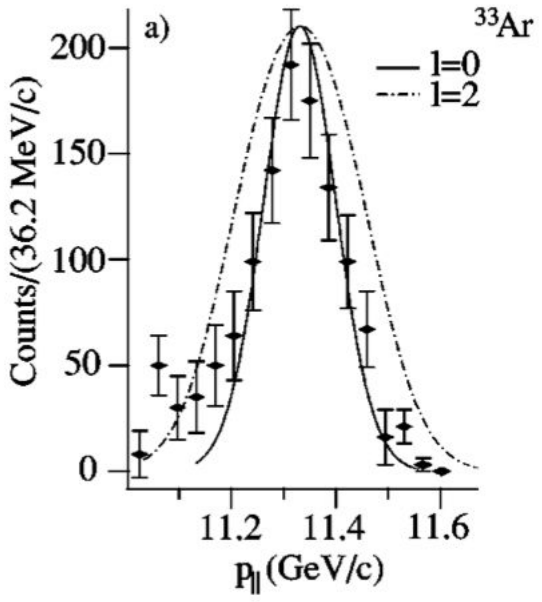
Capel et al., PRC 85, 044604 (2012)

Relative ^{14}C-n energy after the breakup

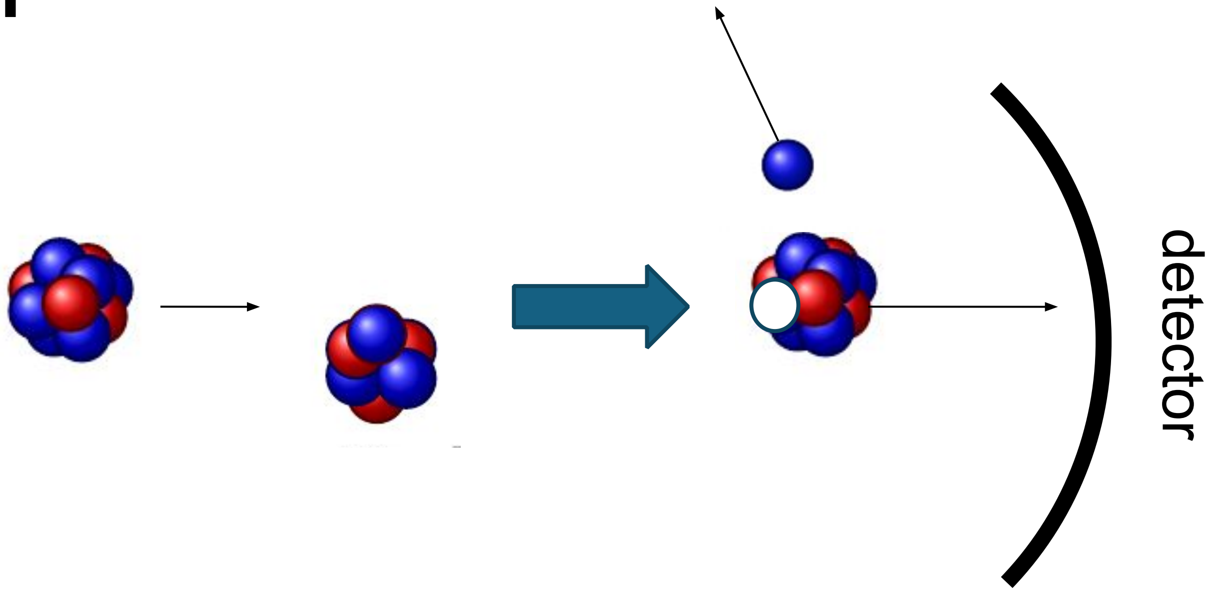


Breakup reactions are useful to study the cluster structure and probe electromagnetic response of nuclei !

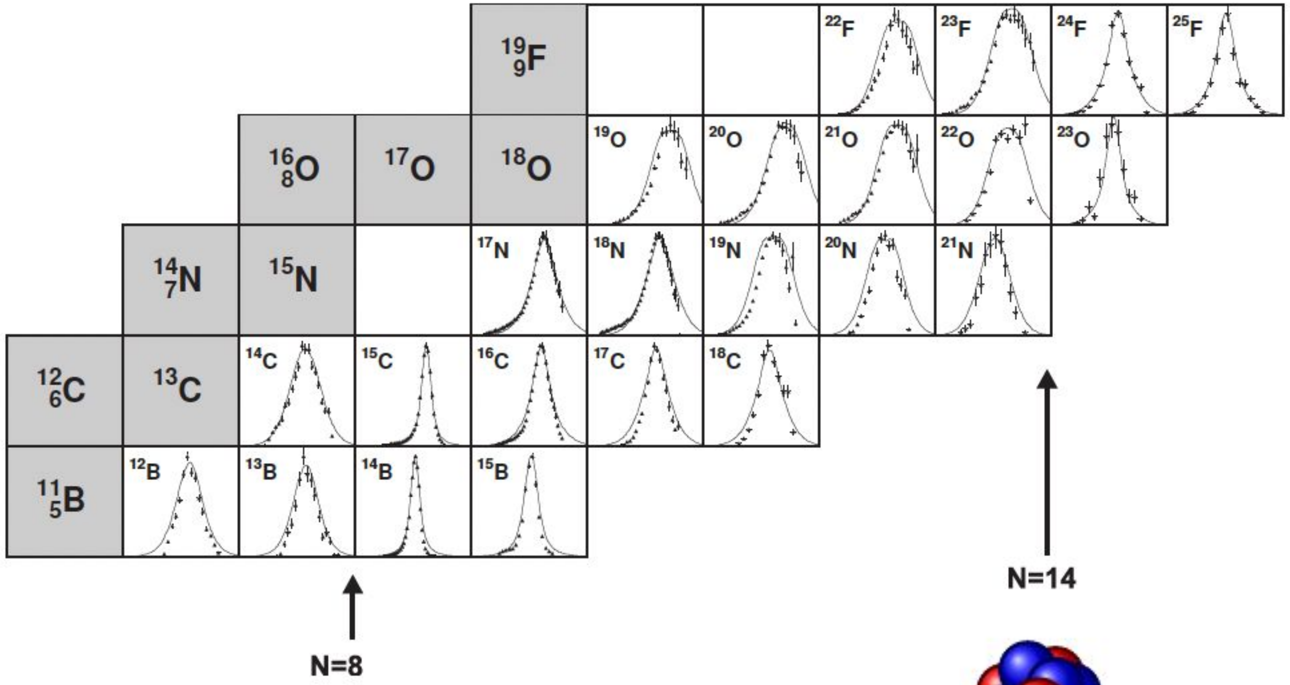
Nucleon removal to study spectroscopic properties of nuclei



Gade et al., PRC 69 034311 (2004)



Nucleon removal also carries information about the nucleus size



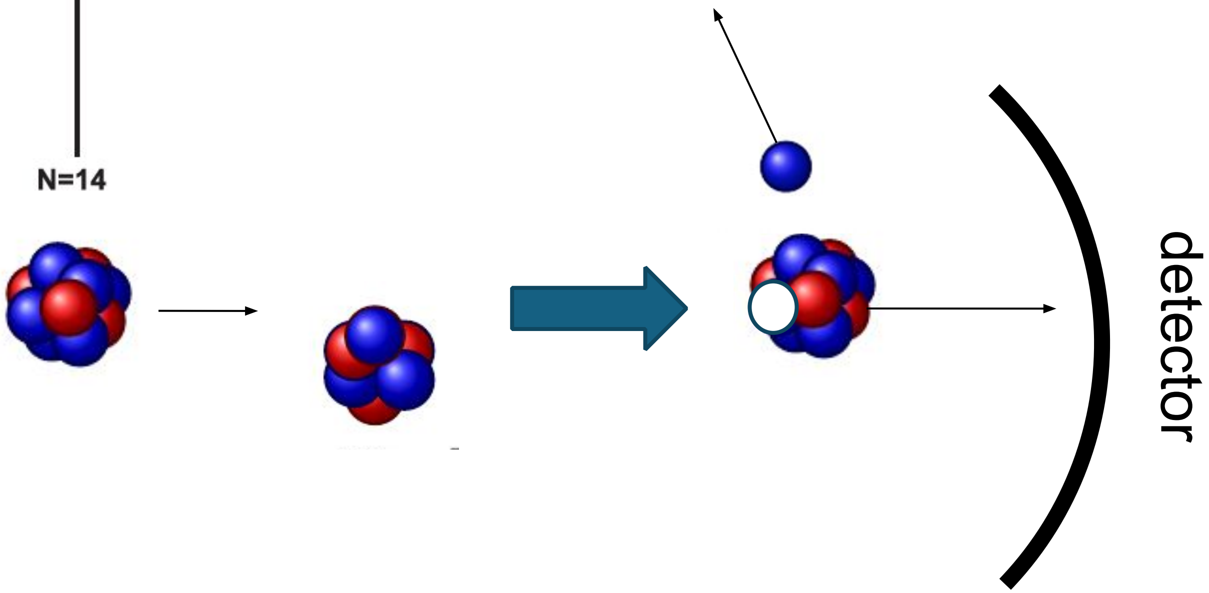
Sauvan et al. PLB 491 1 (2000)

Heisenberg's principle:

Narrow momentum distribution

=

Large spatial extension



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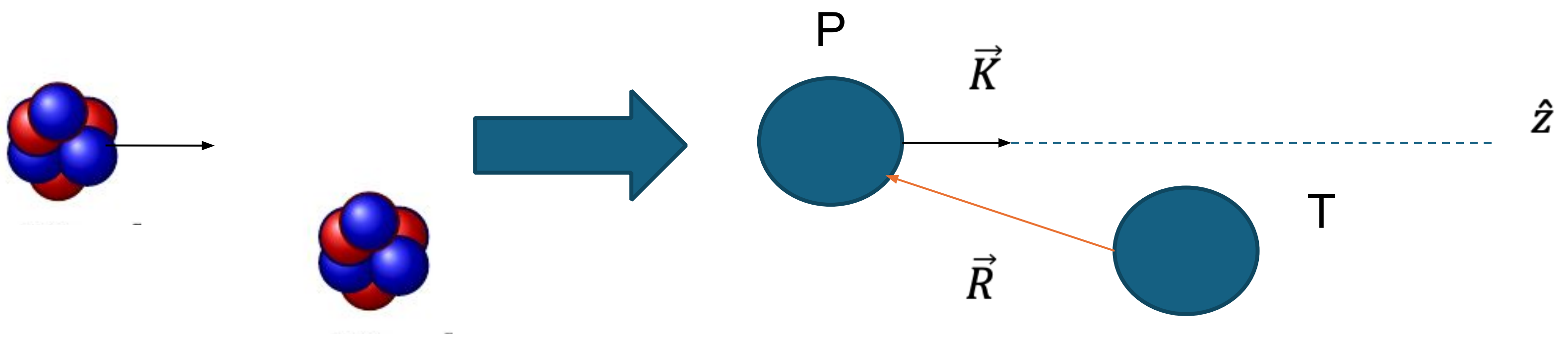
The reaction is often seen as a few-body problem



- Two-body Schrödinger equation

$$\left[T_{\vec{R}} + V_{PT}(R)\right]\Psi(\vec{R}\,) = E\Psi(\vec{R}\,)$$

- Initial condition

$$\Psi(\vec{R}\,) \rightarrow_{Z\rightarrow-\infty} e^{iKZ}$$

With energy of the system $E = \frac{\hbar^2 K^2}{2\mu}$

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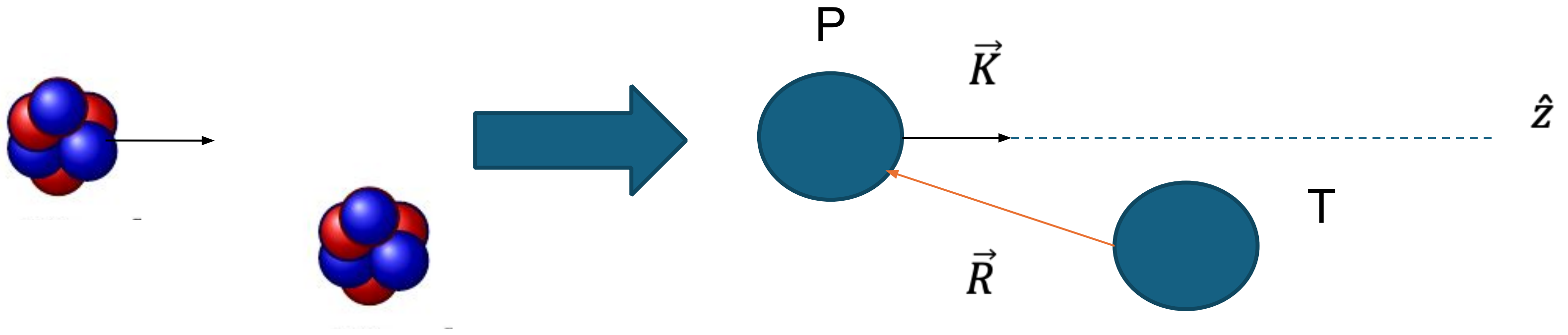

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With energy of the system $E = \dfrac{\hbar^2 K^2}{2\mu}$

Optical potential:

complex effective interaction

with imaginary parts modeling loss of flux in inelastic channels

At high energy, the projectile is not much deflected by the target

- Two-body Schrödinger equation $\left[T_{\vec{R}} + V_{PT}(R)\right]\Psi\left(\vec{R}\,\right) = E\Psi\left(\vec{R}\,\right)$
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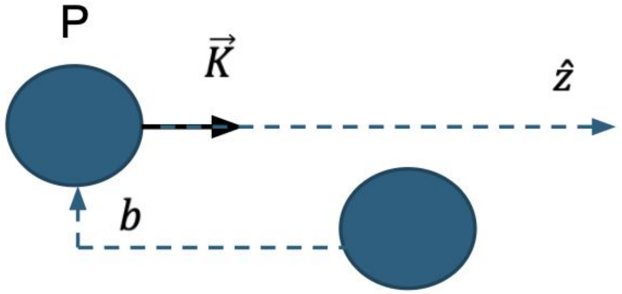

- Eikonal factorization: $\Psi\left(\vec{R}\,\right) = e^{iKZ}\,\widehat{\Psi\left(\vec{R}\,\right)}$

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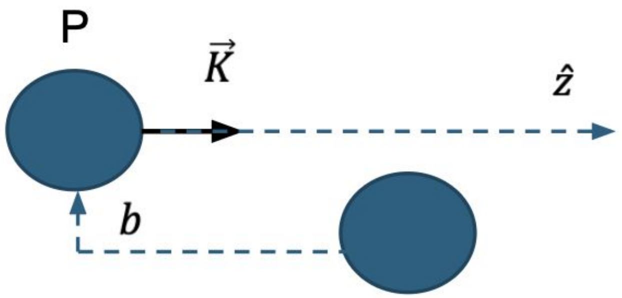

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$$\mathrm{T}_{\vec{R}} e^{iKZ}\,\widehat{\Psi\left(\vec{R}\,\right)} = -\frac{\hbar^2}{2\,\mu}\Delta_{\vec{R}} e^{iKZ}\widehat{\Psi\left(\vec{R}\,\right)} = -\frac{\hbar^2}{2\,\mu}\left(\frac{\delta^2}{\delta X^2} + \frac{\delta^2}{\delta Y^2} + \frac{\delta^2}{\delta Z^2}\right) e^{iKZ}\,\widehat{\Psi\left(\vec{R}\,\right)}$$

$$=?$$

$$= -\frac{\hbar^2}{2\,\mu}\left(-K^2 e^{iKZ}\widehat{\Psi\left(\vec{R}\,\right)} + 2iK\,e^{iKZ}\,\frac{\delta}{\delta Z}\widehat{\Psi\left(\vec{R}\,\right)} + e^{iKZ}\,\Delta_{\vec{R}}\,\widehat{\Psi\left(\vec{R}\,\right)}\right)$$

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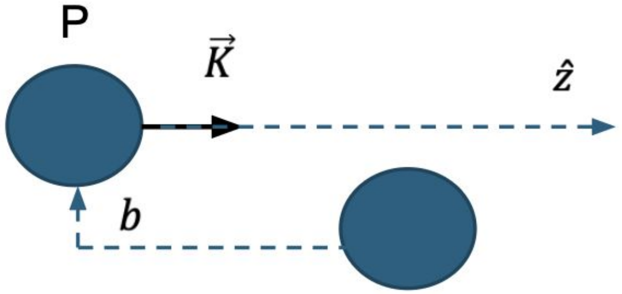

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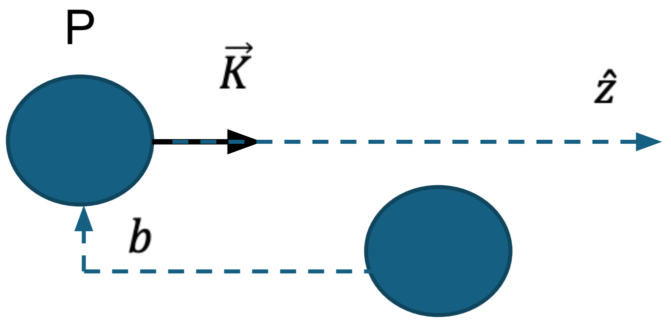
- Eikonal approximation : $\Delta_{\vec{R}}\widehat{\Psi(\vec{R}\,)} \ll K\,\frac{\delta}{\delta Z}\widehat{\Psi(\vec{R}\,)}$

Eikonal equations can be solved analytically!

… and one directly obtains $i\,\hbar\,v\,\frac{\delta}{\delta Z}\widehat{\Psi\left(\vec{R}\,\right)} = V_{PT}(R)\widehat{\Psi\left(\vec{R}\,\right)}$ with $v = \frac{\hbar\quad K}{\mu}$



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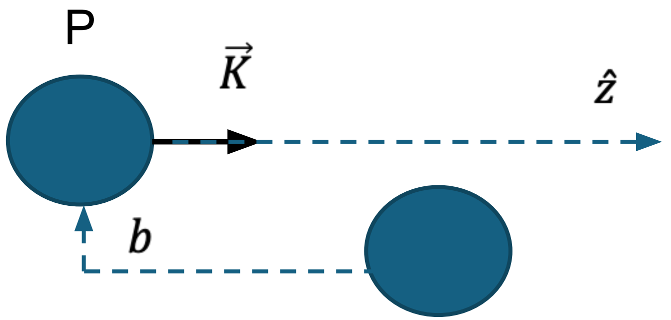
Eikonal Solution: $\Psi(\vec{R}\,) = e^{iKZ}\,\widehat{\Psi(\vec{R}\,)} = e^{iKZ} e^{\frac{-i}{\hbar\ v}\int_{-\infty}^{Z} V_{PT}(R')dZ'}$

Analytical form → Fast



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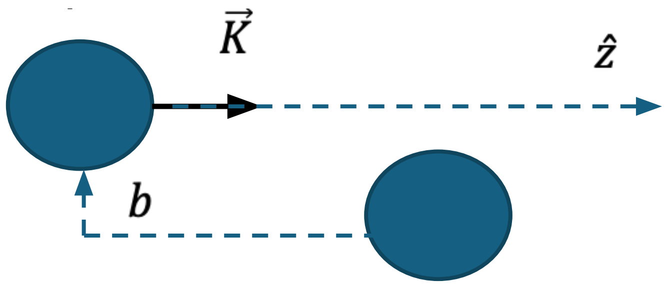
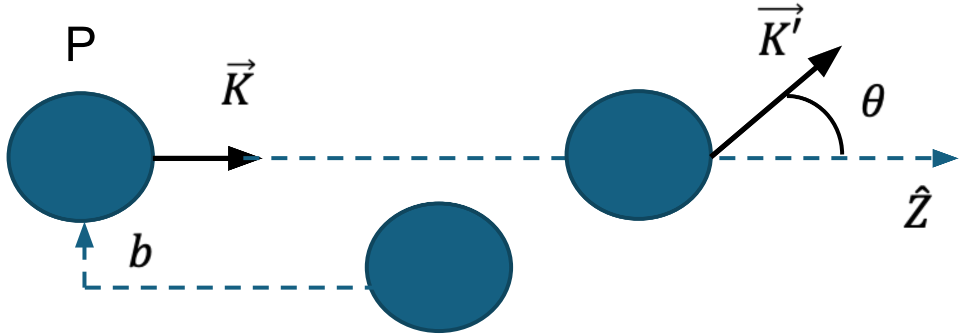
Analytical form → Fast

Semiclassical interpretation : P follows a straightline trajectory at constant impact parameter b along which it accumulates a phase due to its interaction with T



How to obtain elastic scattering cross sections?



Elastic scattering for a certain momentum:

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2$$

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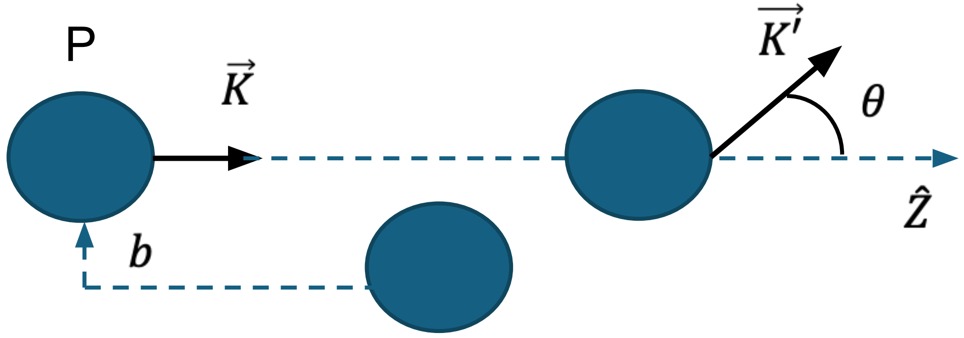


Elastic scattering for a certain momentum:

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2$$

One can show

$$f^{eik}(\theta) = -iK \int_0^\infty db\, b\, J_0(qb)\left[e^{i\chi(b)} - 1\right]$$

with $\vec{q} = \overrightarrow{K'} - \vec{K}$

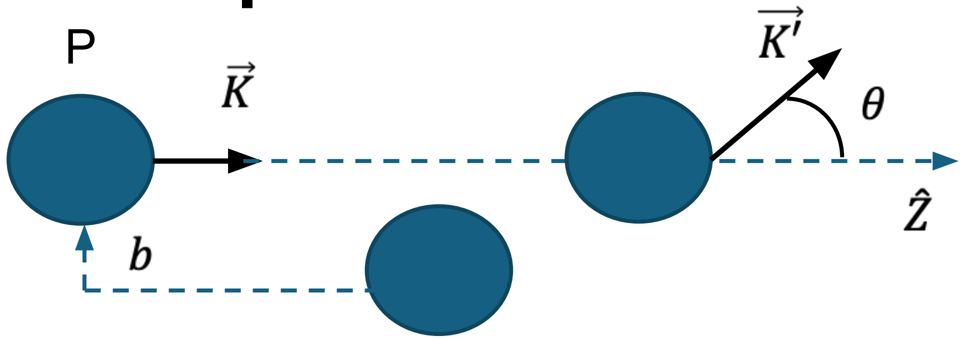
Eikonal phase $\chi(b) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V_{PT}(R)dZ$

All the information is in the eikonal phase!

Integrated cross sections and questions

- Using the eikonal phase

$$\chi(b) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V_{PT}(R)dZ$$



- Elastic scattering cross sections

$$\sigma_{el} = 2\pi \int_0^{+\infty} b\, db\, P_{el}(b) \text{ with } P_{el}(b) = \left|1 - e^{i\chi(b)}\right|^2$$

- Absorption cross section

$$\sigma_{abs} = 2\pi \int_0^{+\infty} b\, db\, P_{abs}(b) \text{ with } P_{abs}(b) = 1 - |e^{i\chi(b)}|^2$$

- Total cross section

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs} = 2\pi \int_0^{+\infty} b\, db\, 2[1 - Re(e^{i\chi(b)})]$$

1) What happens to P_{el} and P_{abs} if the potential is nil?

2) What happens to P_{el} and P_{abs} when b is very large (outside of the range of the potential)?

3) What happens to P_{abs} if the potential is real?

During this hour, we'll try to reply to few questions..

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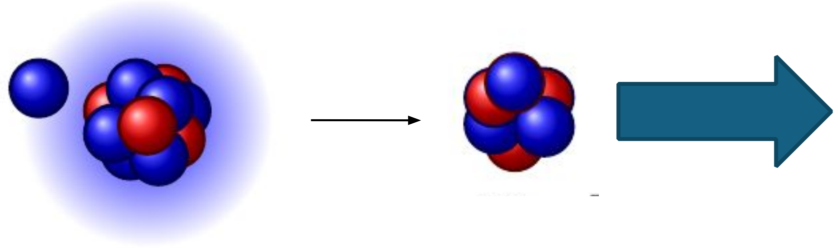
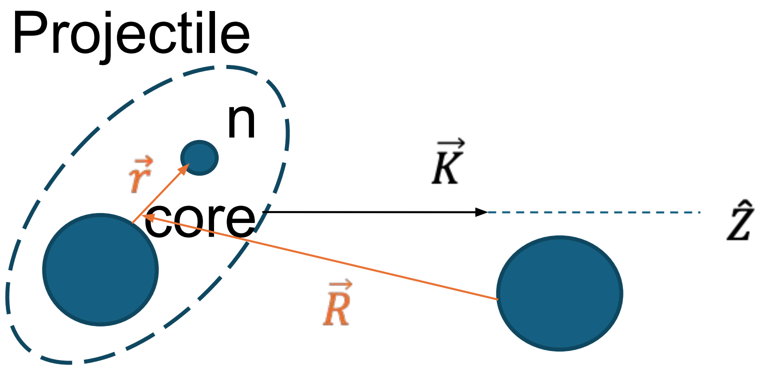
- 



- Three-body scattering problem

$$\left[T_{\vec{R}} + h_{cn} + V_{cT}(R_{cT}) + V_{nT}(R_{nT})\right]\Psi(\vec{R},\vec{r}) = E\Psi(\vec{R},\vec{r})$$

- Initial condition

$$\Psi(\vec{R},\vec{r}) \rightarrow_{Z\rightarrow-\infty} \phi_{gs}(\vec{r})e^{iKZ}$$

With $h_{cn}\phi_{gs}(\vec{r}) = [T_{\vec{r}}+V_{cn}(r)]\phi_{gs}(\vec{r}) = \epsilon_{gs}\phi_{gs}(\vec{r})$

Similar process as before… except that now it depends on the projectile system

- Eikonal factorization: $\Psi(\vec{R},\vec{r}\,) = e^{iKZ}\,\widehat{\Psi(\vec{R}\,,\vec{r})}$ and eikonal approximation $\Delta\Psi \ll K\frac{\partial\Psi}{\partial Z}$
- Using the eikonal factorization in the 3-body Schrödinger equation & energy conservation

$$i\,\hbar\,v\,\frac{\delta}{\delta Z}\widehat{\Psi(\vec{R},\vec{r}\,)} = (h_{cn} - \epsilon_{gs})\Psi(\vec{R},\vec{r}\,) + (V_{cT}(R_{cT}) + V_{nT}(R_{nT}))\,\Psi(\vec{R},\vec{r}\,) \text{ with } v = \frac{\hbar\;\;K}{\mu}$$

(dynamical eikonal approximation)

- Usual Eikonal assumes the adiabatic approximation: $h_{cn} \sim \epsilon_{gs}$

$$i\,\hbar\,v\,\frac{\delta}{\delta Z}\widehat{\Psi(\vec{R},\vec{r}\,)} = (V_{cT}(R_{cT}) + V_{nT}(R_{nT}))\,\Psi(\vec{R},\vec{r}\,)$$

- Eikonal Solution: $\Psi(\vec{R}\,,\vec{r}) = e^{ikz}\,\widehat{\Psi(\vec{R}\,,\vec{r})} = e^{ikz}e^{\frac{-i}{\hbar\,v}\int_{-\infty}^{Z}[V_{cT}(R'_{cT})+V_{nT}(R'_{nT})]dZ'}\;\phi_{gs}(\vec{r})$

  → 2 eikonal phases: one for each fragment of P !

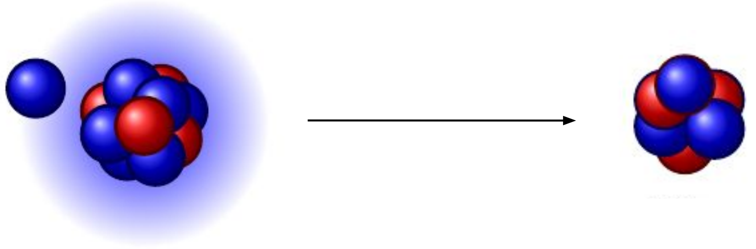
- Semiclassical interpretation : P follows a straight line trajectory at constant impact parameter b along which it accumulates a phase due to its interaction with T

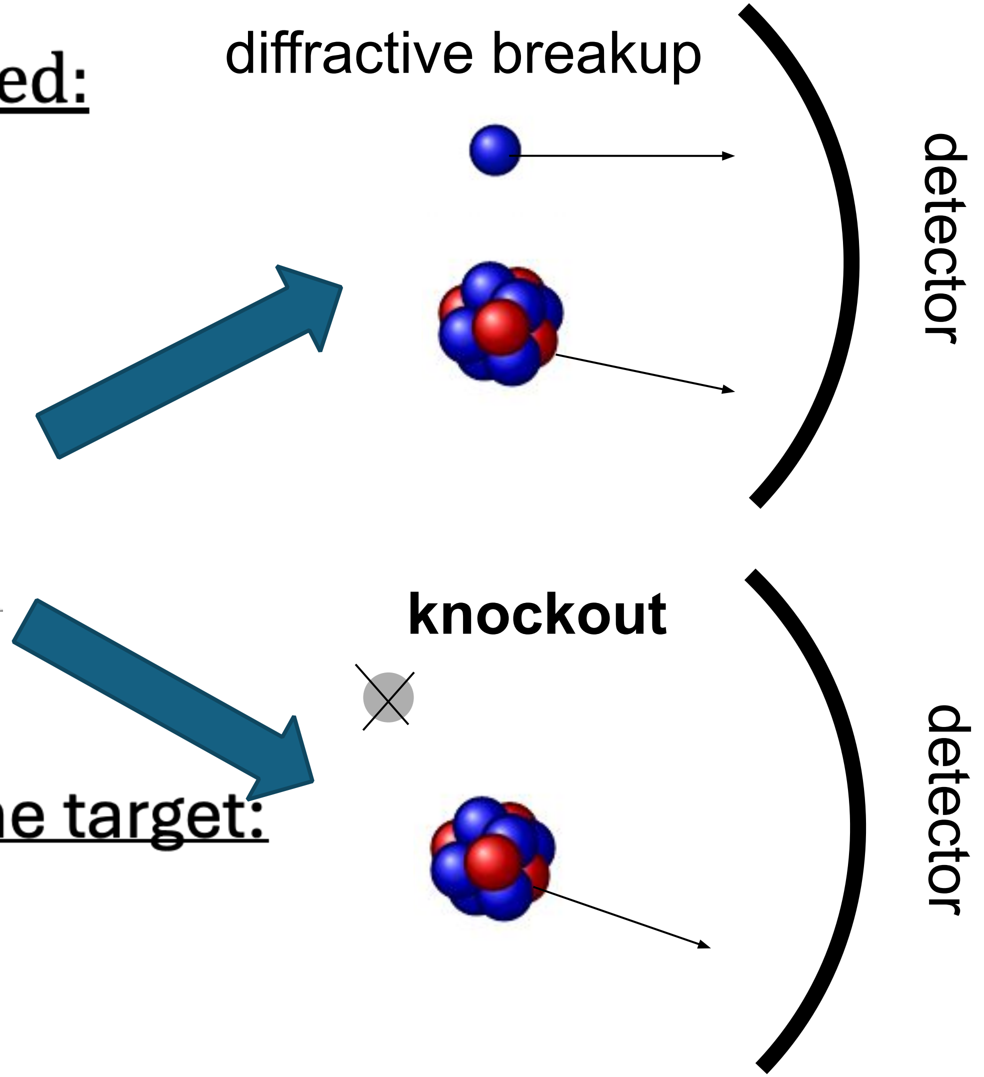
Breakup and knockout cross sections

<u>Diffractive breakup: neutron elastically scattered:</u>

$$\sigma_{diff} = 2\pi \int d\epsilon \int b\, db\, \left|\langle \phi_\epsilon | 1 - S_n S_c | \phi_{gs} \rangle\right|^2$$

with ϕ_ϵ c-n scattering state





<u>Stripping: neutron inelastically scatters with the target:</u>

$$\sigma_{str} = 2\pi \int b\, db\, \left|\langle \phi_{gs} | (1 - |S_n|^2)|S_c|^2 | \phi_{gs} \rangle\right|^2$$

<u>Knockout:</u>$\sigma_{ko} = \sigma_{str} + \sigma_{diff}$

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Few questions to answer

1. Understanding:

- Do you understand what the python notebook ComparisonEikonalExact.ipynb is doing?
- Can you interpret the eikonal S-matrix?
- What happens to the eikonal S-matrix if you take a nil potential? Is it what you expect?
- What happens to the S-matrix & the eikonal phase when $b \to \infty$? Is it what you expect?
- What happens to the S-matrix if the potential is real? Comment on the value of the absorption cross section.

1. Develop a cell to compare the integrated and angular cross sections obtained with different beam energy? (make sure that you results are converged)

1. Develop a cell to compare the integrated and angular cross sections obtained with different rR?

1. Develop a cell to compare the integrated and angular cross sections obtained with different imaginary depths?

Bonus :

1. Considering a knockout reaction, estimate from the S-matrix the knockout probability as a function of b. What can you deduce about the peripherality of knockout reactions

1. What happens if you change the range of the n-T potential?

1. What happens if you change the range of the c-T potential?

Some useful references about eikonal

- Suzuki, Y., Yabana, K., Lovas, R.G., & Varga, K. (2003). Structure and Reactions of Light Exotic Nuclei (1st ed.). CRC Press.

- P.G. Hansen and J. A. Tostevin, Annu. Rev. Nucl. Sci. 53, 219 (2003)

- D. Baye and P. Capel, in Clusters in Nuclei, Vol. 2, edited by C. Beck (Springer, Heidelberg, 2012), vol. 848.

Useful website for those who make reaction calculations

https://sites.google.com/view/opticalpotentials/