Fundamental Concepts of Statistics

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School Of Statistics

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General introduction

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Goals of the lecture

- recap the basics needed for the SOS
- learn how to be critical with statistics (in science, but not only)
- *focus* on meaning and (mis)intuition rather than mathematical rigour

Statistics versus probability (according to Persi Diaconis)

> *The problems considered by probability and statistics are inverse to each other. In probability theory we consider some underlying process which has some randomness [...] and we figure out what happens. In statistics we observe something that has happened, and try to figure out what underlying process would explain those observations.*

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Few personal tips for this lecture

- keywords/concepts will be listed at the end of each section
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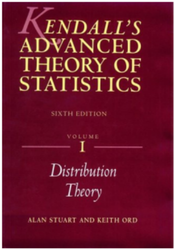
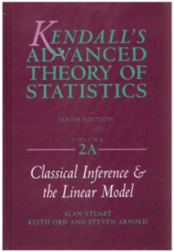
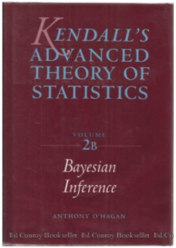
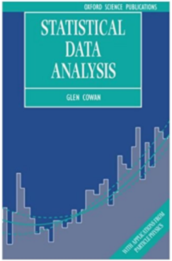
- keywords/concepts will be listed at the end of each section
  → make sure you know the ideas behind them!
- statistics is almost like a language: you need practice to learn it!
  → compute/code as much as simple examples as you can **by yourself!**

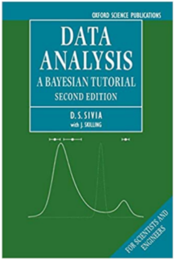
Some references

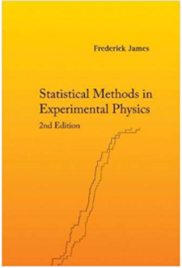


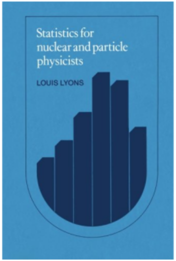


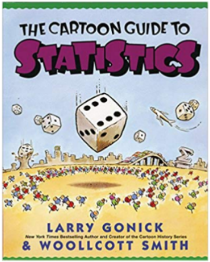












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What this lecture is not:

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Feel free to interrupt any time if you have questions !

Content

Statistics

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  - variance: $v_x = \overline{(x - \overline{x})^2}$; $\sigma_x \equiv \sqrt{v_x}$ - dispersion

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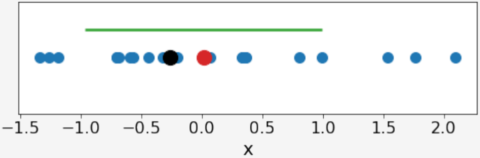
  - Kurtosis: $\beta_x = \overline{\left(\frac{x-\overline{x}}{\sigma_x}\right)^4}$ - importance of tails

Sample characterisation - illustrations



blue: x_i, red: mean. black: median, green: σ_x

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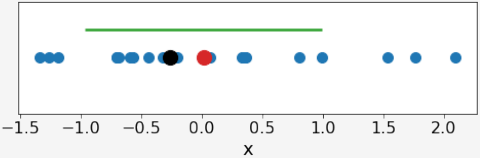

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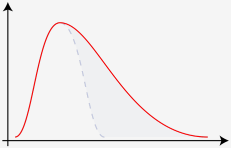
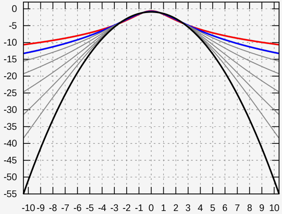
Skewness and Kurtosis (using probability functions)

Negative Skew

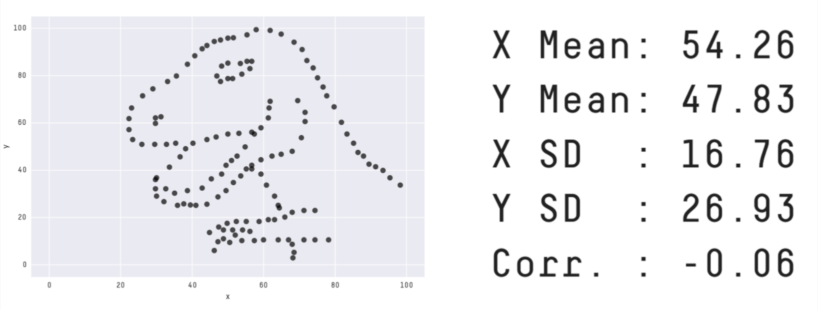


Positive Skew



Right plot: Kurtosis $\gamma = \infty$ (red), 2 (blue), $1, 1/2, 1/4, 1/8$, and $1/16$ (gray), 0 (black)

A single value is NOT the sample



Sample characterisation - comments

Notion of estimator (more on this later)

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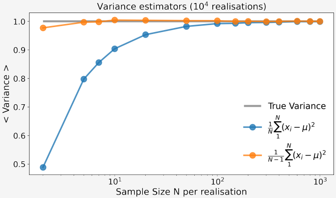

$\rightarrow$ sample variance v_x is a biased estimator of the true variance.

But $\frac{1}{n-\mathbf{1}}\sum(x_i-\overline{x})^2$ is unbiased.

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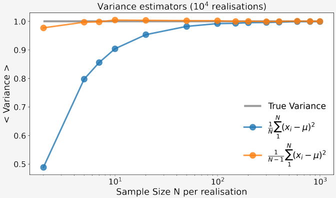

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Statistical moments (more on this later)

- Order-r moment: $m_r = \overline{\left(\frac{x-\overline{x}}{\sigma_x}\right)^r}$ (relates directly to the mean of x^r)
- probability theory: all truth moments $\equiv$ exact underlying probability
- first moments $\equiv$ "main" features of the sample

Multidimensional sample

- single observation i = several numbers: $x_i \rightarrow (x_i^{(1)}, x_i^{(2)}, ...x_i^{(p)})$
- *e.g.* biological dataset: person size, weight, age and genre

Previous description applies to each variable $x_i^{(j)}$ but one can now explore how variables behave wrt each other.

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Covariance and correlations between two variables a and b:

$$\text{cov}_{ab} \equiv \overline{(a - \overline{a})(b - \overline{b})} \quad ; \quad \rho_{ab} \equiv \frac{\text{cov}_{ab}}{\sigma_a \sigma_b}$$

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- probes if fluctuations around the mean are coherent for a and b
- covariance (and correlation) are symetric - fortunate
- covariance of x with itself is the variance
- $\rho_{a,b} \in [-1, 1]$; 0 = uncorrelated ($\neq$ indep!), (-)1 = (anti-)correlated

More on correlations

Covariance matrix or error matrix

- $C_{ij} = \rho_{ij} \times \sigma_i \sigma_j$ - real and symmetric.
- ρ_{ij} is the correlation matrix - symmetric with 1's on diagonal.

Why is this object so important?

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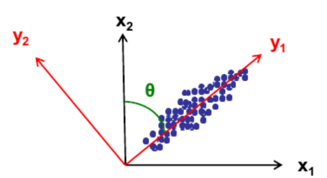

- x_1 and x_2 both have a large σ
- but, they are highly correlated
- most of the information is in y_2 (smallest σ)
  $\rightarrow$ idea of dimension reduction
  $\rightarrow$ idea of pre-processing in ML

Correlation and dependence

Correlation $\equiv$ *linear* dependence $\Rightarrow$ dependence

BUT

Non-correlation *dosen't* imply independence (matter of vocabulary)

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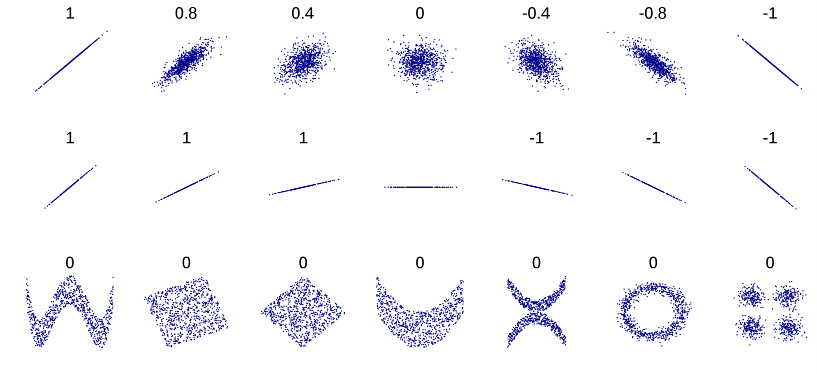

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(Never go to the hospital, people there die 10 times more than at home)

Check this website for funny examples.

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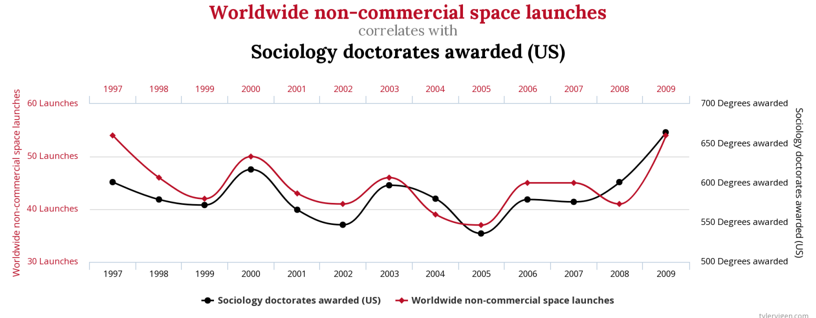

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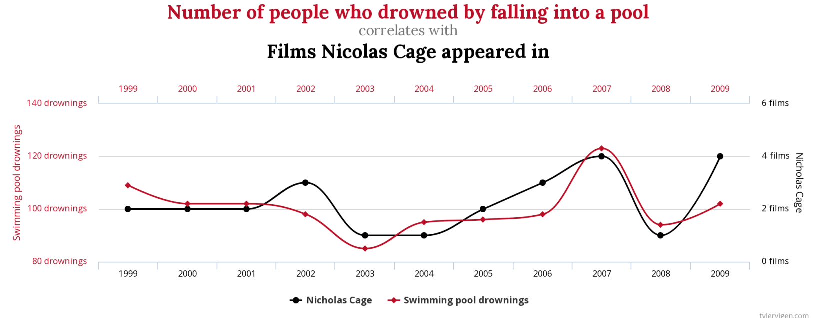

Keywords and concepts

Part I

descriptive statistics – sample – mean – (co)variance – (de)correlation

Probability

Some definitions

Caution: what follows is *not* mathematically rigorous

Random variable and associated probability

- a random variable X describes an observable which is not certain
- all possible outcomes - realisations - of X form a set Ω
- a probability P_i is associated to each realisation i of Ω
- $\{P_i\}$ must satisfy $P_i \in [0,1]$ and $\sum P_i = 1$

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Simple concrete example: a flippin coin

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$\rightarrow$ these notions can be defined and manipulated without any sample

Coming back to estimators - I

Previously: sample mean $\neq$ "true mean". What is the true mean?

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E.g. of the flipping coin

- $\mu = 1/2$, $\sigma = 1/2$,
- $m_r = 1$ if r is even, $m_r = 0$ if r is odd

Conditional probabilities and Bayes theorem

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- *evidence*: what we observed (*e.g.* measurement)

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Comments

- many ways to *understand* this fundamental equation
- in some case, each of these term has a clear meaning
- these two posts are quit interesting post 1 and post 2

Understanding Bayes theorem

Example: *hypothesis* = *fire* and *evidence* = *smoke*

$$P(fire|smoke) = P(fire) \times \frac{P(smoke|fire)}{P(smoke)}$$

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- $P(hypothesis|evidence)$: proba that there is a fire if there is smoke
  $\rightarrow$ difficult to assess (many sources of smoke), that's **the posterior**

Understanding Bayes theorem

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N.B.: $P(evidence)$ is independent from the hypothesis, and is sometime impossible to compute. It is often seen as a "normalization factor" and dropped while comparing different hypothesis.

Everyday life questions are often bayesian

Few examples:

- I'm not feeling so well $\rightarrow$ Am I sick ?
- There are clouds $\rightarrow$ will it rain?
- I attend to a school statistics $\rightarrow$ will I learn something?

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Always the same thinking:

1. you observe a fact
2. you wonder the probability of something, given that this fact happened
3. you have (sometimes rough/wrong) prior, based on past knowledge
4. your brain applies Bayes theorem, even you if don't know it!

Generalization to the continuous case

- There is a whole continuum of outcome (realization) for X
- Probability described by a density probability function (PDF), $f(x)$:

$$P(x \in [x_1, x_2]) = \int_{x_1}^{x_2} f(x)\mathrm{d}x \quad ; \quad \int_{\Omega} f(x)\mathrm{d}x = 1$$

Continous random variables

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- moments are the Taylor expension coefficients: $m_r = (-i)^r \left.\frac{\mathrm{d}^r \varphi_x}{\mathrm{d}t^r}\right|_{t=0}$

Important PDF examples

Binomial law: efficiency, trigger rates, …

$$B(k;n,p) = C_k^n p^k (1-p)^{n-k}, \mu = np, \sigma = \sqrt{np(1-p)}$$

Poisson distribution: counting experiments, hypothesis testing

$$P(n;\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}, \mu = \lambda, \sigma = \sqrt{\lambda}$$

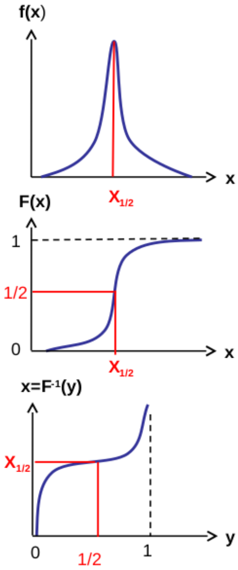
Gauss distribution (aka Normal): many use-case (asymptotic convergence)

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Cauchy distribution (aka Breit-Wigner): particle decay width, ...

$$f(x;x_0,\gamma) = \frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$$

μ and σ not defined (divergent integral)



Probability density function: $f(x)$

Cumulative distribution: $F(x)=y$

Inverse cumulative distribution: $x=F^{-1}(y)$

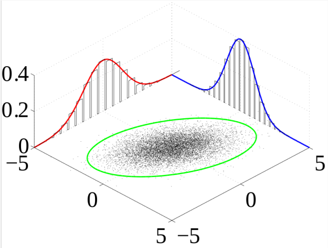
Median: x such that $F(x)=1/2 \rightarrow x_{1/2} = F^{-1}(1/2)$

Quantile of order α: $x_\alpha = F^{-1}(\alpha)$

Multidimensional PDF

How to describe several random variables simulataneously?

- X and Y are two random variables $\rightarrow$ PDF is f_{XY},
- several questions can be asked about X, Y or both.



- Probability that $X \in [x, x + \mathrm{d}x]$ and $Y \in [y + \mathrm{d}y]$:
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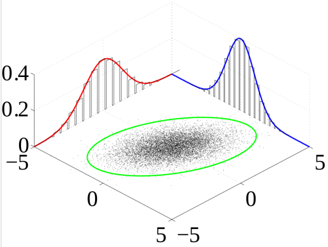

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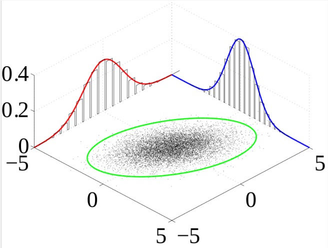

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Multidimensional normal distribution

$$f(\vec{x};\vec{\mu},\Sigma) \;=\; \frac{1}{\sqrt{(2\pi)^n \det\Sigma}}\,\exp\left(-\frac{1}{2}\,(\vec{x}-\vec{\mu})^T\,\Sigma^{-1}\,(\vec{x}-\vec{\mu})\right)$$

- $\vec{\mu}$ mean position of $\vec{x}$, Σ covariance matrix

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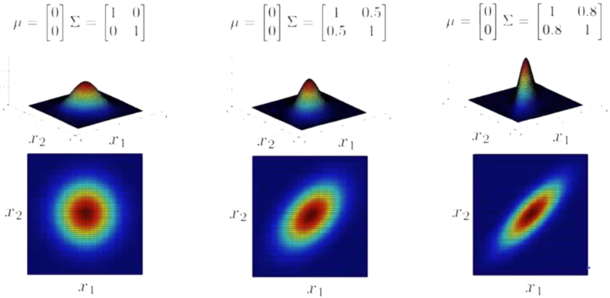

Central limit theorem

Caution: what follows is *not* mathematically rigorous

If n random variables $\{X_i\}$ are distributed according to the same PDF f_X with a defined mean μ_x and a std σ_x, **then** the random variable $Y = \frac{1}{n}(X_1 + ... + X_n)$ is following a normal distribution of mean μ_x and std $\sigma_x/\sqrt{n}$.

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For 2 variables $Y = X_1 + X_2$

- The PDF of Y is $f_Y(y) = \int f_{X_1}(x_1) \times f_{X_2}(y - x_1)\mathrm{d}x_1 \rightarrow$ convolution!
- Characteristic function: $\varphi_Y(t) = \varphi_{X_1}(t) \times \varphi_{X_2}(t) = \varphi_X(t)^2$ - same PDF!
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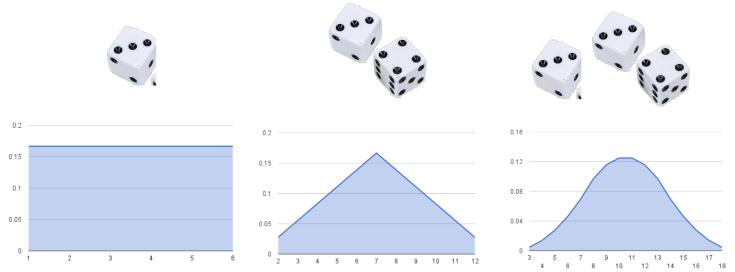
Generalizing for sum of n variables:

- $\varphi_Y(t) = \varphi_x(t)^n \sim \left(1 - \frac{t^2}{n}\right)^n \rightarrow e^{-\frac{1}{2}t^2}$ for $n \rightarrow \infty$
- going back to real space, a normal distribution is obtained

N.B. this reasonning doesn't explain why $\sigma_Y = \sigma_x/\sqrt{n}$, this needs to properly re-scale Y.

One way to understand why it works



Central limit theorem – continued

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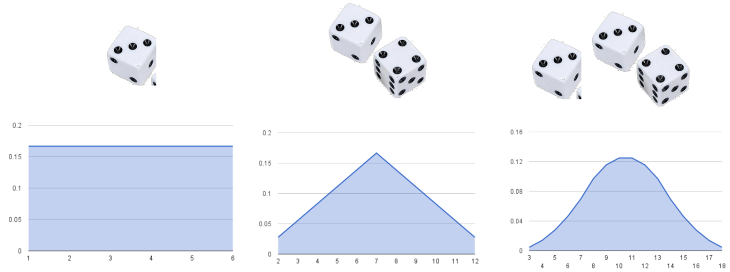

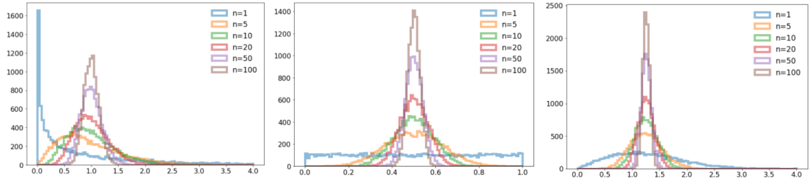

Central limit theorem - homework

Proof

Proove that $\sigma_Y = \sigma_X/\sqrt{n}$ with the proper scalings to define Y.

Application

Proove, using the CLT, that a Poisson distribution $P(n;\lambda)$ tends to a normal distribution for large numbers.

Hint: $N = 1 + 1 + 1.... + 1$ N-times

Function of random variables

Final observable is very often a combination of (random) variable.

- $\mathcal{O} = g(X_1, X_2, ..., X_n) \equiv g(\vec{X})$. $\mathcal{O}$ is also a random variable
- what is the PDF of $\mathcal{O}$, knowing $f_{\vec{X}}$? *Not trival* (think about a sum)!

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Comments:

- these equations are known as error propagation
- **this procedure is not exact** and relies on Taylor expansion
- only 1st and 2nd moments of $\vec{X}$ are needed (or their estimators)

Error propagation formula is not exact

(Counter) example with one variable

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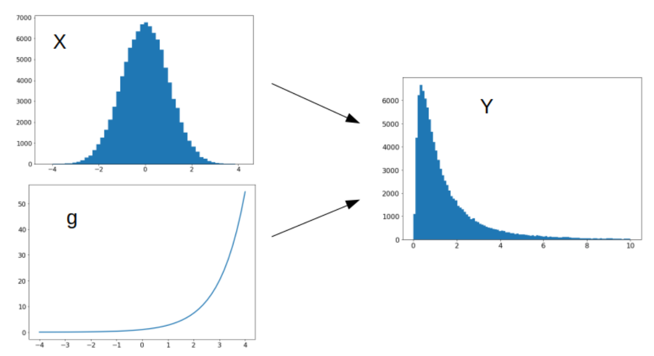


Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

Part II: probability

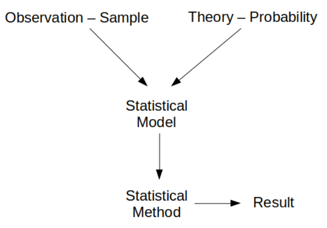
Bayes theorem – prior – posterior – random variable – (marginal) PDF – moments – characteristic function – (in)dependent variables – CLT – error propagation

Content

1. Statistics
2. Probability
3. Statistical model
4. The two big schools
5. Parameter estimation and hypothesis testing

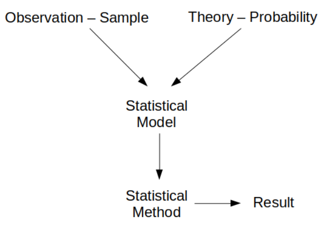
Statistical model

Statistical model: what, why, how



What? missing piece between the "sample" and "probablity"

Statistical model: what, why, how



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Why? because a measurement is always **one** realization of a random variable.

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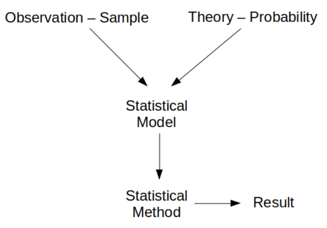

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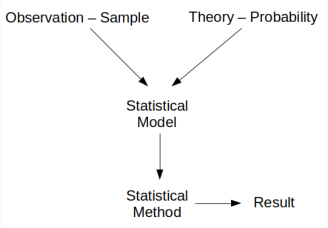

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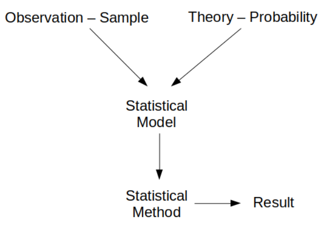

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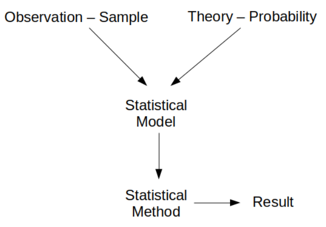

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A statistical model is also called likelihood function $\mathcal{L}(\vec{\mu}, \vec{\theta}; \vec{x})$. It can be seen as the probability that the physical model predicts the observable $\vec{x}$, given the parameters $(\vec{\mu}, \vec{\theta})$.

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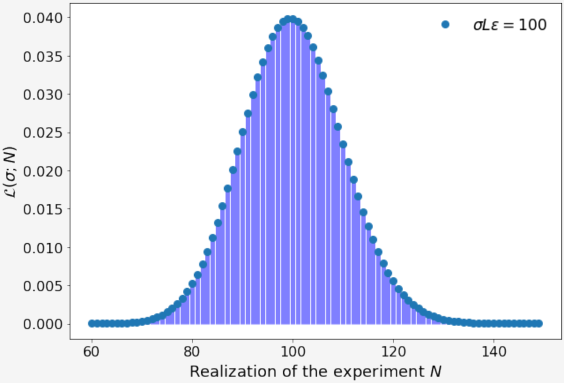
Statistical model

$$\mathcal{L}(\sigma; N) = e^{-\sigma L \epsilon} \frac{(\sigma L \epsilon)^N}{N!}$$

Illustration of the Likelihood

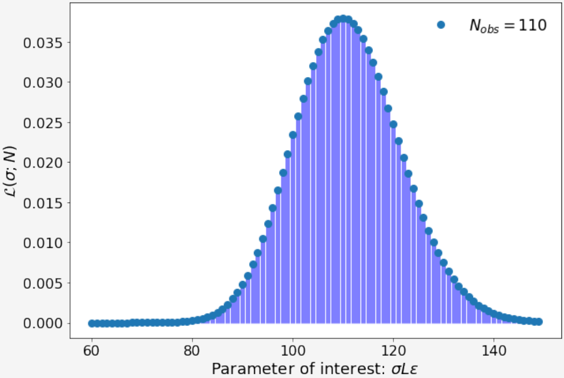
Given a value of σ, what's the "probability" to observe N ?



Anticipation: frequentist "usage" of the likelihood

Illustration of the Likelihood

If we observed a value for N, what's the "probability" that $\sigma = X$?



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Systematic uncertainties turn numbers into new random variables. Their PDFs depend on parameters, we don't really care about: nuisances parameters. *Example* of systematic parametrization:

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In realistic experiment:

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- Many processes are usually needed to describe data
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- Does the order of bins in the histogram matters for the result?
- Why do we multiply terms?

A first discussion on uncertainties

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Systematic uncertainty estimation *and* treatment is not an exact science.

(While statistics deals with the non-certain, systematic uncertainties says we don't exactly know the PDF quantifying the non-certain)

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Implications:

- arbitrariness (and a loooot of discussion that go with it)
- always check the robustness of the conclusion wrt to those
- that's the way it is, no choice! → be *smartly* practical!

Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

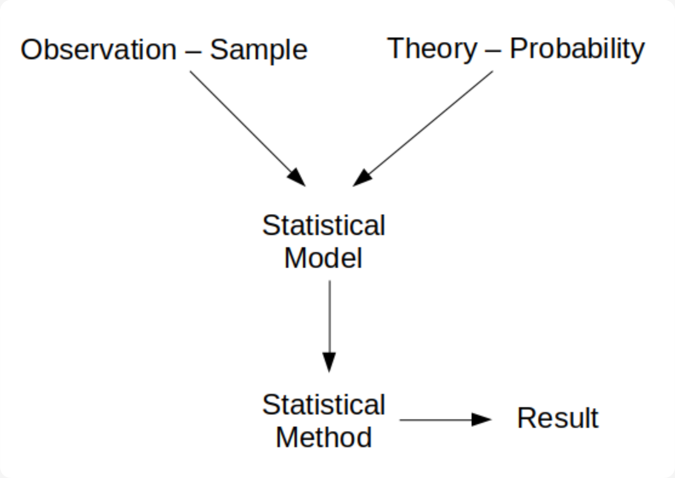
Part II: probability

Bayes theorem – prior – posterior – random variable – (marginal) PDF – moments – characteristic function – (in)dependent variables – CLT – error propagation

Part III: statistical model

Likelihood – nuisance parameter – parameter of interest – systematic uncertainties

Overview


Observation – Sample
Theory – Probability
Statistical
Model
Statistical
Method
Result

Content

The two big schools

Fequentist versus bayesian

	Frequentist	**Bayesian**
probability	frequency of occurence	degree of belief

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- *bayesian:* exploits the Bayes theorem to compute the posterior $P(para|obs)$, using the prior $P(para)$ and $P(obs|para)$ - the **likelihood**

Is a flipping coin tricked?

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We toss a coin 113 times and we got 'tail' 68 times. Is the coin tricked?

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Statistical Model assuming $N = 113$ is large enough to apply CLT

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- N (known parameter): number of tosses
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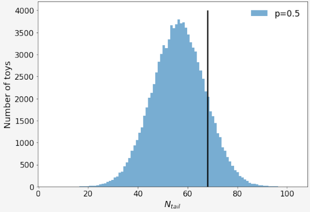
Let's try to analyze this same experiment with both frequentist and bayesian approaches

Is a flipping coin tricked? Frequentist approach



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14.1% of pseudo-experiments using an normal coin would lead to $N_{tail} \geq 68$

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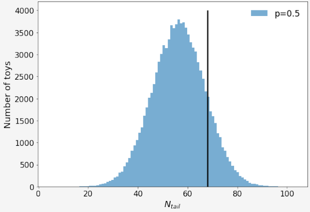

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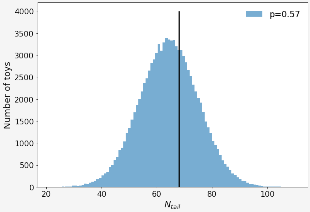

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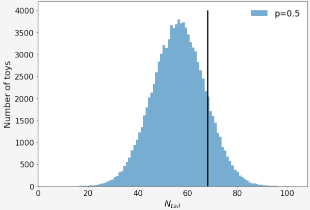

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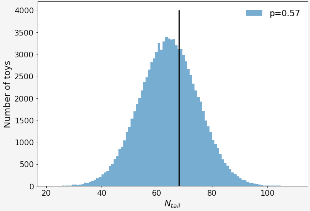

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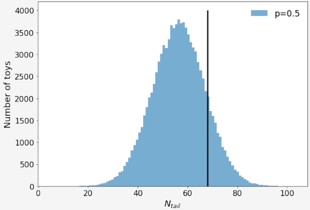

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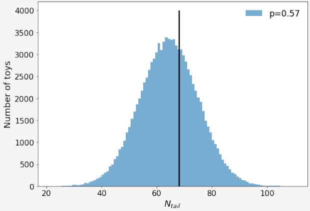

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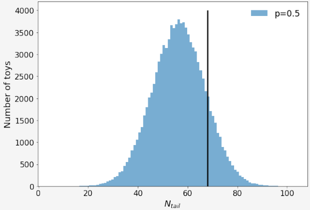

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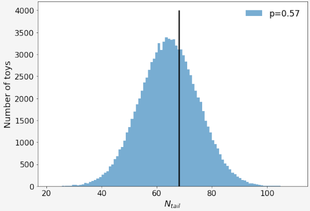

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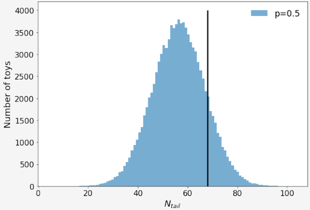

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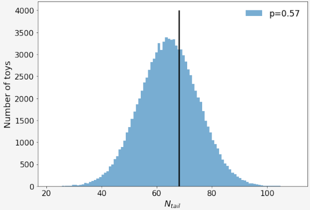

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  $\rightarrow$ this question has no sense in frequentist

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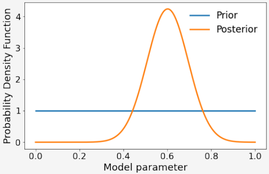

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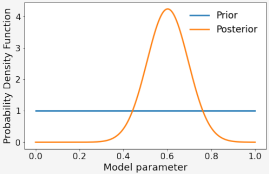

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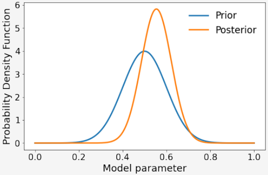

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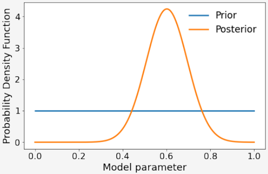

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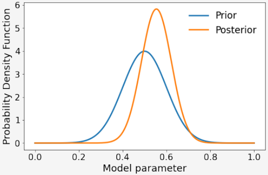

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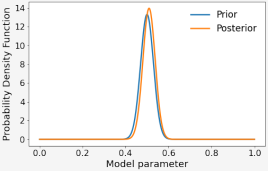

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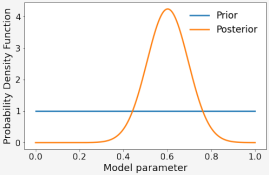

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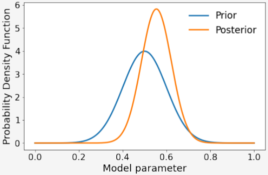

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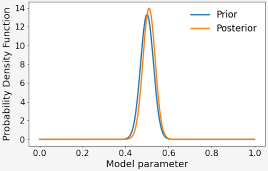

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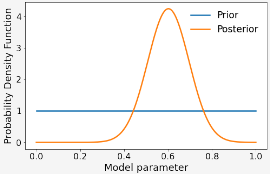

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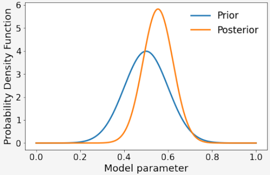

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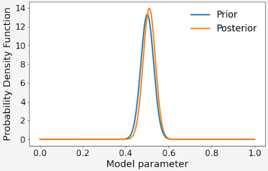
Narrow centered prior



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In the end, is the coin tricked?

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Is a flipping coin tricked? Bayesian approach

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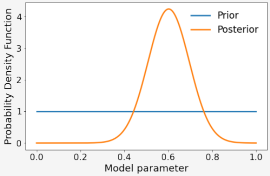
Flat prior



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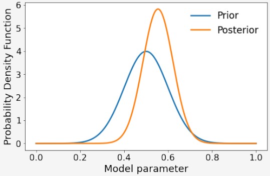
Wide centered prior



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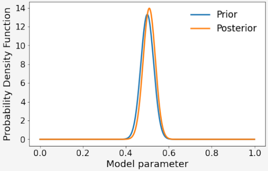

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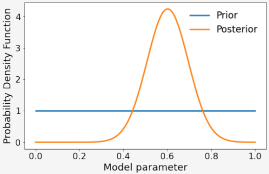

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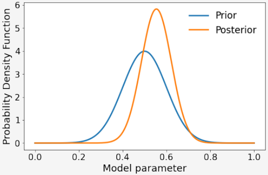

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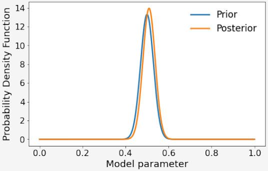

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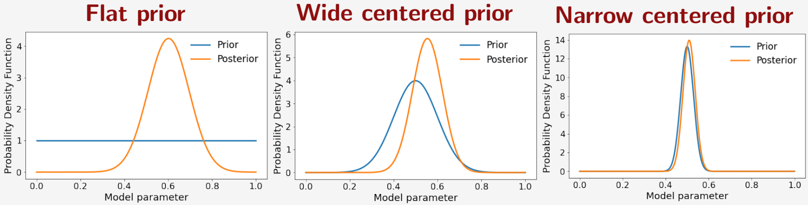

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  $\rightarrow$ expect it depends on the choice of the prior ...

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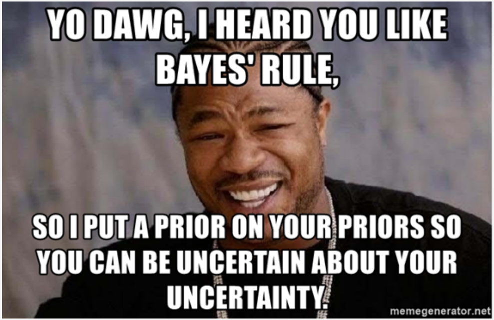

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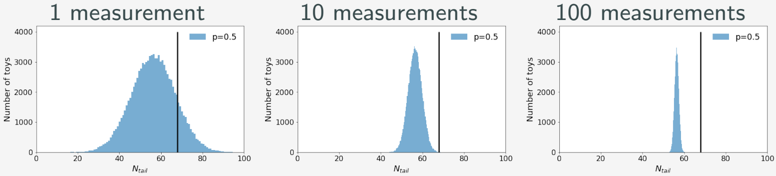

Frequentists say "Yes, the coin is tricked!"

Certainty comes from the extremely low fraction of pseudo-experiments of a normal coin, that would lead the observed result.

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Handling many measurements in Bayesian

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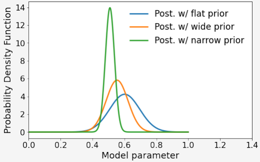
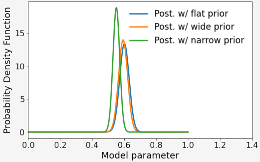
Bayesian, posterior for various priors

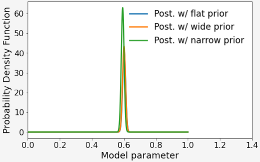
$N_{meas} = 1$



$N_{meas} = 10$



$N_{meas} = 100$



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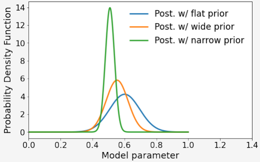

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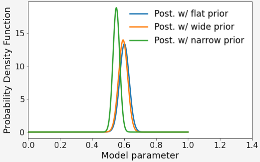

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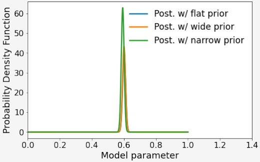

Bayesians also say "Yes, the coin is tricked!"

Frequentist v.s. Bayesian: what to take away

1. Both approaches handle differently the "non fully certain"

2. Final conlusions should be compatible, even if the question they address are not exaclty the same.

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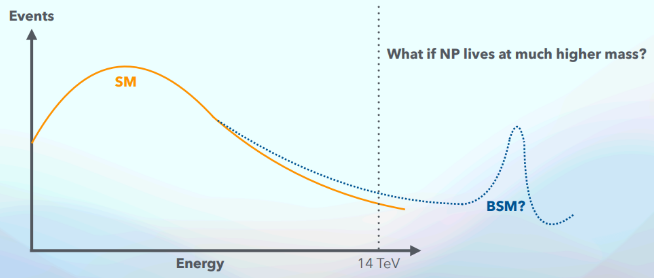
One thing I like from each approach

- probability intepretation from the frequentist
- ranking two theories using their probability, called Bayes factors

Aparte: A Recent Example of Bayes Factors

Courtesy of Adrien Auriol



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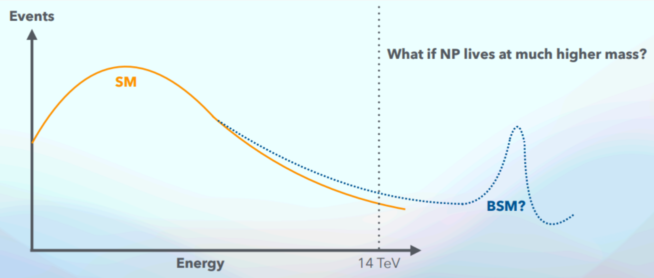
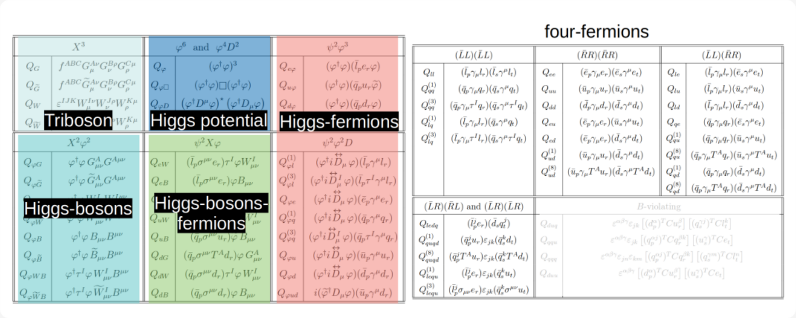


Effective Fields Theory (EFT):

- Systematic way to extend the SM (fixing the field content, symmetries, and dimension)
- Conceptually:
  → (almost) UV-independant parametrization of new phenomena
- Technically:
  → a set of coefficients (strength of each new term in the lagrangian)
  → to be fitted using many measurements

Aparte: A Recent Example of Bayes Factors

“Dimension-Six Terms in the Standard Model Lagrangian” – arXiv:1008.4884



X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar l_p e_r \varphi)$
$Q_{\widetilde G}$	$f^{ABC} \widetilde G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p u_r \widetilde\varphi)$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	$Q_{\varphi D}$	$(\varphi^\dagger D^\mu\varphi)^\star (\varphi^\dagger D_\mu\varphi)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p d_r \varphi)$
$Q_{\widetilde W}$	[illegible] $W_\rho^{K\mu}$				

$X^2\varphi^2$		$\psi^2 X\varphi$		$\psi^2\varphi^2 D$	
$Q_{\varphi G}$	$\varphi^\dagger\varphi\, G^A_{\mu\nu} G^{A\mu\nu}$	Q_{eW}	$(\bar l_p \sigma^{\mu\nu} e_r)\tau^I \varphi W^I_{\mu\nu}$	$Q_{\varphi l}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar l_p \gamma^\mu l_r)$
$Q_{\varphi\widetilde G}$	$\varphi^\dagger\varphi\, \widetilde G^A_{\mu\nu} G^{A\mu\nu}$	Q_{eB}	$(\bar l_p \sigma^{\mu\nu} e_r)\varphi B_{\mu\nu}$	$Q_{\varphi l}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}^I_\mu \varphi)(\bar l_p \tau^I \gamma^\mu l_r)$
[illegible]	[illegible]	[illegible]	[illegible]	$Q_{\varphi e}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar e_p \gamma^\mu e_r)$
$Q_{\varphi W}$	[illegible]	Q_{uW}	[illegible] $W^I_{\mu\nu}$	$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar q_p \gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger\varphi\, B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\bar q_p \sigma^{\mu\nu} u_r)\widetilde\varphi B_{\mu\nu}$	$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i\overleftrightarrow{D}^I_\mu \varphi)(\bar q_p \tau^I \gamma^\mu q_r)$
$Q_{\varphi\widetilde B}$	$\varphi^\dagger\varphi\, \widetilde B_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\bar q_p \sigma^{\mu\nu} T^A d_r)\varphi G^A_{\mu\nu}$	$Q_{\varphi u}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar u_p \gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger\tau^I\varphi\, W^I_{\mu\nu} B^{\mu\nu}$	Q_{dW}	$(\bar q_p \sigma^{\mu\nu} d_r)\tau^I \varphi W^I_{\mu\nu}$	$Q_{\varphi d}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu \varphi)(\bar d_p \gamma^\mu d_r)$
$Q_{\varphi\widetilde WB}$	$\varphi^\dagger\tau^I\varphi\, \widetilde W^I_{\mu\nu} B^{\mu\nu}$	Q_{dB}	$(\bar q_p \sigma^{\mu\nu} d_r)\varphi B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\widetilde\varphi^\dagger D_\mu\varphi)(\bar u_p \gamma^\mu d_r)$

four-fermions

$(\bar LL)(\bar LL)$		$(\bar RR)(\bar RR)$		$(\bar LL)(\bar RR)$	
Q_{ll}	$(\bar l_p\gamma_\mu l_r)(\bar l_s\gamma^\mu l_t)$	Q_{ee}	$(\bar e_p\gamma_\mu e_r)(\bar e_s\gamma^\mu e_t)$	Q_{le}	$(\bar l_p\gamma_\mu l_r)(\bar e_s\gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\bar q_p\gamma_\mu q_r)(\bar q_s\gamma^\mu q_t)$	Q_{uu}	$(\bar u_p\gamma_\mu u_r)(\bar u_s\gamma^\mu u_t)$	Q_{lu}	$(\bar l_p\gamma_\mu l_r)(\bar u_s\gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\bar q_p\gamma_\mu\tau^I q_r)(\bar q_s\gamma^\mu\tau^I q_t)$	Q_{dd}	$(\bar d_p\gamma_\mu d_r)(\bar d_s\gamma^\mu d_t)$	Q_{ld}	$(\bar l_p\gamma_\mu l_r)(\bar d_s\gamma^\mu d_t)$
$Q_{lq}^{(1)}$	$(\bar l_p\gamma_\mu l_r)(\bar q_s\gamma^\mu q_t)$	Q_{eu}	$(\bar e_p\gamma_\mu e_r)(\bar u_s\gamma^\mu u_t)$	Q_{qe}	$(\bar q_p\gamma_\mu q_r)(\bar e_s\gamma^\mu e_t)$
$Q_{lq}^{(3)}$	$(\bar l_p\gamma_\mu\tau^I l_r)(\bar q_s\gamma^\mu\tau^I q_t)$	Q_{ed}	$(\bar e_p\gamma_\mu e_r)(\bar d_s\gamma^\mu d_t)$	$Q_{qu}^{(1)}$	$(\bar q_p\gamma_\mu q_r)(\bar u_s\gamma^\mu u_t)$
		$Q_{ud}^{(1)}$	$(\bar u_p\gamma_\mu u_r)(\bar d_s\gamma^\mu d_t)$	$Q_{qu}^{(8)}$	$(\bar q_p\gamma_\mu T^A q_r)(\bar u_s\gamma^\mu T^A u_t)$
		$Q_{ud}^{(8)}$	$(\bar u_p\gamma_\mu T^A u_r)(\bar d_s\gamma^\mu T^A d_t)$	$Q_{qd}^{(1)}$	$(\bar q_p\gamma_\mu q_r)(\bar d_s\gamma^\mu d_t)$
				$Q_{qd}^{(8)}$	$(\bar q_p\gamma_\mu T^A q_r)(\bar d_s\gamma^\mu T^A d_t)$

$(\bar LR)(\bar RL)$ and $(\bar LR)(\bar LR)$		B-violating	
Q_{ledq}	$(\bar l_p^j e_r)(\bar d_s q_t^j)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$
$Q_{quqd}^{(1)}$	$(\bar q_p^j u_r)\varepsilon_{jk}(\bar q_s^k d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{quqd}^{(8)}$	$(\bar q_p^j T^A u_r)\varepsilon_{jk}(\bar q_s^k T^A d_t)$	Q_{qqq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(1)}$	$(\bar l_p^j e_r)\varepsilon_{jk}(\bar q_s^k u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{lequ}^{(3)}$	$(\bar l_p^j \sigma_{\mu\nu} e_r)\varepsilon_{jk}(\bar q_s^k \sigma^{\mu\nu} u_t)$		

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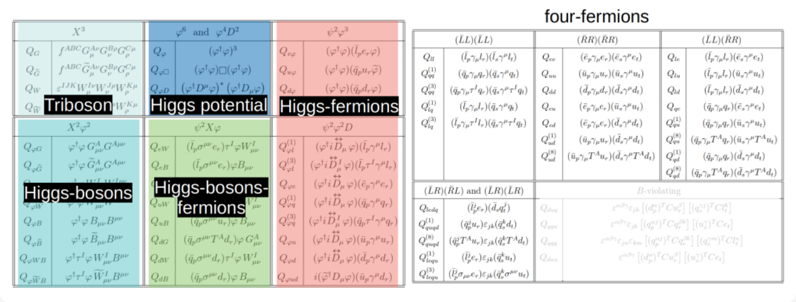
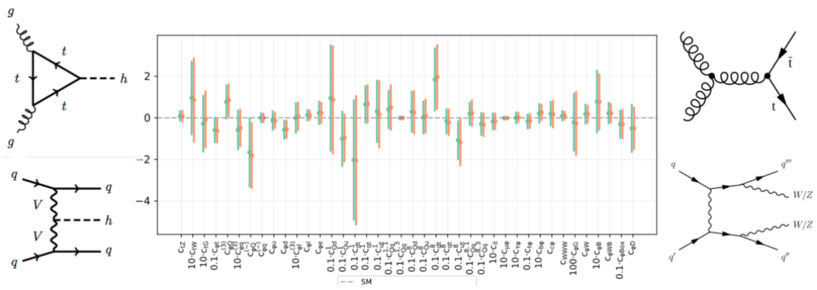


For comparison, QED in his full glory is written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e j^\mu A_\mu$$

Example of global fit:
Courtesy of Lucas Mantani



Aparte: A Recent Example of Bayes Factors

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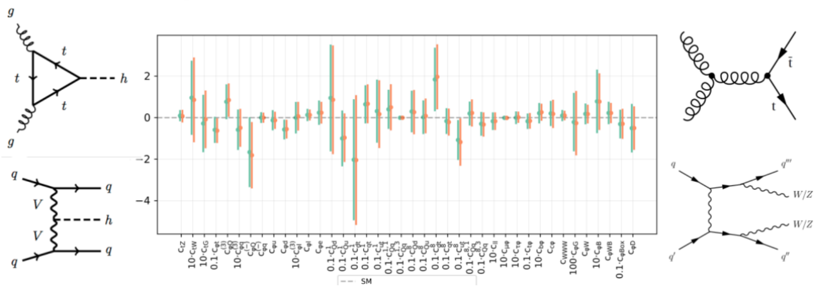

The dream: fitting many measurements and see non-zero coefficient where new interactions are present.

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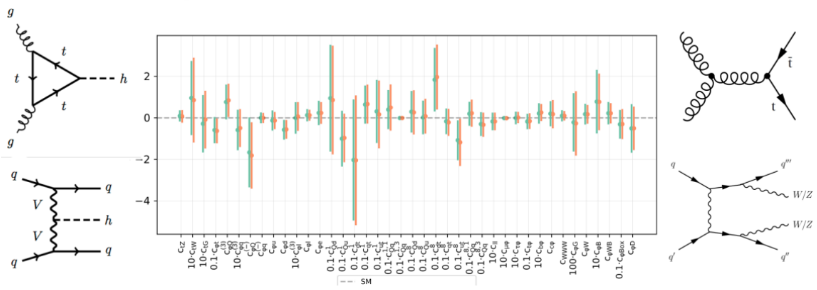

The dream: fitting many measurements and see non-zero coefficient where new interactions are present.

The reality: too many degrees of freedom to fit fluctuations, and even if new physics is here, all coefficients might remain null. But restricting to a subset of coefficients, *i.e.* an alternative theory, then new physics can be measured. arXiv:2507.11109, arXiv:2605.21594

Aparte: A Recent Example of Bayes Factors

How to formally define one theory among all possible ones ?

Aparte: A Recent Example of Bayes Factors

How to formally define one theory among all possible ones ?

Define a theory as a vector in a theory space, one can define/compute posterior in that space.

We specify models with binary vectors

$\vec{b} = (b_1, \ldots, b_d), \qquad b_i \in \{0, 1\}$

$\vec{c}_{\vec{b}} = \{c_i \mid b_i = 1\}$ Parameters of each model

$\vec{b} = \vec{0}$ SM

$\vec{b} = \vec{1}$ Traditional global fit

$$p(\vec{b} \mid D) = \frac{p(D \mid \vec{b})\, p(\vec{b})}{\sum_{\vec{b}'} p(D \mid \vec{b}')\, p(\vec{b}')}$$

Fundamental target:
posterior distribution in model space

$$p(D \mid \vec{b}) = \int d\vec{c}_{\vec{b}}\, p(D \mid \vec{c}_{\vec{b}}, \vec{b})\, p(\vec{c}_{\vec{b}} \mid \vec{b})$$

Courtesy of Lucas Mantani

Aparte: A Recent Example of Bayes Factors

Extraction of the probability that a given (single) new interaction is realized in data.

Aparte: A Recent Example of Bayes Factors

Extraction of the probability that a given (single) new interaction is realized in data.

$$p_{\mathcal{O}_i} = \sum_{\{b_j\}_{j \neq i} \in \{0,1\}^{N-1}} p(b_1, b_2, \ldots, b_i = 1, \ldots, b_N)$$

Fix the one we are interested in

Joint probability distribution of models

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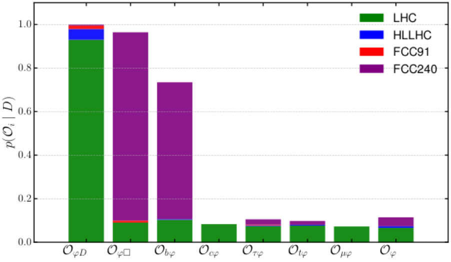


Good evidence for $c_{\varphi D}$ already at LHC+LEP

The tera-Z program nails $c_{\varphi D}$ for good

ZH production gives us evidence for two more WCs

Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

Part II: probability

Bayes theorem – prior – posterior – random variable – (marginal) PDF – moments – characteristic function – (in)dependent variables – CLT – error propagation

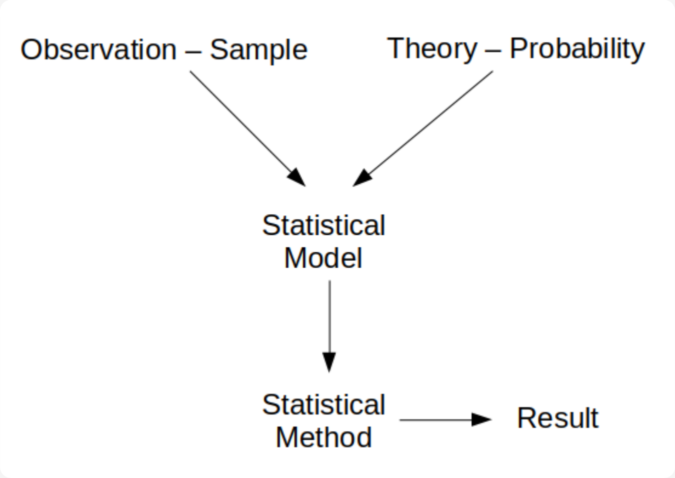
Part III: statistical model

Likelihood – nuisance parameter – parameter of interest – systematic uncertainties

Part IV: The two big school

Frequentist – occurence frequency – pseudo-data (toys) – bayesian – degree of belief

Overview


Observation – Sample
Theory – Probability
Statistical Model
Statistical Method
Result

Content

Parameter estimation, hypothesis testing

Program of this section

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3. **Coming back on nuisance parameters** (*i.e.* uncertainties on the model)

Frequentist approach: estimators

Definition: random variable which gives a 'good' estimate of your parameter of interest ($\hat{\mu} = \frac{1}{N}\sum_i x_i$ as estimator of $\mathbb{E}[X]$). Estimator depends on observation $\hat{\mu}(x_1, ..., x_n)$ and is *not* constant. N_{meas} needed to assess its quality.

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Two important examples of estimators

1. Maximum likelihood estimator (MLE): $\hat{\mu}$ which maximizes $\mathcal{L}(\mu; x)$
   $\to$ numerically easier to minimze $-2\ln\mathcal{L}(\mu; x)$ - negative log likelihood (NLL)
2. χ^2 estimator: $\hat{\mu}$ which minimizes $\chi^2(\mu) \equiv \sum_i w_i \left(X_i^{pred}(\mu) - x_i\right)^2$

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- consistency $+$ finite variance $\rightarrow$ no bias
- consistency $\rightarrow$ no bias $+$ vanishing variance

More information on this post.

Example: linear fit

Model $N^{pred}(p_0, p_1; t) = p_0 + p_1 t$

4 estimators (or "cost function") are used:

$$-2 \log \mathcal{L}_{poisson}$$

$$\chi^2(p_0, p_1) = \sum_i (N_i^{pred}(p_0, p_1) - N_i)^2$$

$$\chi^2_{Pearson}(p_0, p_1) = \sum_i \left(\frac{N_i^{pred}(p_0, p_1) - N_i}{\sqrt{N_i^{pred}(p_0, p_1)}} \right)^2$$

$$\chi^2_{Neyman}(p_0, p_1) = \sum_i \left(\frac{(N_i^{pred}(p_0, p_1) - N_i)^2}{\sqrt{N_i}} \right)^2$$

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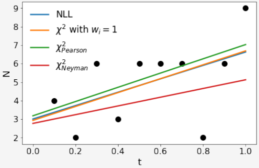

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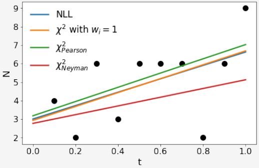

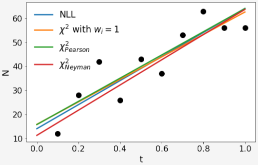

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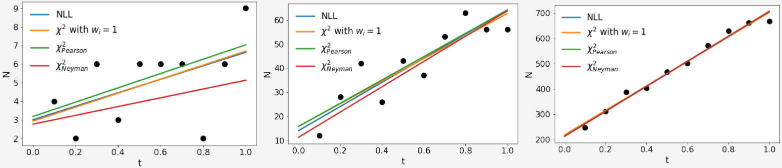

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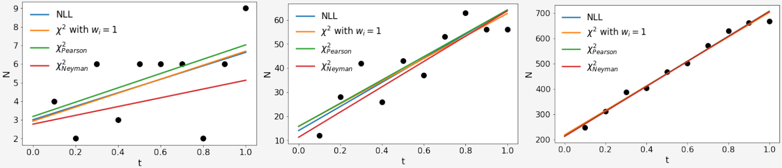

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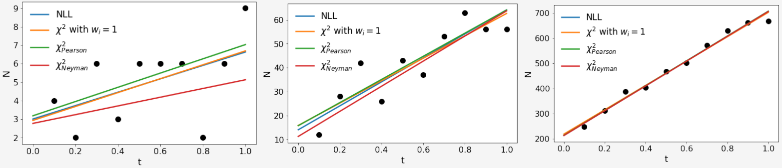

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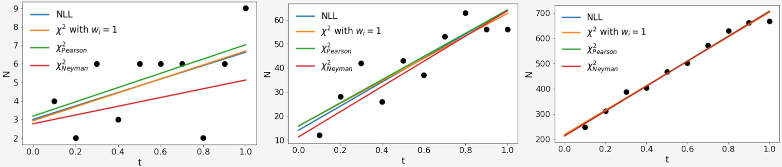

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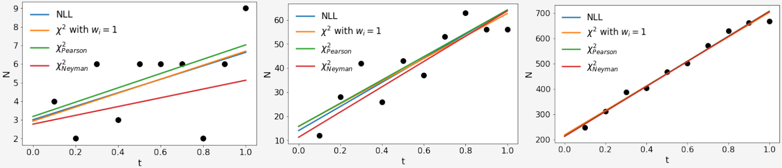

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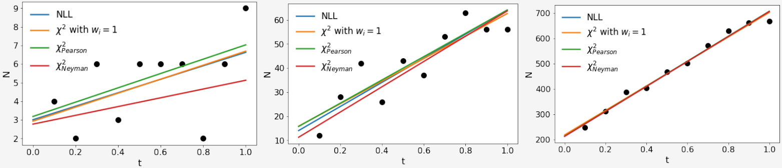

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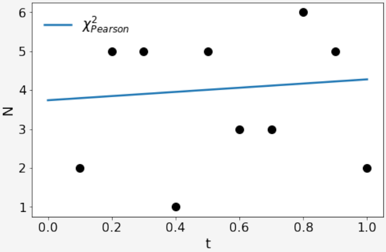
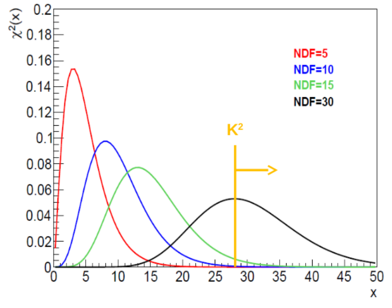
  $\rightarrow$ goodness-of-fit is possible to evaluate since χ^2 PDF is known

The basics of goodness-of-fit





$\chi^2_{min} = 6.7$ with 10 data points ($nDoF = 10$) $\rightarrow$ blue PDF tells us this is a good fit, *even if* not a point is on the line.

We can actually compute the fraction of pseudo-data that would lead to a higher χ^2 (p-value), to quantify this statement.

Food for thought

1. Perform a fit of an histogram in ROOT, with quite wide binning. Do you recover the true value? Does the result depends on the number of bins? How to solve it?

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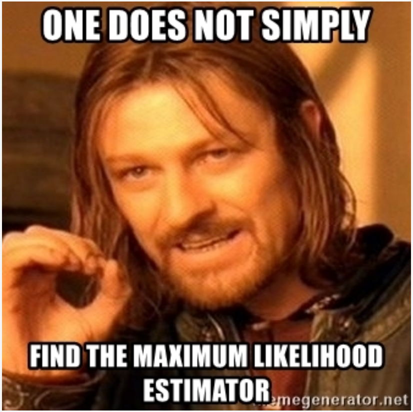
2. Imagine you have one dataset, but you want to fit simultaneously two distributions of these events. How to write the χ^2?

Frequentist parameter uncertainty



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Confidence interval and level $\mu \in [\mu_{min}, \mu_{max}]$ @ α CL

- $\equiv$ the true value is in $[\mu_{min}, \mu_{max}]$ in α% of all possible realisations
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n is called "number of σ" and $\alpha(n)$ is known for a normal PDF:

- $\alpha(1) = 68\%$
- $\alpha(1.64) = 90\%$
- $\alpha(1.95) = 95\%$
- $\alpha(2) = 95.4\%$
- $\alpha(3) = 99.7\%$
- $\alpha(5) = 99.99994\%$

Frequentist parameter uncertainty

Quality of a given confidence interval (CI)

- CI $\equiv$ random variable: consider the limit of ∞ number of meas.
- Coverage $\equiv$ probability P that the true parameter *actually is* in C
- "Confidence level = what we target" while "coverage = what we get"

The 3 cases

1. $P = \alpha$: perfect coverage $\rightarrow$ ideal
2. $P > \alpha$: over-coverage $\rightarrow$ acceptable (conservative conclusions)
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In practice: estimating coverage can be done using toys experiment (CPU-intensive for realistic models).

Example: binomial distribution, with parameter of interest p

$$P(k;N,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\hat{p} = \frac{k}{N}$$

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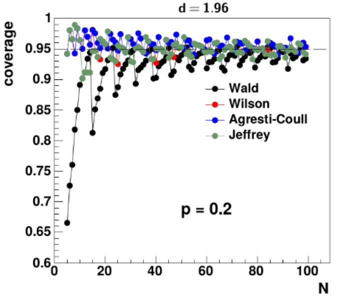

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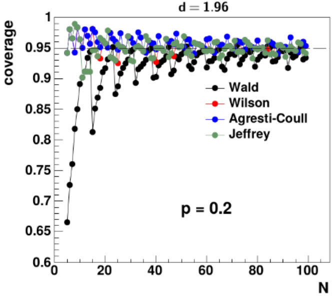

Take away messages:

- notation $\mu = X^{+Y}_{-Z}$ (assuming 68% C.L.) is sometimes only indicative
- only object which contains the full information is the likelihood
- OK to use these approximate quantities - just know what they are(n't)

Bayesian parameter estimation

From the posterior to the final value: given $f(\mu) \equiv P(\mu|data)$

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  - Replace $\mathbb{E}[\mu]$ by the mode, or the median ...

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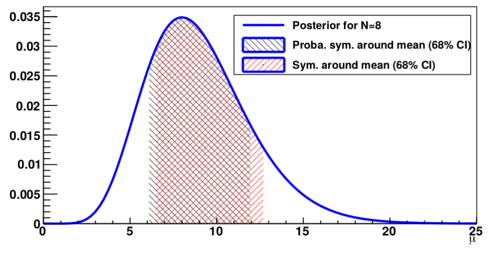

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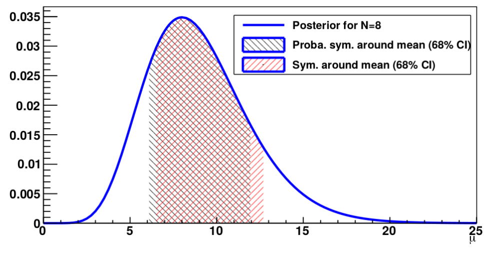

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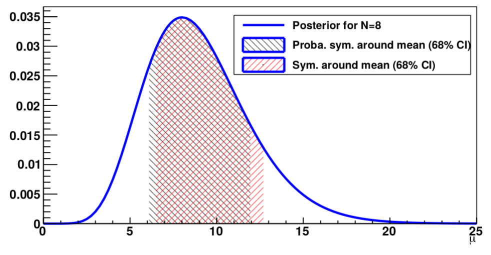

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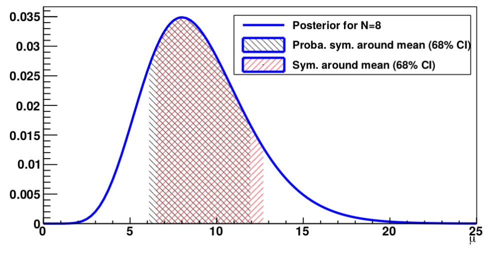

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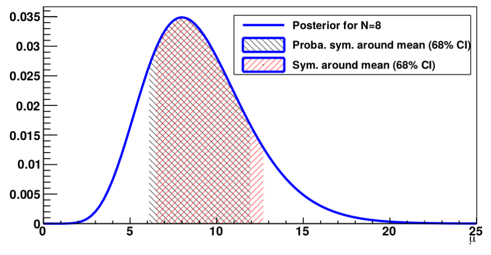
- questions: (1) why there is no coverage in bayesian?

Bayesian parameter estimation



Take away messages:

- as in frequentist, the notation $\mu = X^{+Y}_{-Z}$ is sometimes only indicative
- the only object which contains the full information is the posterior

Few reminders

- impact of the prior decreases with the number of measurements
- frequentist $\approx$ bayesian with flat prior (numbers are $=$ but meaning is $\neq$)
- questions: (1) why there is no coverage in bayesian?
  (2) Why the 3 properties of freq. estimator are not defined in bayesian?

Coming back to model uncertainties - I

Frequentist approach. Imagine your measurement depends on the detector energy response r_E. This response is measured using a dedicated dataset d_E.

- measurement described by a likelihood $\mathcal{L}_{energy}(r_E\,;\,d_E)$

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- this is the notion of auxiliary measurement.
- including exact LHs of all aux. meas. is usually too complicated
- approx by its 2nd order Taylor expension around the minimum obtained at $r_E = \hat{r_E}$ ($\equiv$ gaussian likelihood)

$$\mathrm{NLL}(r_E) \approx \mathrm{NLL}(\hat{r}_E) + \frac{(r_E - \hat{r}_E)^2}{2\sigma_E^2}$$

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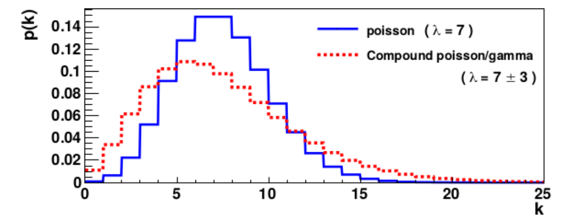
$$\mathcal{L}(\mu, \theta; data) = \mathcal{L}(\mu; data)\, \pi(\theta)$$

- the final likelihood is marginalized over θ:

$$\mathcal{L}_m(\mu; data) = \int \mathcal{L}(\mu, \theta; data)\, \pi(\theta) \mathrm{d}\theta$$

- Interpretation: average all possible situations (defined by a θ value), accounting for the probability of occurence of each of them.

Example of marginalization



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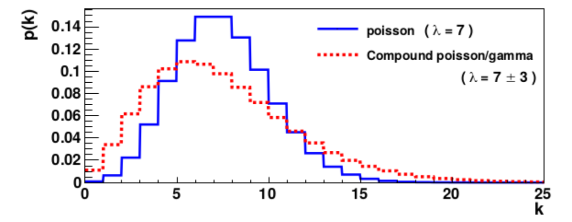

Uncertainty implementation : frequentist or bayesian ?

- no absolute answer to this question → arbitrariness
- choice depending on the context (interpretation, calculation, ...)
- always check the robustness of your conclusion wrt these choices

Why it is relevant

Most emblematic question: is there a signal in my data?

Test of Hypothesis

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Formalism

- 2 hypothesis: H_1 =there is signal and H_0: there is no signal

$\rightarrow$ test statistics $t \equiv$ random variable, discrimating H_1 from H_0

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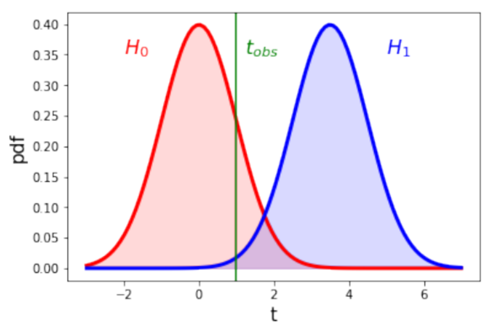
Most naive approch: event count as test statistics $t = N$

- $e.g.$ H_1 predicts $N_1 = 110$, while H_0 predicts $N_1 = 100$
- observation $N_{obs} = 112$: do I reject the signal hypothesis?
- Steps of test hypothesis
  - find distribution of t in both hypothesis $f(t|H_0)$ and $f(t|H_1)$
  - check where t_{obs} fall wrt to $f(t|H_0)$ and $f(t|H_1)$
  - conclude with a confidence level ($p-$value)

Test of Hypothesis



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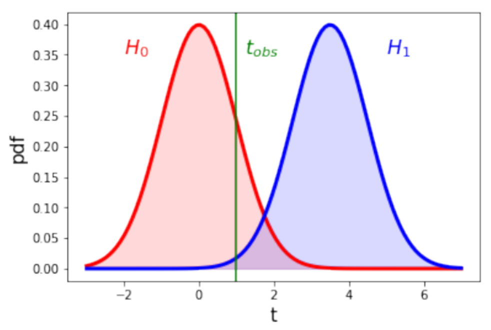
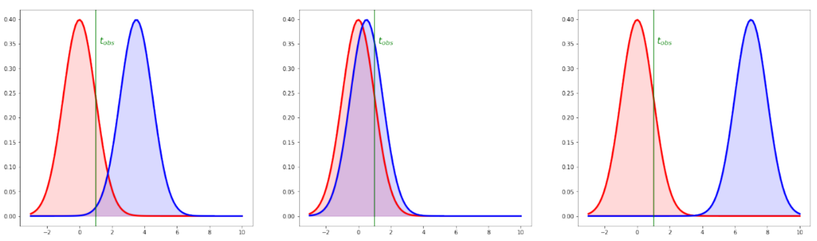


Quantitative agreement with an hypothsis: *p*-value

p-value = probability to observe what you observed in measurement or "more extreme" values

How to find exclusion limit



→ Increase the signal until the signal hypothesis get rejected (at a given confidence level).

Test of Hypothesis

Egon Pearson

Jerzy Neyman

Pearson-Neyman Lemma (1933)

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$$NLLR \equiv -2 \log \frac{\mathcal{L}(H_1|data)}{\mathcal{L}(H_0|data)}$$

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In practice: hunders or thousands of event counts!

Keywords and concepts

Part I: statistics

descriptive statistics – sample – mean – (co)variance – (de)correlation

Part II: probability

Bayes theorem – prior – posterior – random variable – (marginal) PDF – moments – characteristic function – (in)dependent variables – CLT – error propagation

Part III: statistical model

Likelihood – nuisance parameter – parameter of interest – systematic uncertainties

Part IV: The two big school

Frequentist – occurence frequency – pseudo-data (toys) – bayesian – degree of belief

Part VI: Parameter estimation & hypothesis testing

estimator and its properties – χ^2 – confidence/credibility level/interval – coverage – p-value – LLR

Conclusions

Concluding remarks

Statistics deals with the 'not fully known'
$\rightarrow$ not a single way $\rightarrow$ some arbitrariness

1. Statistics $\equiv$ link between measurement and conclusion

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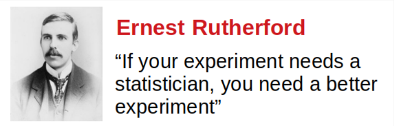

Thanks for you attention !