Cold Quark Matter

Renormalization group improvement of the perturbative series

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Based on

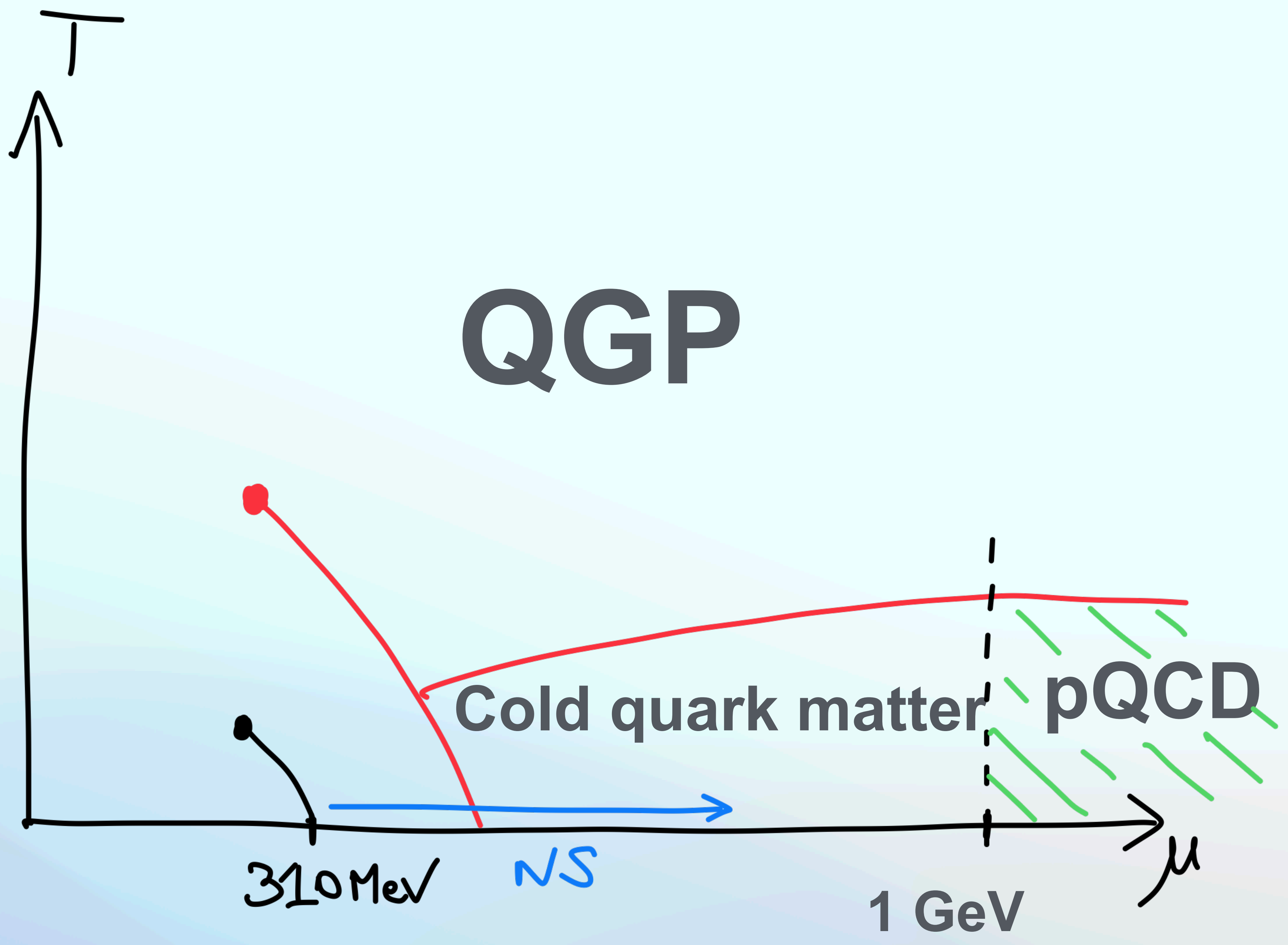
LF and Kneur PRL 129 (2022)

LF and Kneur PRD 111(2025)

Loïc Fernandez

Context

Context

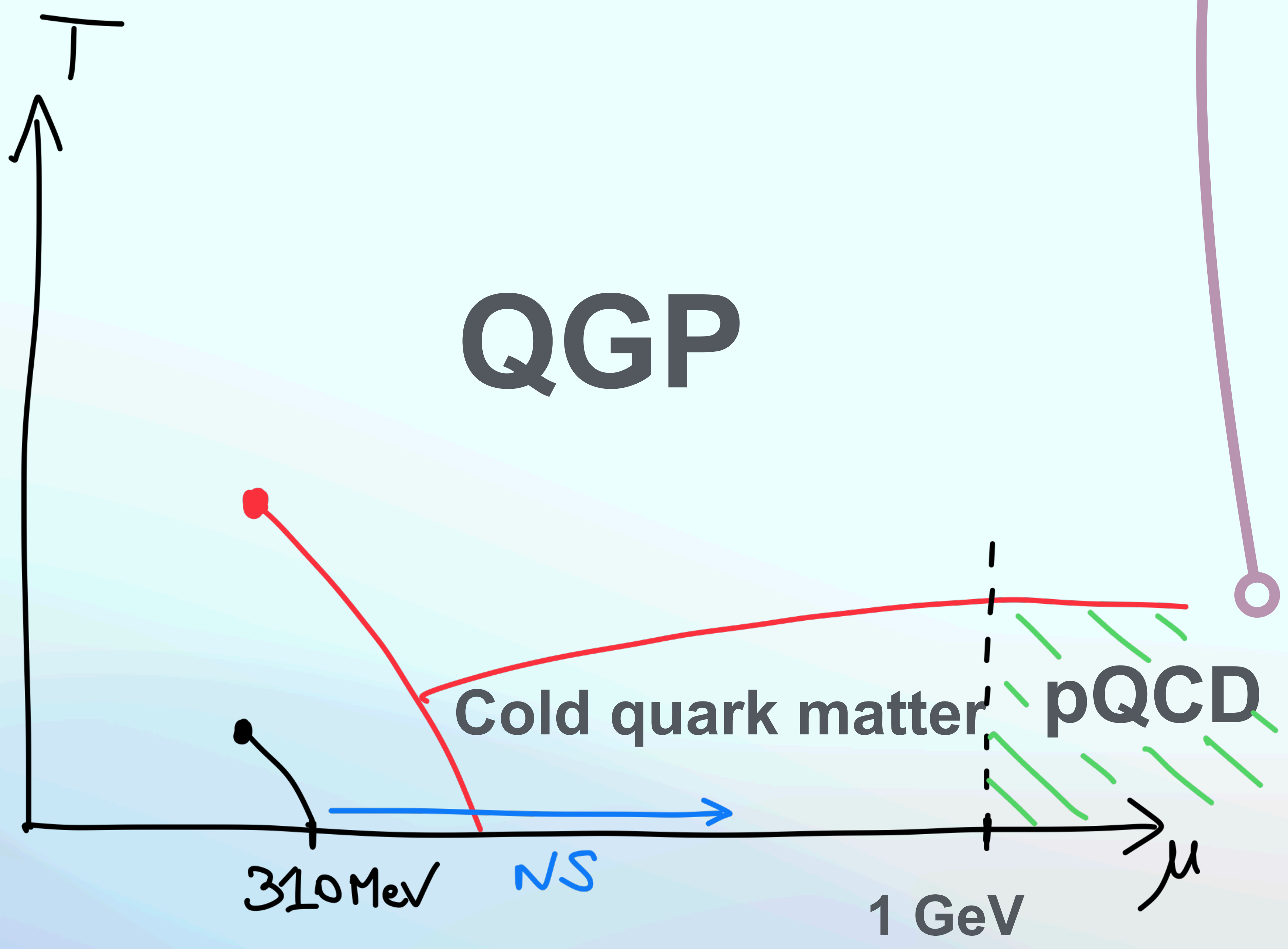


Large uncertainties in pQCD

Context

known

$$\frac{P_{QCD}(T=0,\mu)}{P_f} = 1 + \alpha_s A_1 + \alpha_s^2 (A_2 + B_1 \ln(\alpha_s)) + \ldots$$



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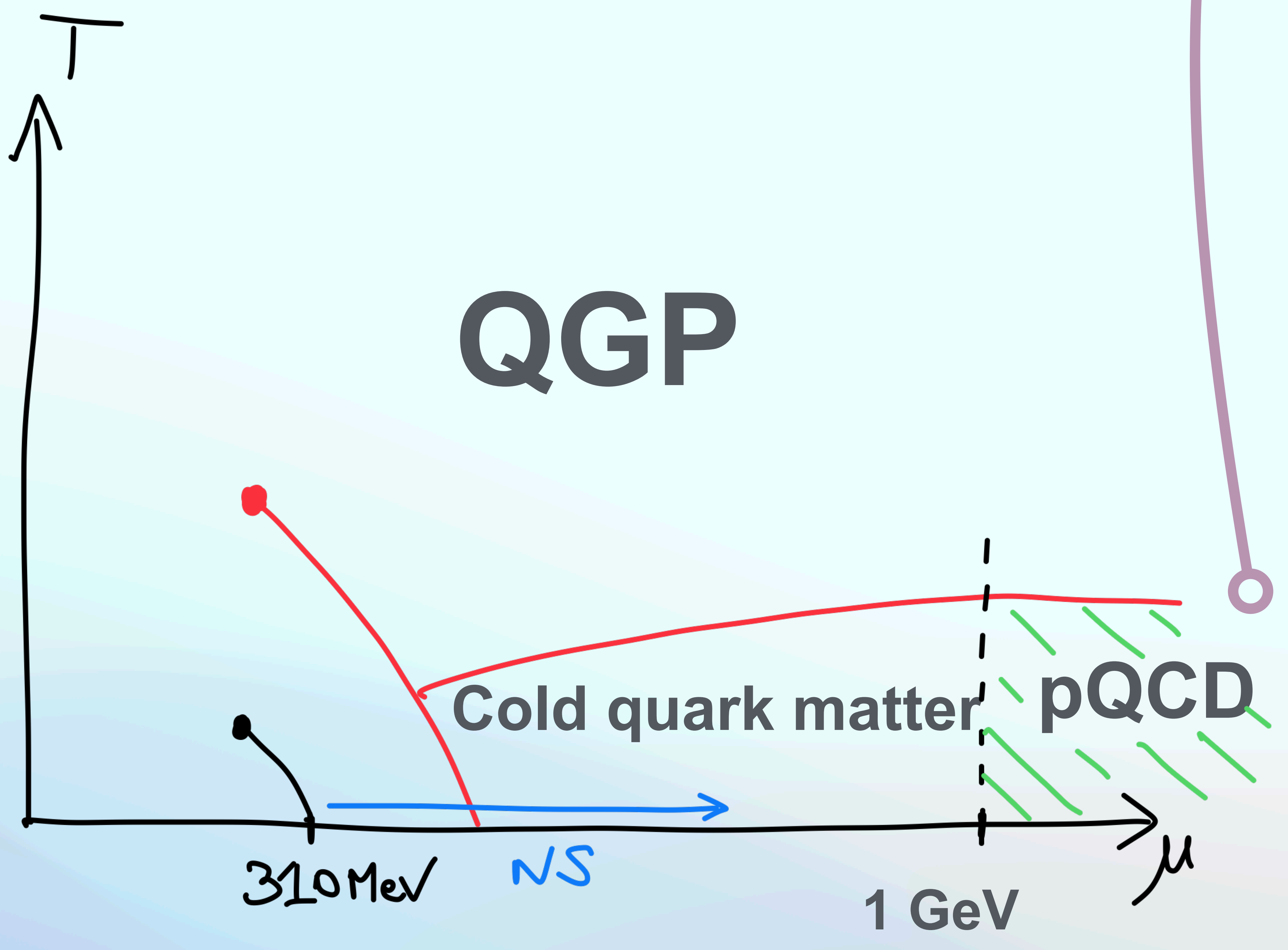

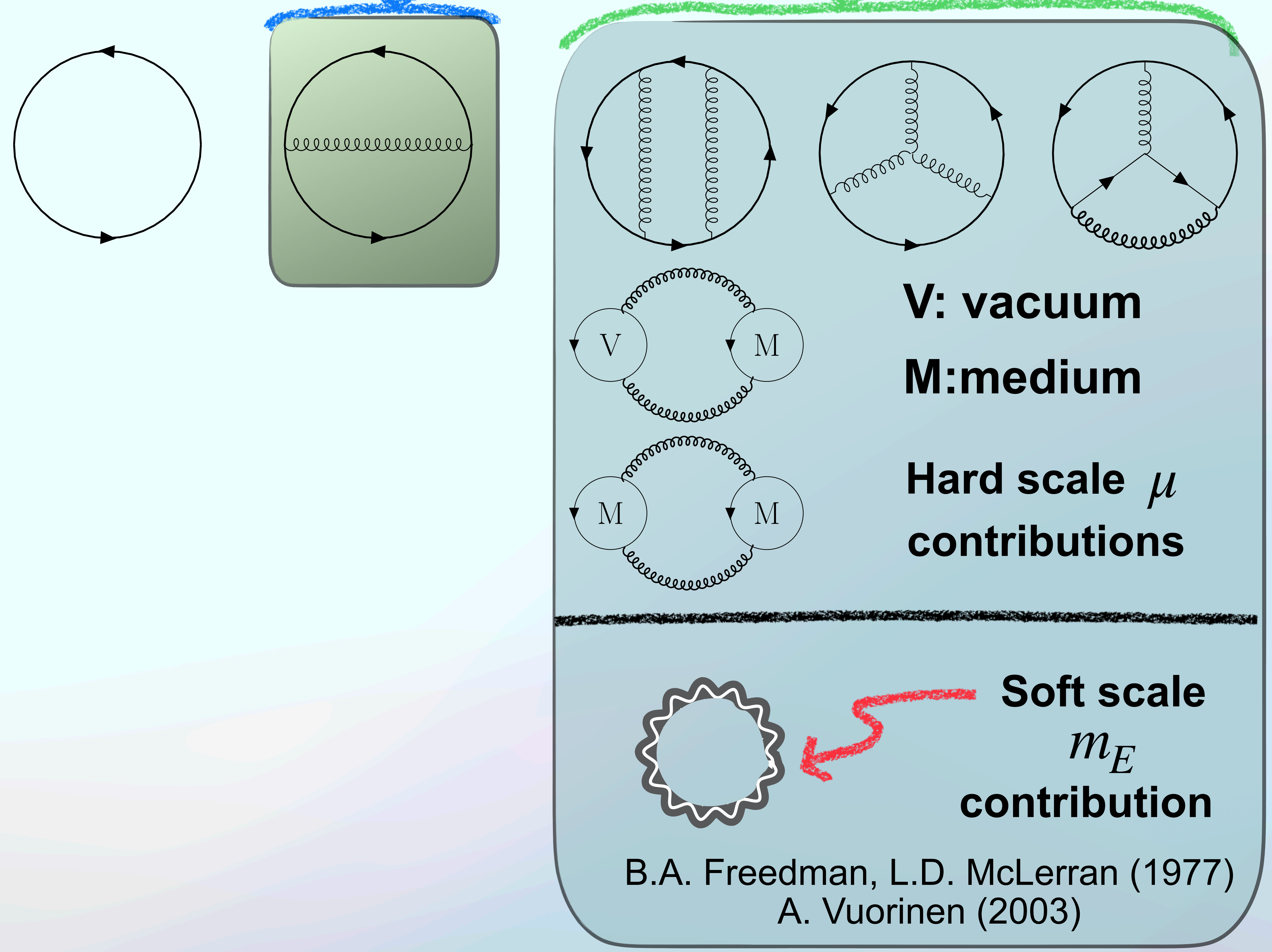

Large uncertainties in pQCD

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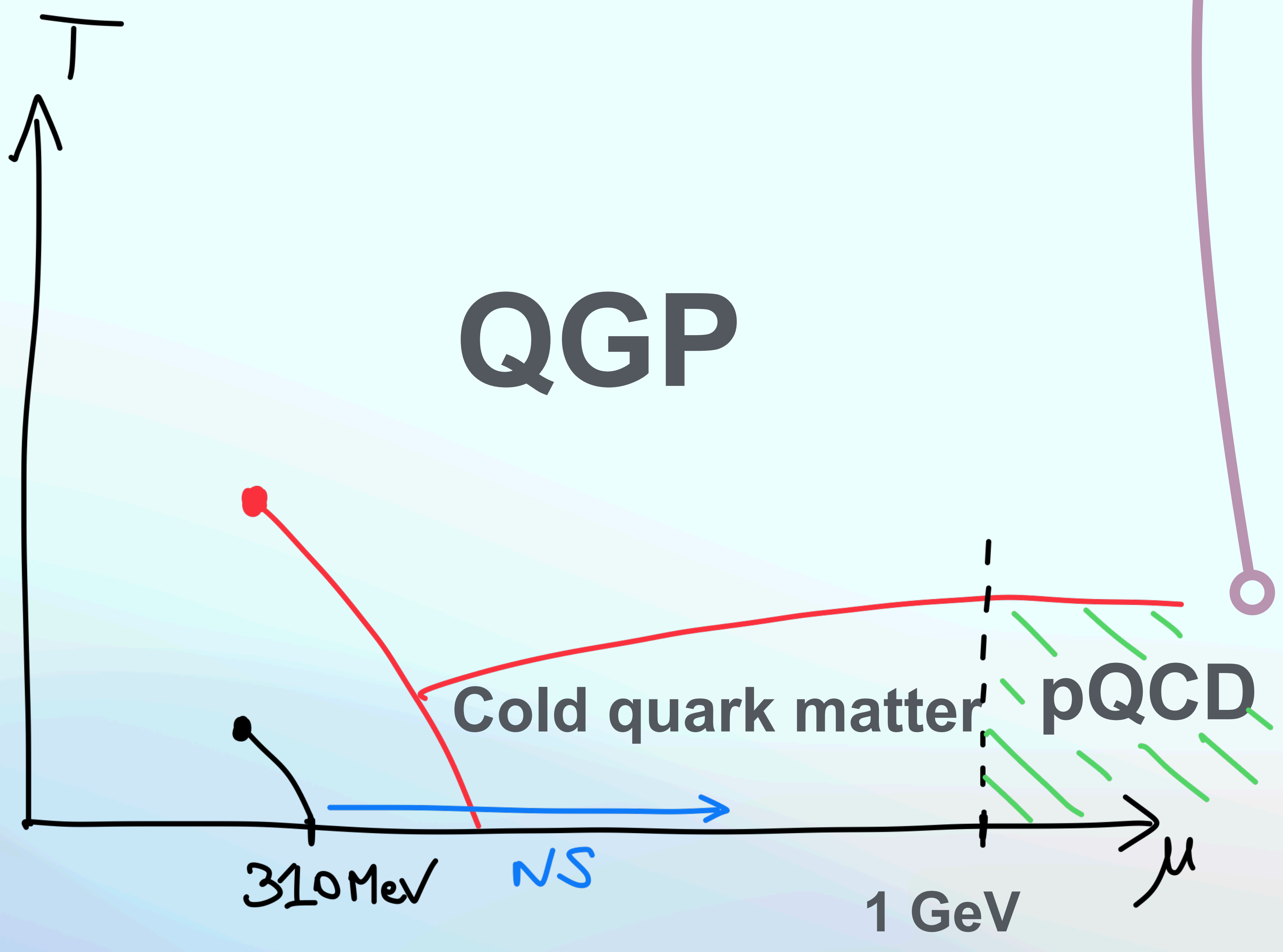

Large uncertainties in pQCD

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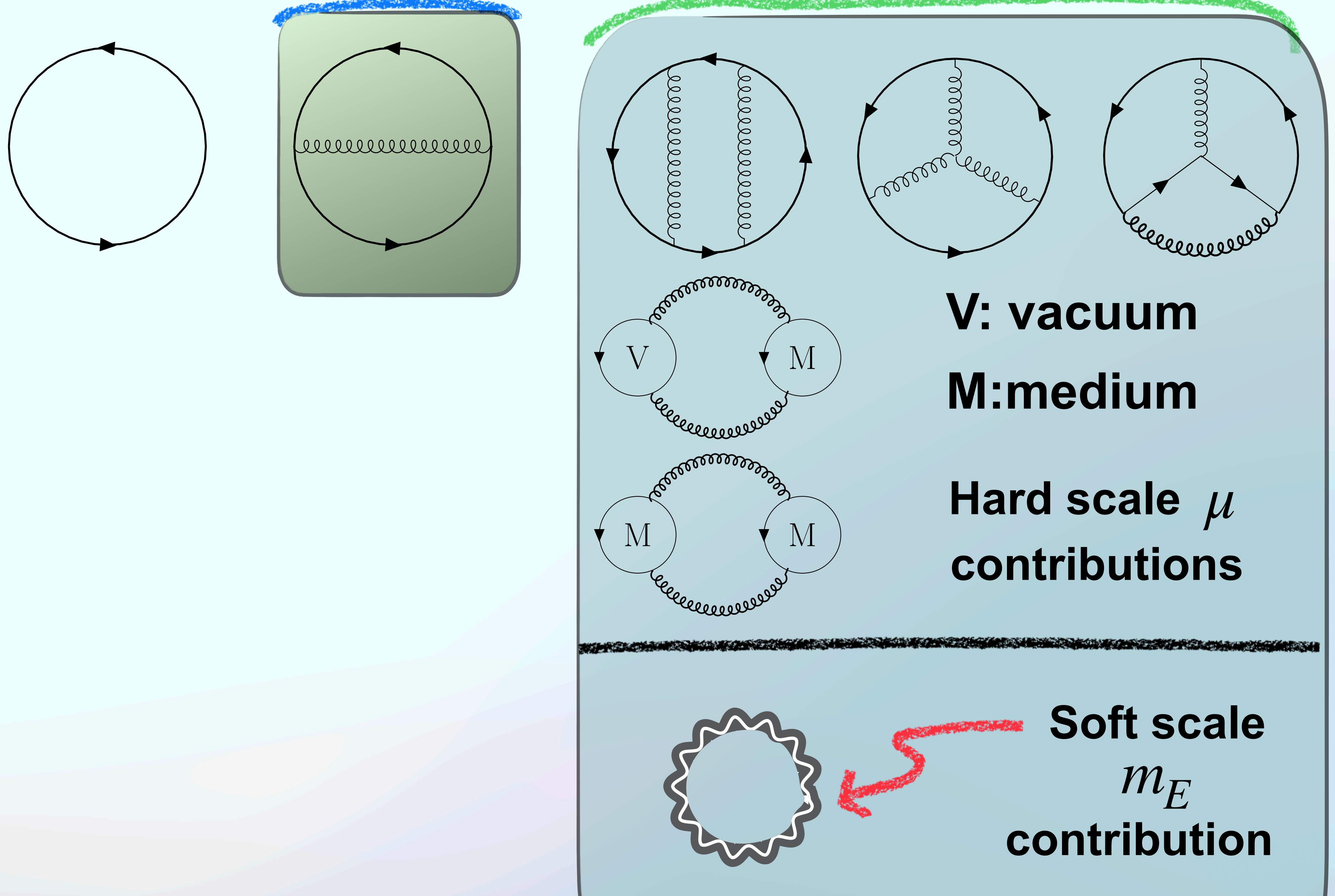
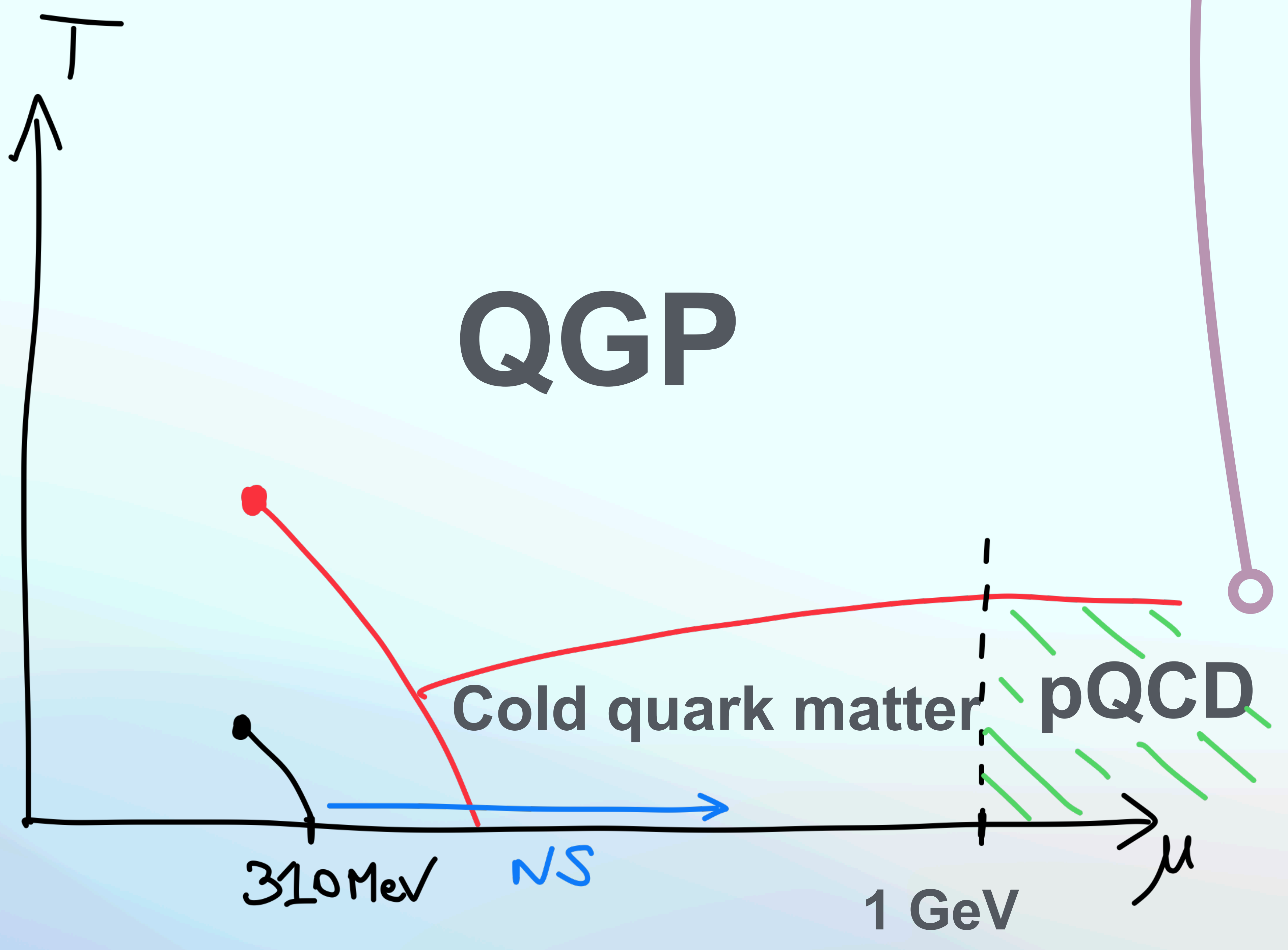


B.A. Freedman, L.D. McLerran (1977)
A. Vuorinen (2003)

$$m_E \sim \sqrt{\alpha_s}\mu$$

Context



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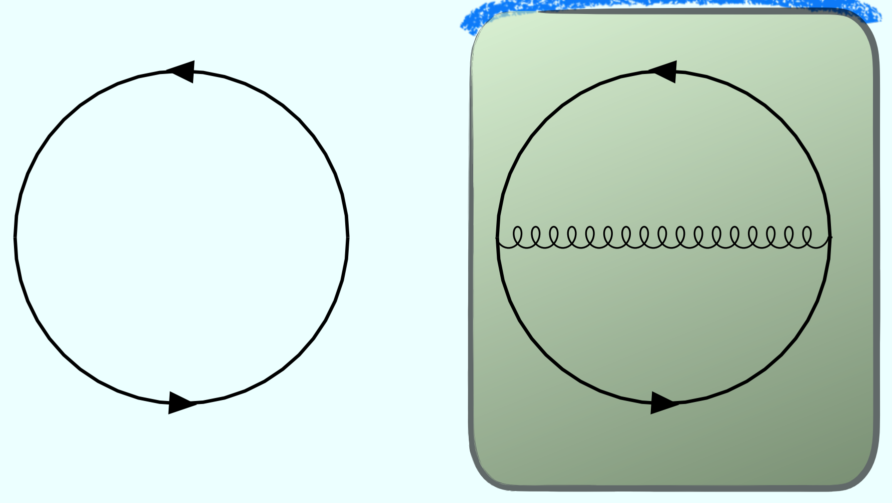
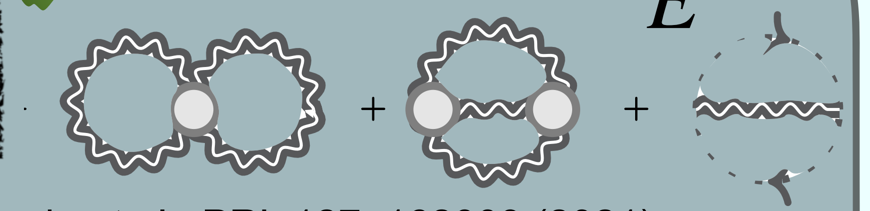


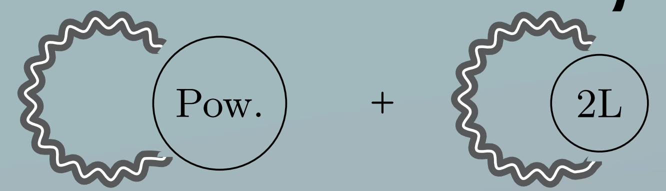
α_s^3

Soft sector m_E



Gorda et al., PRL 127, 162003 (2021)

Mixed sector μ, m_E

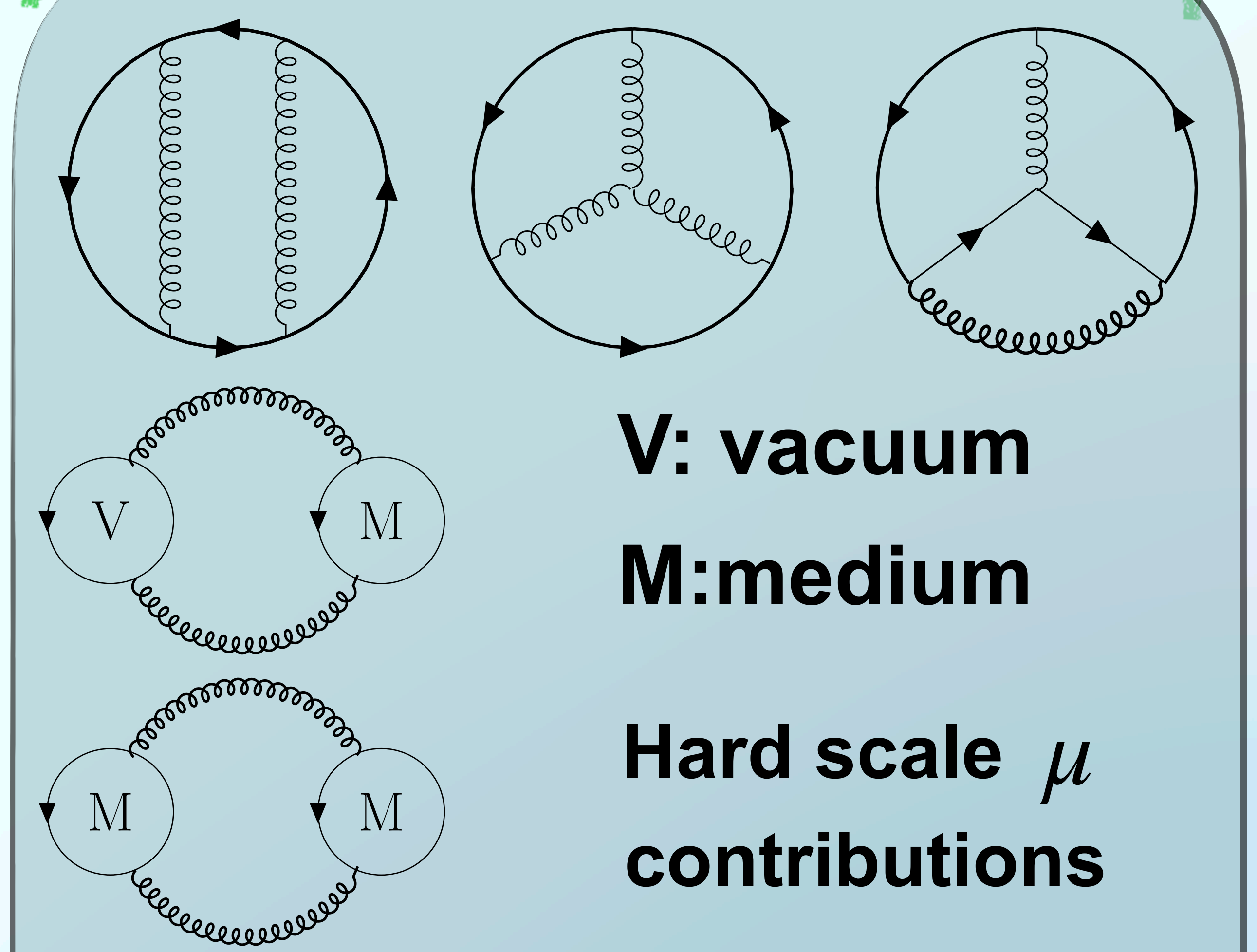


Gorda et al., PRL 131, 181902 (2023)

Hard sector μ

Kärkkäinen et al. 250117921

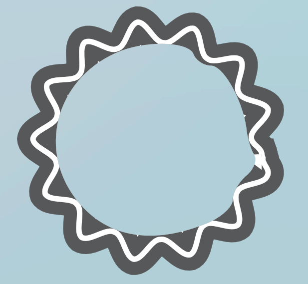
2026?



V: vacuum

M:medium

Hard scale μ contributions



Soft scale m_E contribution

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Going beyond the perturbative expansion

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$$\frac{P_{QCD}(T=0,\mu)}{P_f} = \underbrace{1 + \alpha_s\ A_1 + \alpha_s^2(\ A_2 + \ B_1 \ln(\alpha_s)) + \ldots}_{}$$

Renormalization scale dependent and missing higher orders (correlated)

Uncertainty estimate for missing higher orders

<=>

renormalization scale variation

$\alpha_s(\Lambda) \rightarrow \Lambda \in \{\mu, 4\mu\}$

$g^2 = 4\pi\alpha_s$

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↓ **Resummation**

Perturbative renormalization group

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Soft scale

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Resummation of logarithms in HTL effective theory

LF & J-L. Kneur, PRL 129 (2022)
And on-going work

Going beyond the perturbative expansion

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Resummation

Perturbative renormalization group

(2) **Hard scale**

Renormalization group optimized perturbation theory (RGOPT)

Main topic of this talk
LF & J-L. Kneur, PRD 111, 034020 (2025)

(1) **Soft scale**

Resummation of logarithms in HTL effective theory

LF & J-L. Kneur, PRL 129 (2022)
And on-going work

Resummation of logarithms in HTL effective theory

With massless quarks for simplicity

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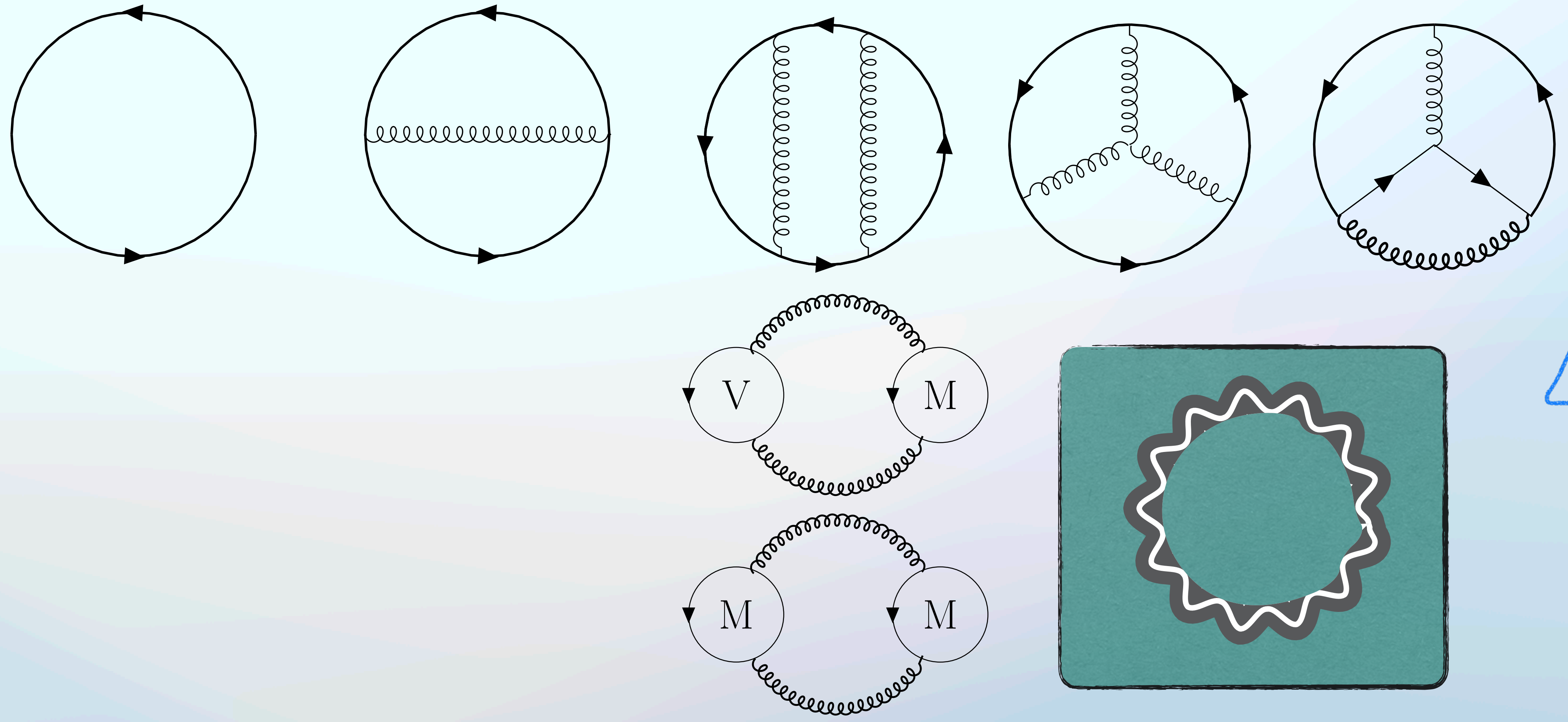

Physics at the soft scale

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$$m_E \sim \sqrt{\alpha_s}\mu$$

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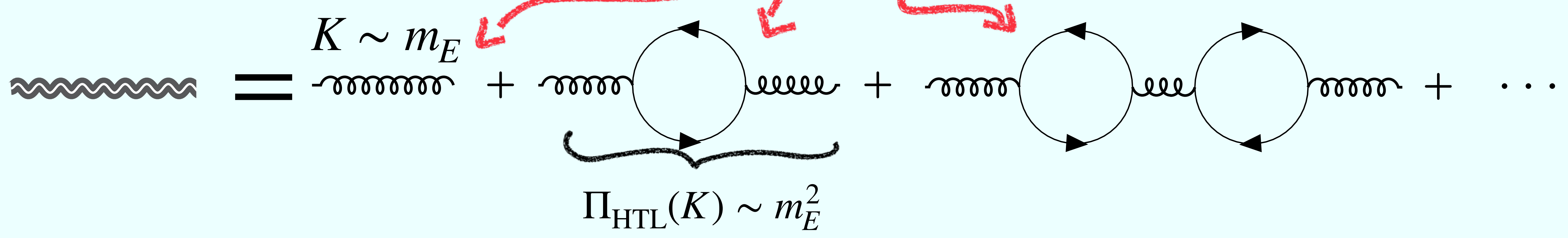
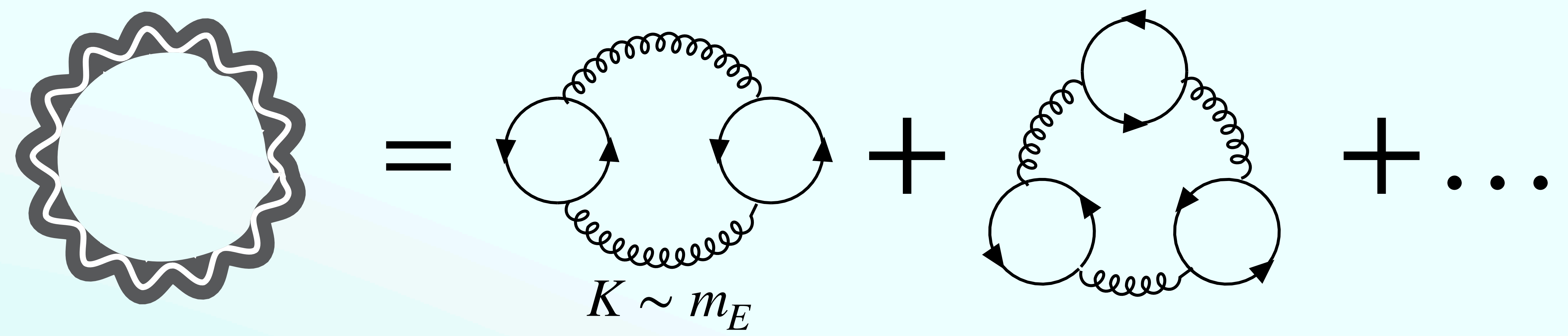
Parametrically of the same order



$$= \frac{P_{\mathrm{T}}^{\mu\nu}}{K^2 + \Pi_{\mathrm{T}}(K)} + \frac{P_{\mathrm{L}}^{\mu\nu}}{K^2 + \Pi_{\mathrm{L}}(K)} + \xi \frac{K^\mu K^\nu}{(K^2)^2}$$



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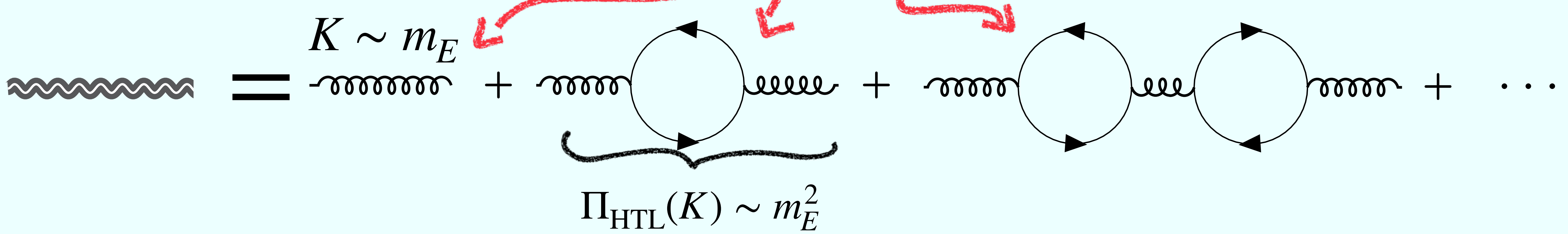

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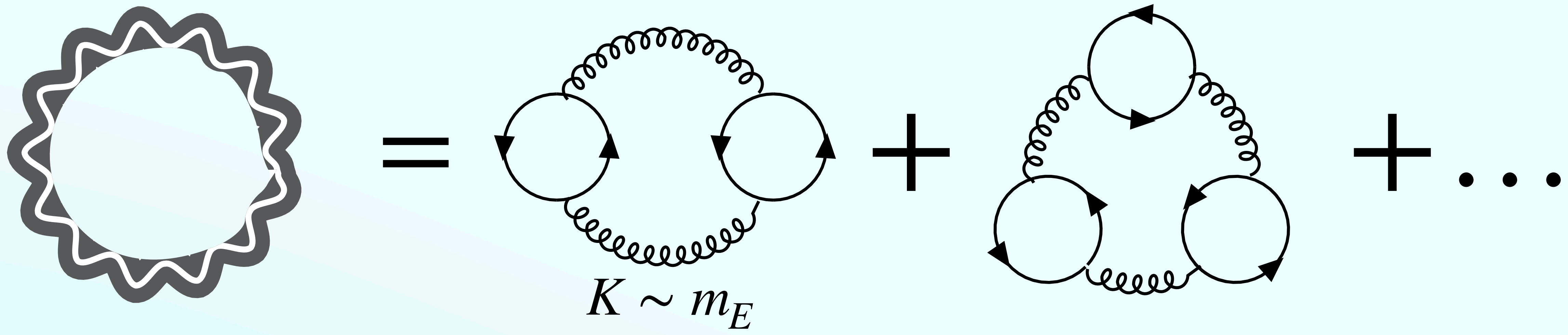

Hard Thermal Loop resummation.

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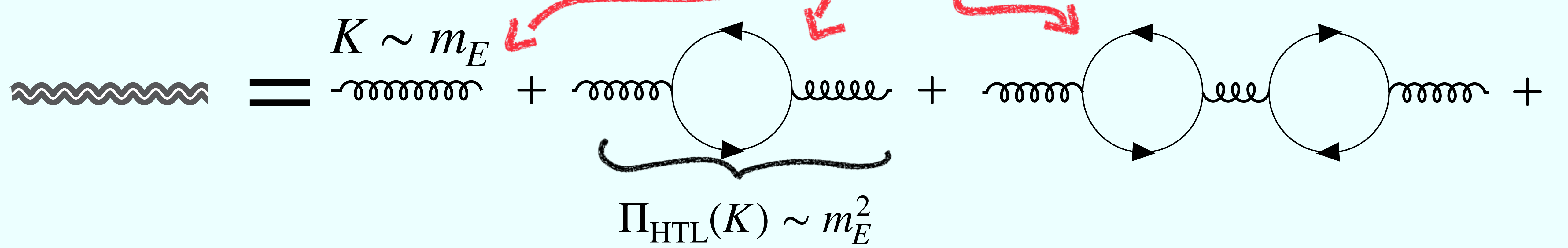

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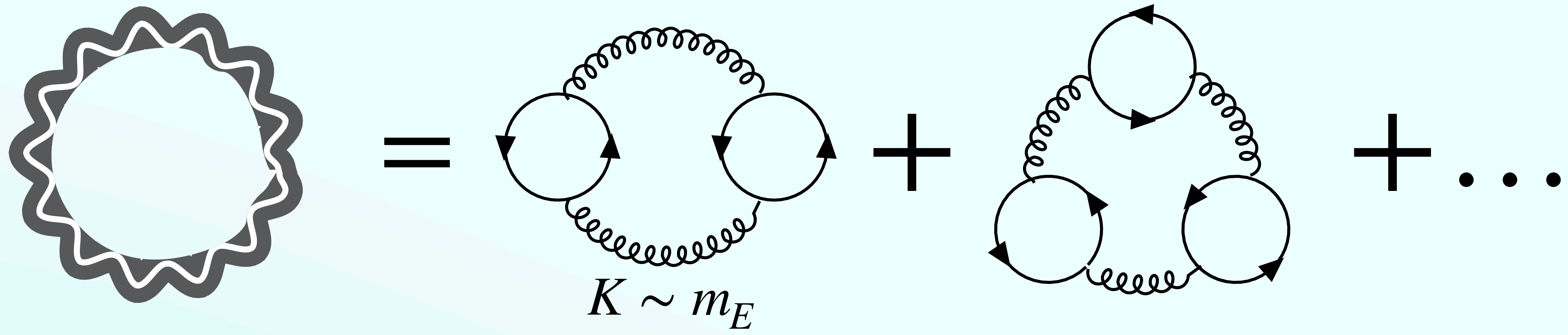


Hard Thermal Loop resummation.

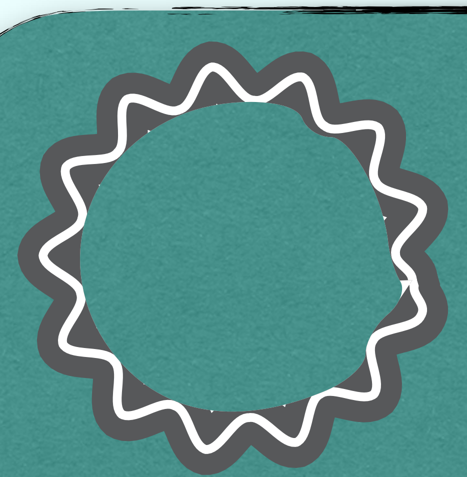
The soft contributions are encoded in the HTL effective Lagrangian !

$$L_{\mathrm{YM}}^{\mathrm{HTL}} = \frac{m_E^2}{2} \mathrm{Tr} \int_{\hat{v}} F^{\alpha\beta} \frac{v_\beta v^\gamma}{(v \cdot D_{\mathrm{adj.}})^2} F_{\gamma\alpha}$$

E. Braaten, R.D. Pisarski, PRD 45(1992)

HTL effective theory and resummation of logs

$$\frac{P_{QCD}(T=0,\mu)}{P_f} = 1 + \alpha_s\, A_1 + \alpha_s^2(\, A_2 + B_1 \ln(\alpha_s)) + \ldots$$



$$= \frac{d_A m_E^2}{(8\pi)^2}\left(C_{11} - \ln\left(\frac{m_E}{\Lambda_s}\right)\right)$$

$$\Lambda_s \sim \mu \to \ln\frac{m_E}{\Lambda_s} \sim \ln\alpha_s$$

In this EFT: $L_{\rm YM} + L_{\rm YM}^{\rm HTL}$ **effective gluon mass.**

Anomalous mass dimension : γ_m^g

m_E **interpreted at first as blind to its physical origin.**

Therefore the RG operator reads: $$\Lambda_s\frac{d}{d\Lambda_s} = \Lambda_s\frac{\partial}{\partial\Lambda_s} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} - m_E\gamma_m^g\frac{\partial}{\partial m_E}$$

HTL effective theory and resummation of logs

$$P^{\text{soft}} = m_E^4 \sum_i (\alpha_s)^{i-1} \sum_{k=0}^{i} a_{i,k} \ln^{i-k}\left(\frac{m_E}{\Lambda_s}\right)$$

(2018) (postdiction)

$$\frac{P^{soft}}{m_E^4} = a_{1,0} L + \alpha_s a_{2,0} L^2 + \dots \quad \text{LL}$$

$$+ a_{1,1} + \alpha_s a_{2,1} L + \dots \quad \text{NLL}$$

$$+ \alpha_s a_{2,2} + \dots$$

(input) (2022)

$$L = \ln\left(\frac{m_E}{\Lambda_s}\right)$$

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Requiring $\Lambda_s \frac{dP^{\rm soft}}{d\Lambda_s} = 0$ $\mapsto$

$$-p\, a_{i,0} = \left(4\gamma_0^g + 2b_0^g(i-2)\right) a_{i-1,0}$$

recurrence relation between leading soft logarithms (LL)

HTL effective theory and resummation of logs

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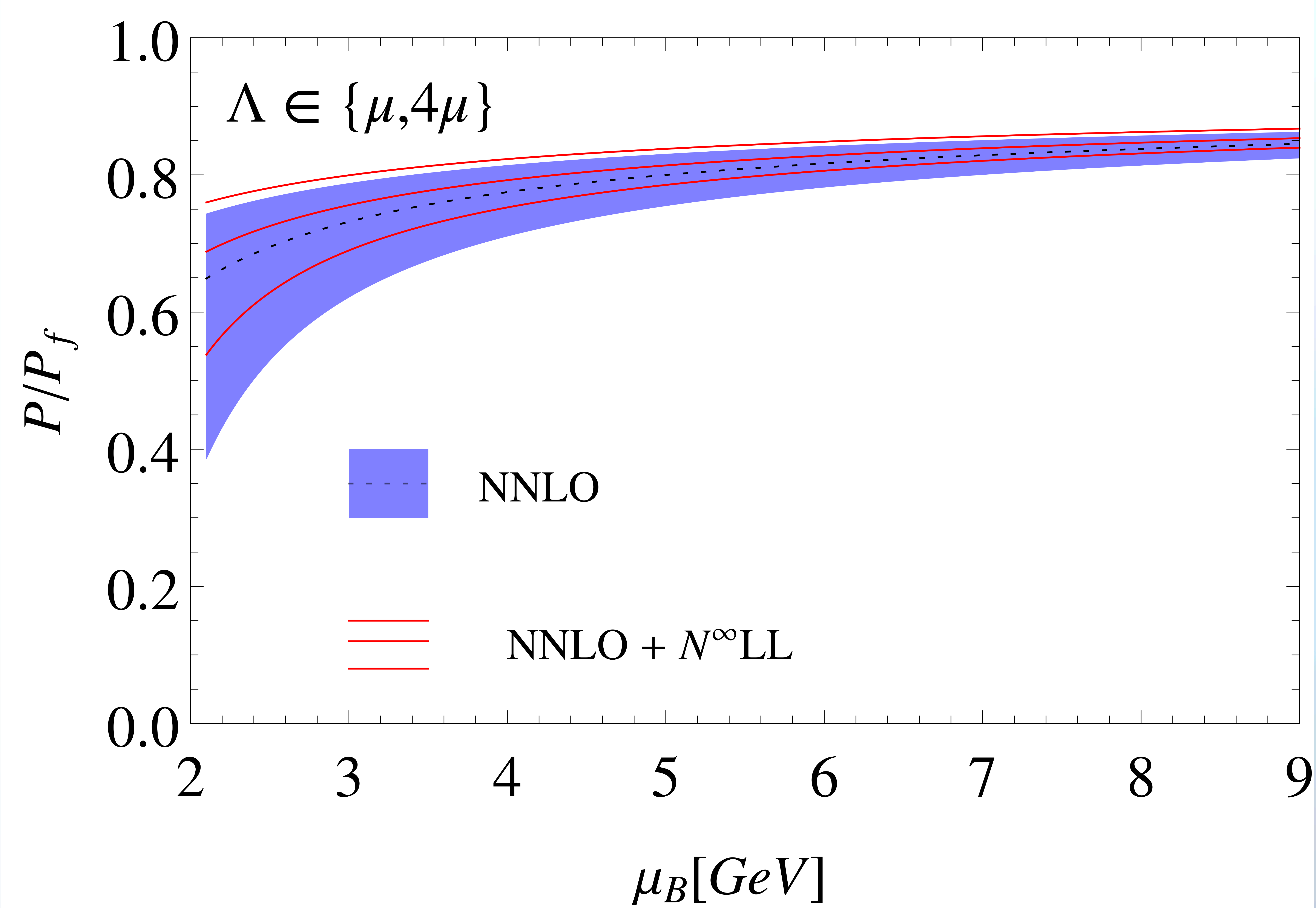
$$+ \ldots$$

"Resummation of a resummation"
(or resummation in two steps)

Resuming the LL



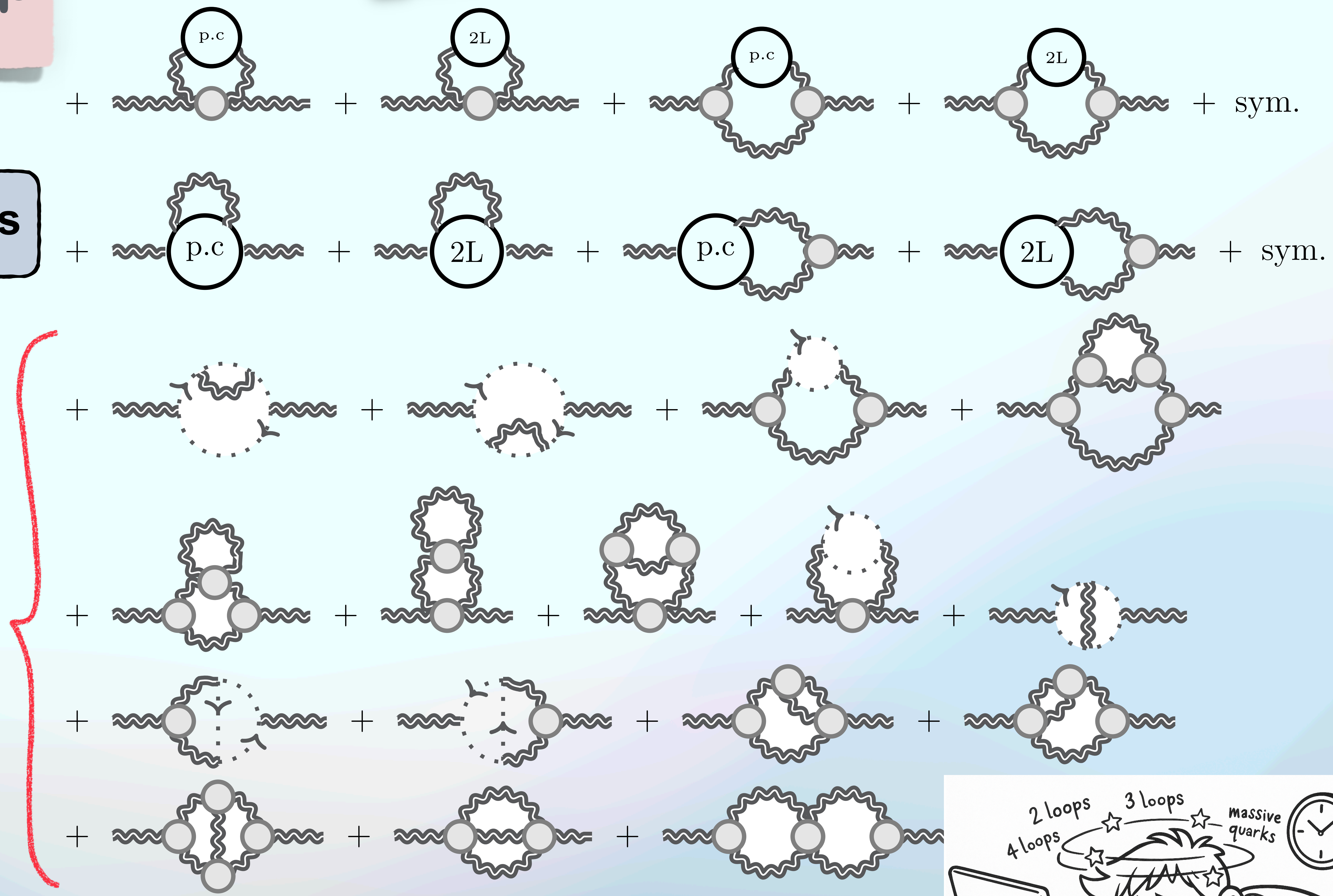
Significant reduction in renormalization scale dependence from resumming the leading soft logarithms.

LF and JL. Kneur PRL 129 (2022)

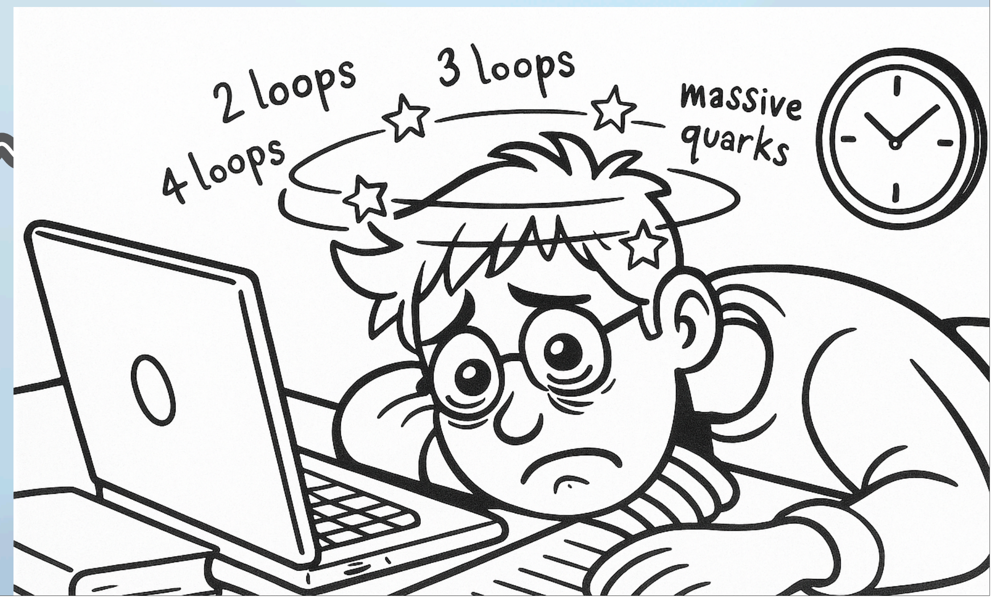
Glimpse of the next step

WORK IN PROGRESS

Full resummation of the NLL series



Need new anomalous dimension coefficients. A lot of fun ahead. :)
Completion of soft ones within reach



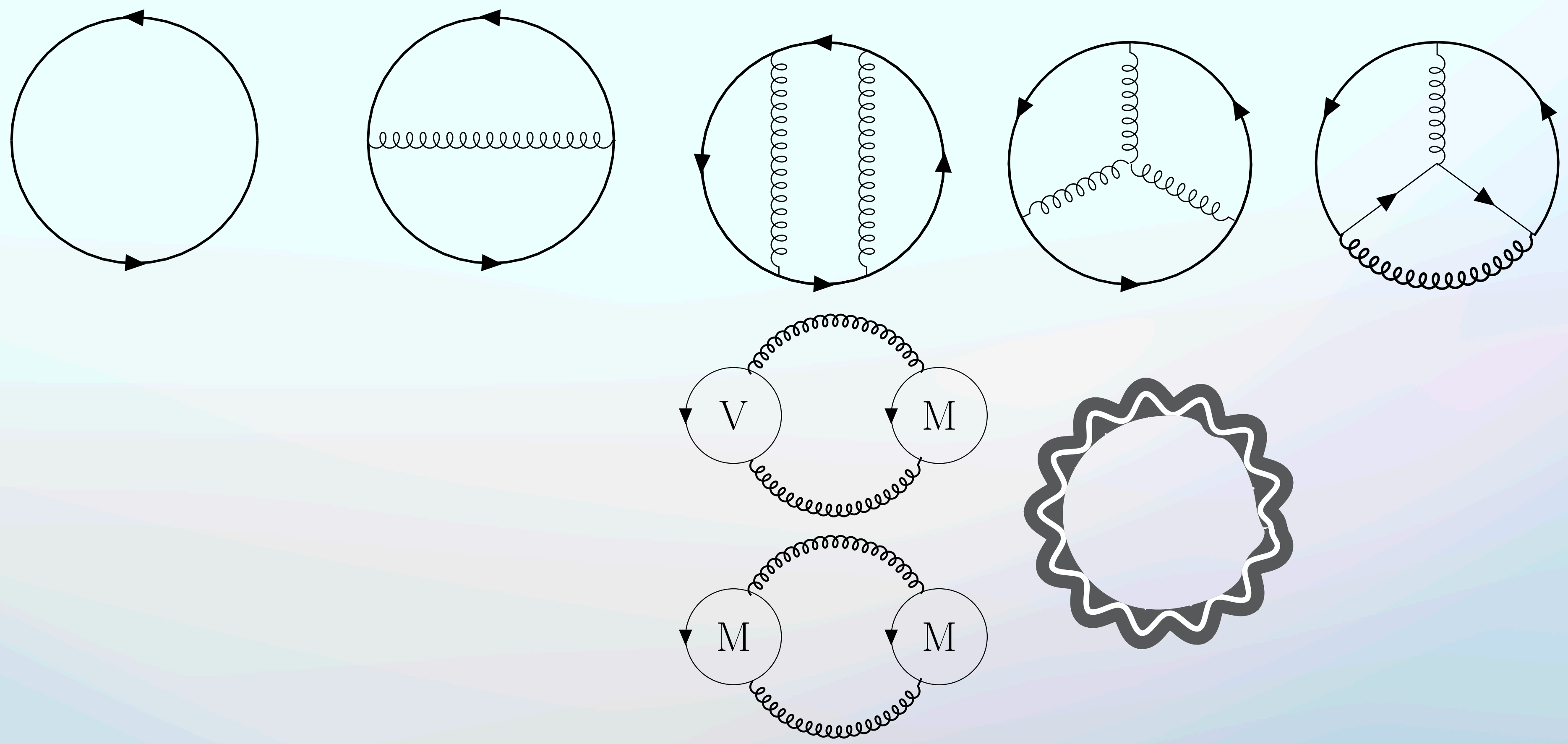
Renormalization Group Optimized Perturbation Theory For Cold Quark matter at NNLO

Now we will focus on the hard sector with masses for the quarks !

(No Hard thermal Loops effective theory anymore)

All diagrams included

$$\frac{P_{QCD}(T=0,\mu)}{P_f} =$$



Previous work at NLO: Kneur, Pinto, Restrepo 2019

Motivations

Renormalization Group Optimized Perturbation Theory

RGOPT

Kneur, Neveu, QCD T=0, (2013)

Kneur, Neveu , QCD T=0, (2015)

LF, Kneur, $\lambda\phi^4$, NNLO, (2021)

Kneur, Pinto, Restrepo Hot QCD(2021)

HTLpt = OPT + HTL

Andersen, Braaten, Petitgirard, Strickland (2002)

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Motivations

Renormalization Group Optimized Perturbation Theory

1 In hot/dense QCD, the fields develop screening masses.
It seems best to start from a massive action.

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Renormalization Group Optimized Perturbation Theory

Motivations

1 **In hot/dense QCD, the fields develop screening masses. It seems best to start from a massive action.**

2 **For cold and dense QCD, the mass of the quark is mostly responsible for the large renormalization scale dependence**

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$$\mathcal{L}(\alpha_s) \rightarrow \mathcal{L}(\delta\alpha_s, m(1-\delta))$$

variational parameter, not The genuine quark mass !

Interpolation

3

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Renormalization Group Optimized Perturbation Theory

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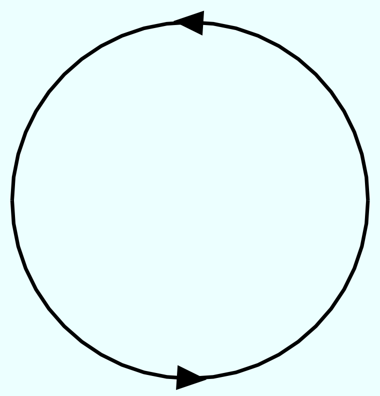
Haque, Andersen, Mustafa, Strickland, Su,(2014)

3 **For high temperature and no chemical potential, RGOPT has been performing extremely well. Let's look into cold and dense QCD with RGOPT**

Walkthrough at leading order

Walkthrough at leading order

Renormalization Group Optimized Perturbation Theory (RGOPT)



Walkthrough at leading order

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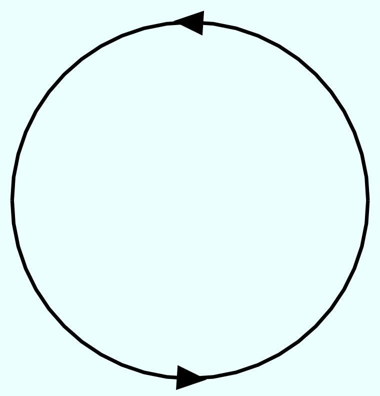
$\mathcal{P}_{LO} =$ 

Walkthrough at leading order

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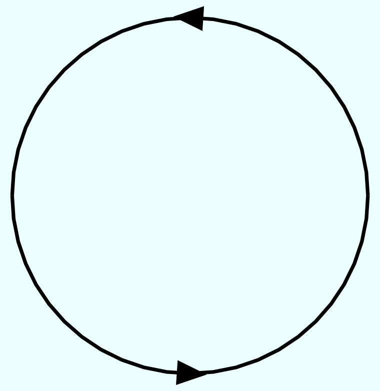
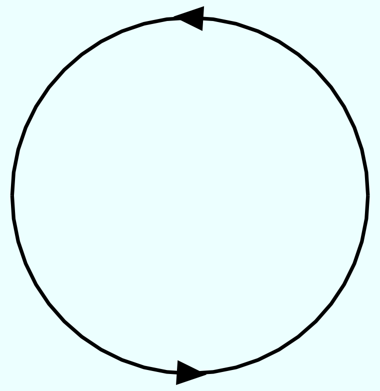
$P_{LO} =$  $= \dfrac{N_c N_f \mu^4}{12\pi^2}$

Walkthrough at leading order

Renormalization Group Optimized Perturbation Theory (RGOPT)

$P_{LO} =$  $= \frac{N_c N_f \mu^4}{12\pi^2} \xmapsto{\text{generalize}} P_{LO}(m) = \text{vacuum}(m) + \text{matter}(m,\mu)$

Walkthrough at leading order

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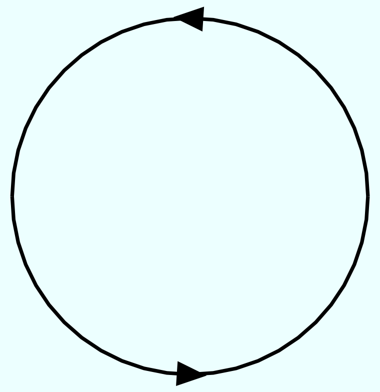
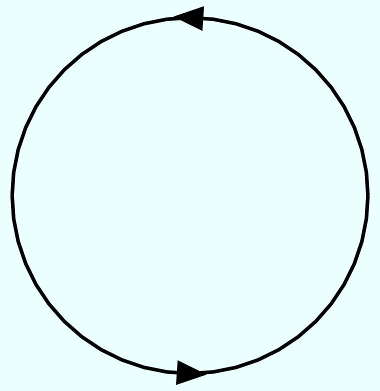
$P_{LO} =$  $= \frac{N_c N_f \mu^4}{12\pi^2} \xmapsto{\text{generalize}} P_{LO}(m) = \text{vacuum}(m) + \text{matter}(m,\mu)$

$m \longmapsto m(T,\mu)$

Walkthrough at leading order

Renormalization Group Optimized Perturbation Theory (RGOPT)

$P_{LO} =$  $= \dfrac{N_c N_f \mu^4}{12\pi^2} \xmapsto{\text{generalize}} P_{LO}(m) = \underbrace{\text{vacuum}(m)} + \text{matter}(m,\mu)$

Usually neglected because not dependent of the thermodynamic variables. But in RGOPT, we deal with medium-dressed masses

$m \longmapsto m(T,\mu)$

Walkthrough at leading order

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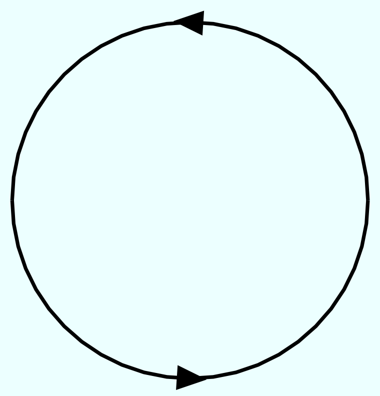


$$P_{LO} = \bigcirc = \frac{N_c N_f \mu^4}{12\pi^2} \quad \xmapsto{\text{generalize}} \quad P_{LO}(m) = \underbrace{\text{vacuum}(m)} + \text{matter}(m,\mu)$$

Usually neglected because not dependent of the thermodynamic variables. But in RGOPT, we deal with medium-dressed masses

$$m \longmapsto m(T,\mu)$$

Not RG invariant on its own !

Walkthrough at leading order

Renormalization Group Optimized Perturbation Theory (RGOPT)

$$P^{RGI} = -\frac{s_0 m^4}{\alpha_s} + \text{Vacuum}(m) + \text{matter}(m, \mu)$$

RG-invariant (RGI)

Then, proceed to the delta re-expansion

$$P^{RGI}_{(m,\mu,\alpha_s)} \longmapsto P^{RGI}_{\delta}\left(m(1-\delta)^{a}, \mu, \delta\alpha_s\right)\Big|_{\delta\approx 0;\ \delta\mapsto 1}$$

13

Walkthrough at leading order

RG eq. $$\left[\Lambda\frac{\partial}{\partial\Lambda}\mathcal{P}+\beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right]=0$$

Solved once and for all !

Solve at leading order for any m

$$a=\frac{\gamma_0}{2b_0},$$ **Universal**

$$\left.\frac{\partial P}{\partial m}\right|_{m=\tilde{m}}=0$$ $\mapsto \tilde{m}_{LO}(\mu,\alpha_s)$

Walkthrough at leading order

RG eq. $\left[\Lambda \frac{\partial}{\partial \Lambda}\mathcal{P} + \beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right] = 0$

Solve at leading order for any m

Solved once and for all !

$a = \frac{\gamma_0}{2b_0}$, **Universal**

$\left.\frac{\partial P}{\partial m}\right|_{m=\tilde m} = 0 \mapsto \tilde m_{LO}(\mu, \alpha_s)$

Generalization at higher orders

Solve "m" through OPT *or* RG equation which incorporates the *all order resummation.*

$\left[\Lambda \frac{\partial}{\partial \Lambda}\mathcal{P} + \beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right]_{\overline{m}_{RG}} = 0$ $\quad \left.\frac{\partial P}{\partial m}\right|_{m=\tilde m} = 0$

$$\begin{pmatrix} \overline{m}_{\mathrm{RG},1} & \tilde m_1 \\ \overline{m}_{\mathrm{RG},2} & \tilde m_2 \\ \overline{m}_{\mathrm{RG},3} & \tilde m_3 \\ \dots & \dots \end{pmatrix}$$

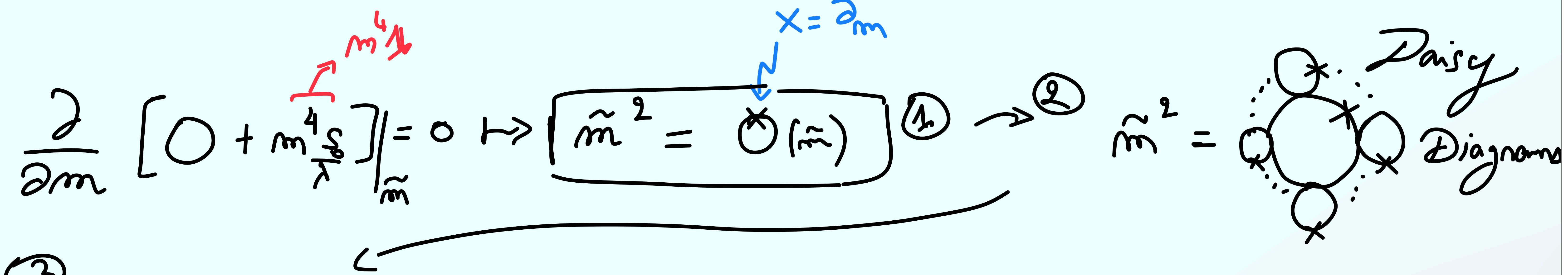
NB: at NNLO only RG gives physically consistent screening mass

Multiple solutions, but <u>*only one*</u> satisfies asymptotic freedom thanks to "a".

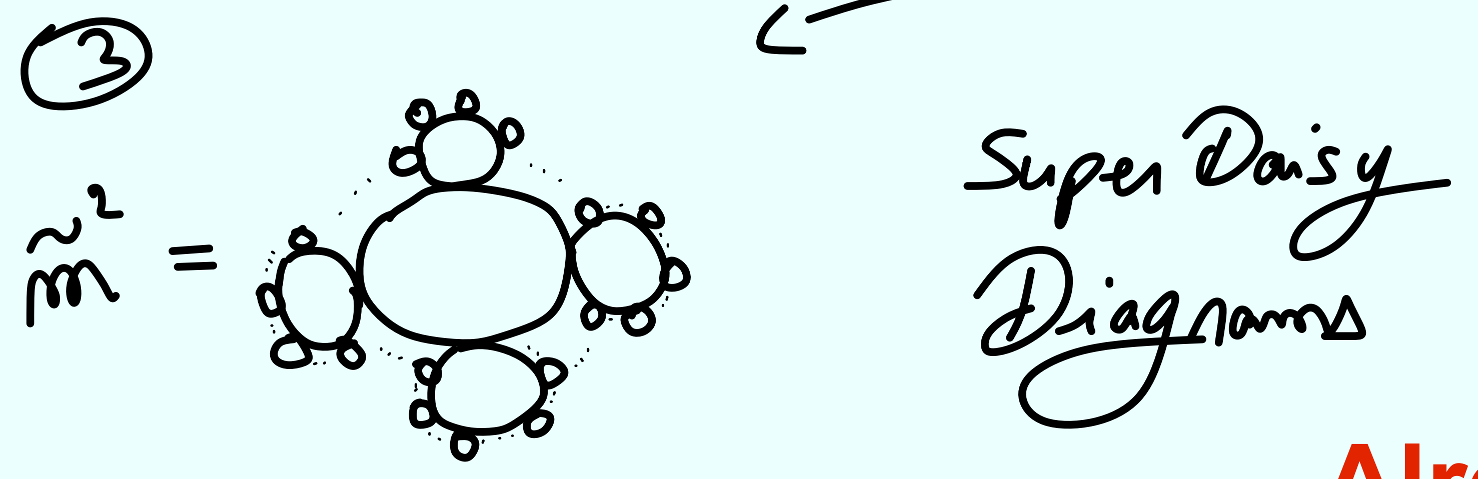
$\mathcal{L}(\alpha_s, m) \to \mathcal{L}(\delta\alpha_s, m(1-\delta)^a)$

Visual representation of the resummation in RGOPT

In $\lambda\phi^4$ with simplified picture:

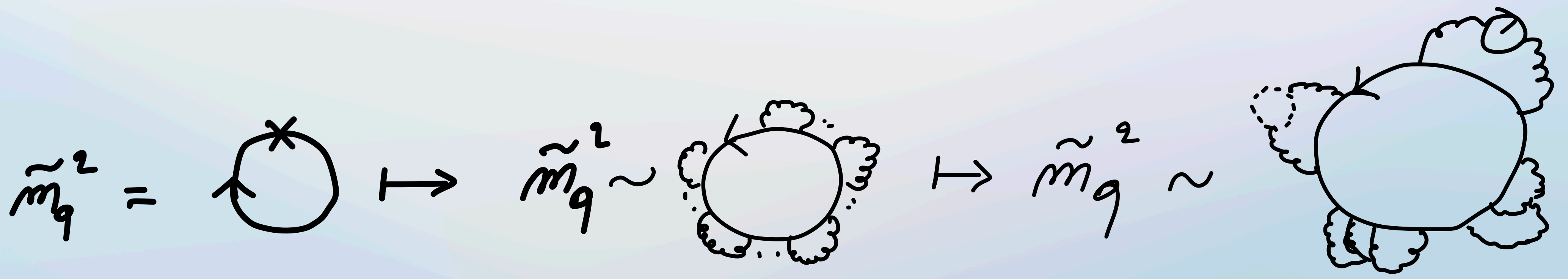


Only a certain subclass of diagrams are resumed



Already at LO, RGOPT reproduces (and go beyond) the standard textbook resummation !

In QCD it is more complicated to visualize:



Renormalization Group Optimized Perturbation Theory (OPT)

What if this solution is not real ?

Use the arbitrariness of the renormalization scheme to change to a new one where the coefficients in the pressure lead to a real solution.

Renormalization scheme change (RSC)

$$m \to m\,(1 + B_2\, g^4)$$

$$m \in \mathbb{C} \longrightarrow m \in \mathbb{R}$$

B2 is chosen as a minimal deviation from $\overline{MS}$.

Solving RG & Det. : $(\overline{m}_{RG}, \tilde{B}_2)$

For Cold Quark Matter , the prescription is unique !

Cold Quark Matter At NNLO

Flashing expressions

Kurkela, Romatschke, Vuorinen, PRD 81, 105021 (2010)

Setup:

$$(0,\mu_u)\quad(0,\mu_d)\quad(m_s,\mu_s)$$

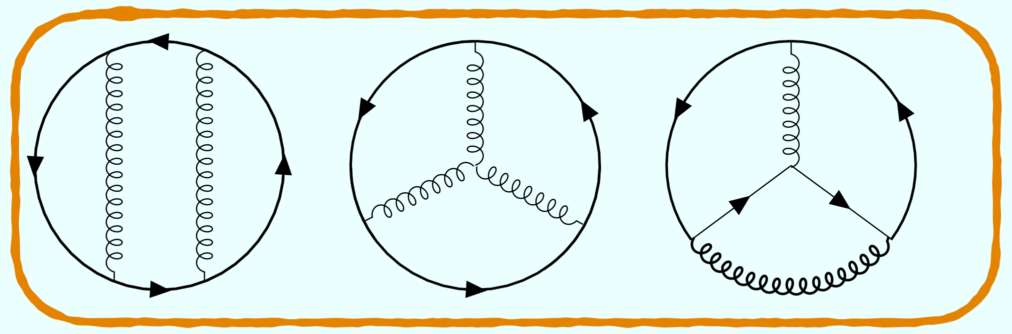
massive quark pressure

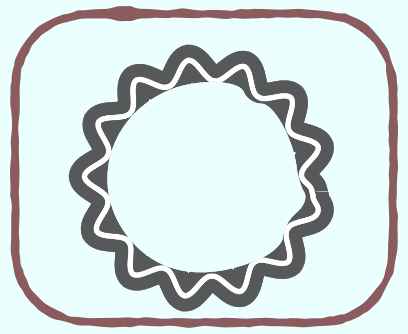
$$P(m,\mu)=P_{\rm LO}(m,\mu)+P_{\rm NLO}(m,\mu)+P_{\rm NNLO}(m,\mu)$$

$$P_{\rm LO}(m,\mu)=\Theta(\mu^2-m^2)\frac{N_c}{12\pi^2}\left[\mu\,p_F\left(\mu^2-\frac{5}{2}m^2\right)+\frac{3}{2}m^4\ln\left(\frac{\mu+p_F}{m}\right)\right]$$

$$P_{\rm NLO}(m,\mu)=-\Theta(\mu^2-m^2)\frac{d_Ag^2}{4(2\pi)^4}\left\{3\left[m^2\ln\left(\frac{\mu+p_F}{m}\right)-\mu\,p_F\right]^2-2p_F^4\right\}$$
$$-\Theta(\mu^2-m^2)\frac{d_Ag^2}{4(2\pi)^4}m^2\left(4-6\ln\frac{m}{\Lambda}\right)\left[\mu\,p_F-m^2\ln\left(\frac{\mu+p_F}{m}\right)\right]$$

α_s





Generalized for arbitrary massive quarks

$$\mathcal{M}_3^{N_f=2+1^*}=\frac{d_A\mu^4}{2\pi^2}\Big\{-\hat{m}^2[(11C_A-2N_f)z+18C_F(2z-\hat{u})](L_m)^2$$
$$-\frac{1}{3}\Big[C_A\Big(22\hat{u}^4-\frac{185}{2}z\hat{m}^2-33z^2\Big)+\frac{9C_F}{2}(16\hat{m}^2\hat{u}(1-\hat{u})-3(7\hat{m}^2-8\hat{u})z-24z^2)$$
$$-N_f(4\hat{u}^4-13z\hat{m}^2-6z^2)\Big]L_m+C_A\left(-\frac{11}{3}\ln\frac{\hat{m}}{2}-\frac{71}{9}+G_1(\hat{m})\right)+C_F\left(\frac{17}{4}+G_2(\hat{m})\right)$$
$$+N_f\left(\frac{2}{3}\ln\frac{\hat{m}}{2}+\frac{11}{9}+G_3(\hat{m})\right)+G_4(\hat{m})\Big\},$$

α_s^2

$$\Omega_{\rm Ring}^{N_f=2+1^*}=\frac{d_A\,g^4}{512\pi^6}\Big\{(\vec{\mu}^2)^2\Big[\left(2\ln\left(\frac{g}{4\pi}\right)-\frac{1}{2}\right)+\frac{1}{2}\left(-\frac{19}{3}+\frac{2\pi^2}{3}+\frac{I_{15}(\vec{\mu})}{(\vec{\mu}^2)^2}+\frac{16}{3}(1-\ln 2)\ln(2)+I_{16}(\hat{m},\vec{\mu}^2)\right)\Big]$$
$$+2\mu^2\sum_{i=1}^{N_l=2}\mu_i^2\left[I_{14}\left(2\ln\left(\frac{g}{4\pi}\right)-\frac{1}{2}\right)+\frac{1}{2}\left(I_{17}(\hat{m},\mu_i)+\frac{16}{3}(1-\ln 2)\ln(2)\,I_{18}+I_{19}(\hat{m},\vec{\mu}^2)\right)\right]$$
$$+\mu^4\left[I_{13}\left(2\ln\left(\frac{g}{4\pi}\right)-\frac{1}{2}\right)+\frac{1}{2}\left(I_{20}(\hat{m})+\frac{16}{3}(1-\ln 2)\ln(2)I_{21}+I_{22}(\hat{m},\vec{\mu}^2)\right)\right]\Big\},\quad \vec{\mu}=(\mu_1,\mu_2)$$

$$\Omega_{{\rm VM},x}^{N_f=2+1^*}=d_A\frac{g^4}{(4\pi)^2}\frac{m^4}{12\pi^2}\sum_{i=1}^{N_l=2}I_x\left(\frac{\mu_i}{m+\mu_i}\right)$$

$\alpha_s^2\ln\alpha_s$

(Without exhaustive detail)

Cold Quark Matter At NNLO

With RGOPT

Setup:

(m, μ) (m, μ) $(m + m_s, \mu)$

$\underbrace{m + m_s}_{m_3}$

Need generalization for three massive quarks with different masses !

We reproduced* existing results based on their detailed calculation and derived the generalization to 3 massive quarks including mixing terms

*We identified a mistake in G2 but numerically irrelevant for physical quark masses.

LF & J-L. Kneur, PRD 111, 034020 (2025)

Flashing expressions for the curious

Vacuum contributions with two masses $m_3 = m + m_s$

$$P^{v,N_f=2^*+1^*}_{\rm NNLO} = \left(\frac{g^4}{\pi^2 (4\pi)^4}\right)\Big((N_f-1)m^4\left(-156.833+253.775L_m-243L_m^2+100L_m^3\right)$$
$$+2(N_f-1)m^2m_3^2\left(-4.75802-6\left(L_m+L_{m_3}\right)\right)$$
$$+m_3^4\left(-152.075+265.775L_{m_3}-243L_{m_3}^2+100L_{m_3}^3\right)\Big),$$

Contribution from the vac. anomalous dimension

$$P^{N_f=2^*+1^*}_{\rm sub} \equiv -(N_f-1)\frac{m^4}{g^2}\left(s_0+s_1 g^2\right) - \frac{m_3^4}{g^2}\left(s_0+s_1 g^2\right)$$
$$-(N_f-1)s_{2,1}m^4 g^2 - s_{2,3}\, m_3^4 g^2 - 2(N_f-1)s_2^{nd}\, m^2 m_3^2\, g^2.$$

Modify $\mathcal{M}_3^{N_f=2+1^*}$ *to account for multiple masses*

$$N_f\left(\frac{2}{3}\ln\frac{\hat m}{2}+\frac{11}{9}+G_3(\hat m)+G_4(\hat m)\right) \to N_f\left(\frac{2}{3}\ln\frac{\hat m_i}{2}+\frac{11}{9}+G_3^{\rm bis}(\hat m_i)\right)+\hat m_i^2 z(m_i)\sum_j^{N_f}\Big\{{\rm Li}_2(\nu_{ij})\left(1+\nu_{ij}^{-2}\right)$$
$$-\Phi\left(\nu_{ij},2,\frac{3}{2}\right)(1+\nu_{ij})-\nu_{ij}^{-2}+\mathcal{F}(\nu_{ij})\Big\} - \sum_j^{N_f}\frac{4}{3}I_{12}^{\rm bis}(\hat m_i,\nu_{ij}),$$

(Without exhaustive detail)

Cold Quark Matter At NNLO

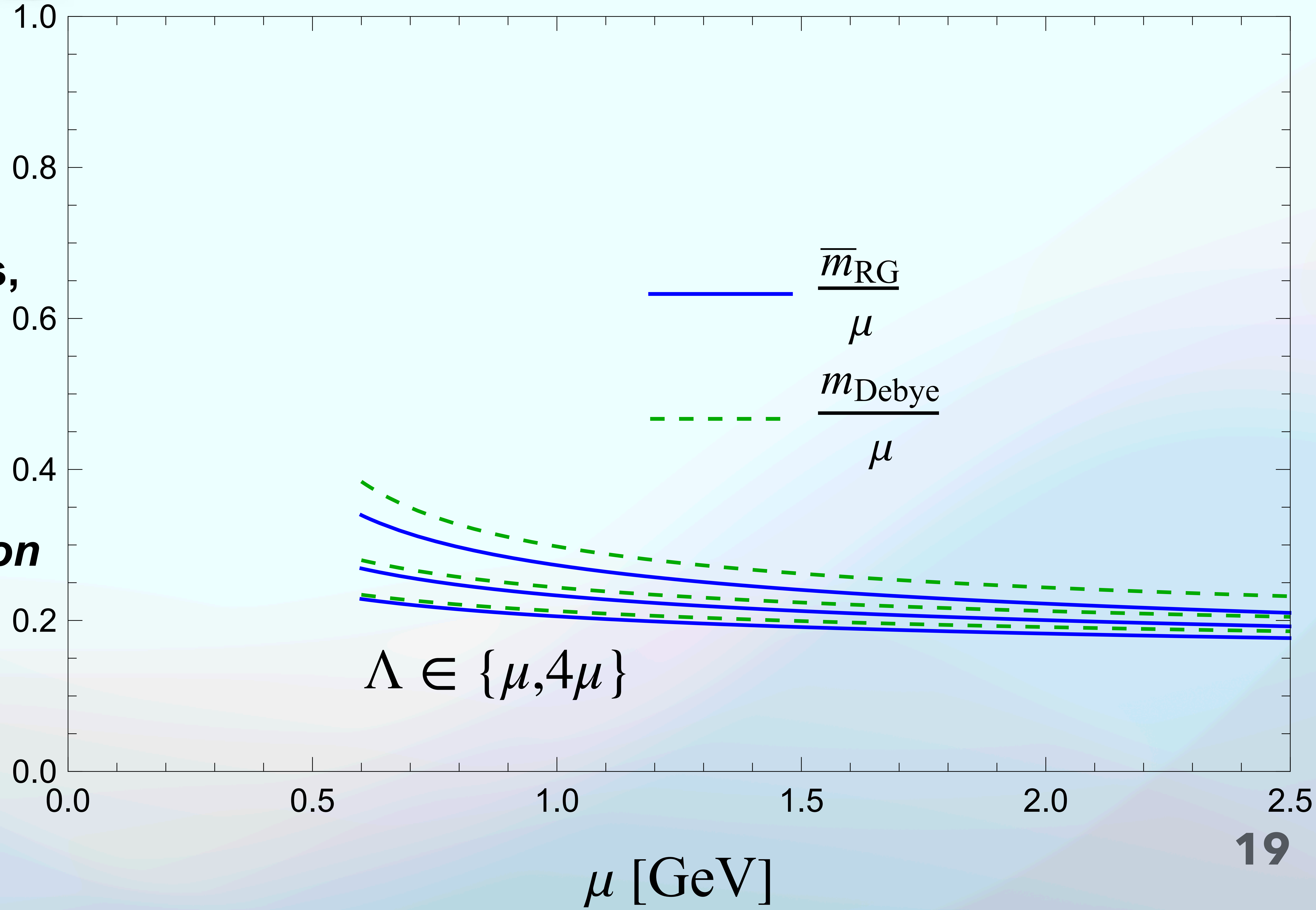
With RGOPT

Setup: Up (m,μ) d (m,μ) s $(m+\cancel{m_s},\mu)$

Solving the RG (and det.) equations, we get $\overline{m}_{\rm RG}$

This solution agrees with the Debye mass at large densities (small α_s) but encodes an *all order resummation*

$$m_{\rm D}^2 = \frac{g^2 C_F}{8\pi^2}\mu^2.$$



Cold Quark Matter At NNLO

With RGOPT

Setup:

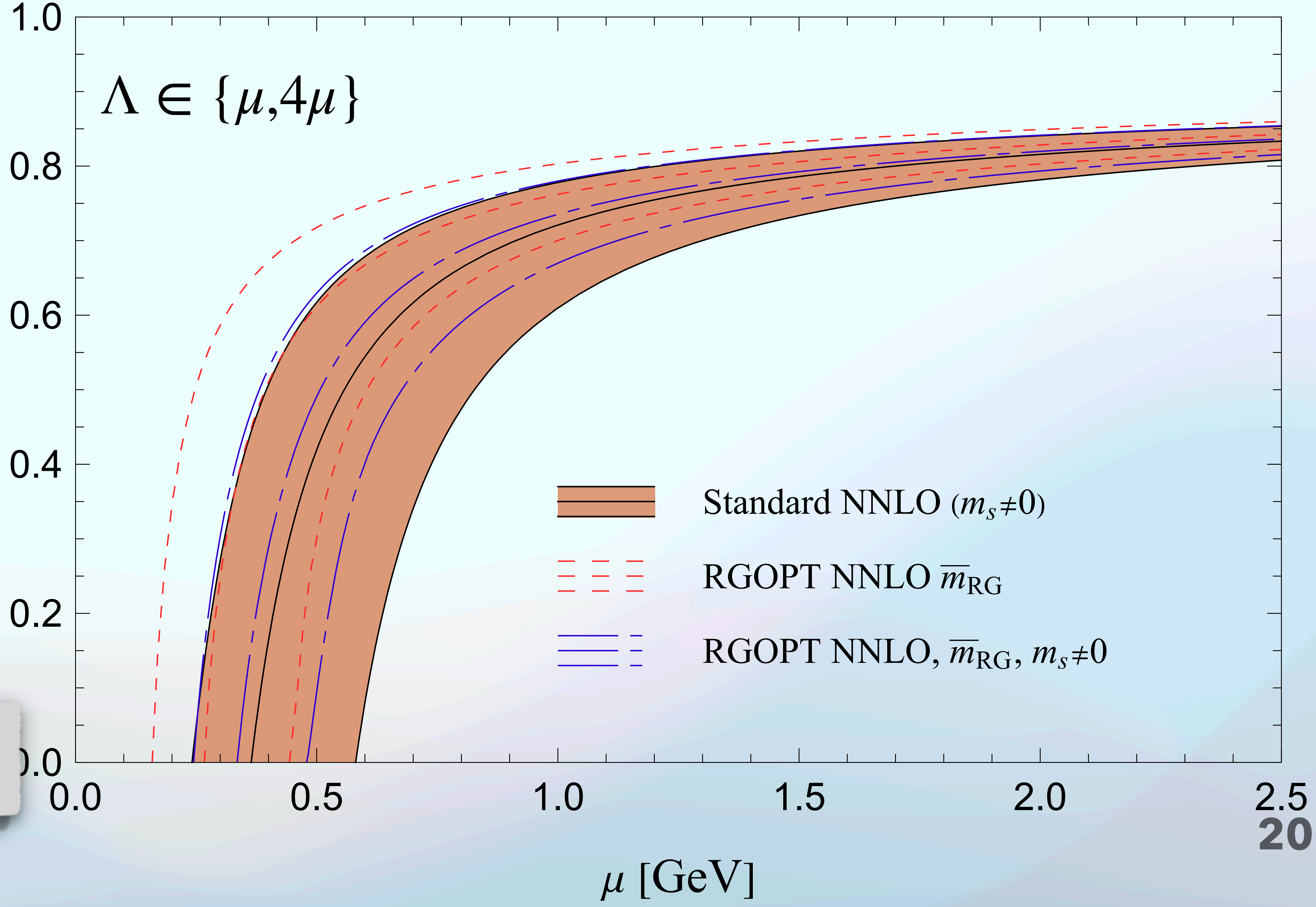
(m, μ) (m, μ) $(m + m_s, \mu)$

Implementing the solutions

Deviation from $\overline{\text{MS}}$

$$\{\overline{m}_{\text{RG}}, \tilde{B}_2\}$$

$$\frac{P}{P_f}$$



Noticeably reduced dependence with respect to renormalization scale variations.

$$\frac{\mathcal{P}^{N_f=2^*+1^*}_{\text{RGOPT}}(\mu, \Lambda = X\,\mu)}{\mathcal{P}_f(\mu, N_f = 3)} = (c_1 + c_2 X^{\nu_3}) - \frac{d_1 (3\tilde{\mu})^{\alpha_1} X^{\nu_1}}{(3\tilde{\mu} - d_2\, X^{-\nu_2})},$$

Pocket formula

Cold Quark Matter At NNLO

With RGOPT

Setup:

(m,μ) (m,μ) $(m+m_s,\mu)$

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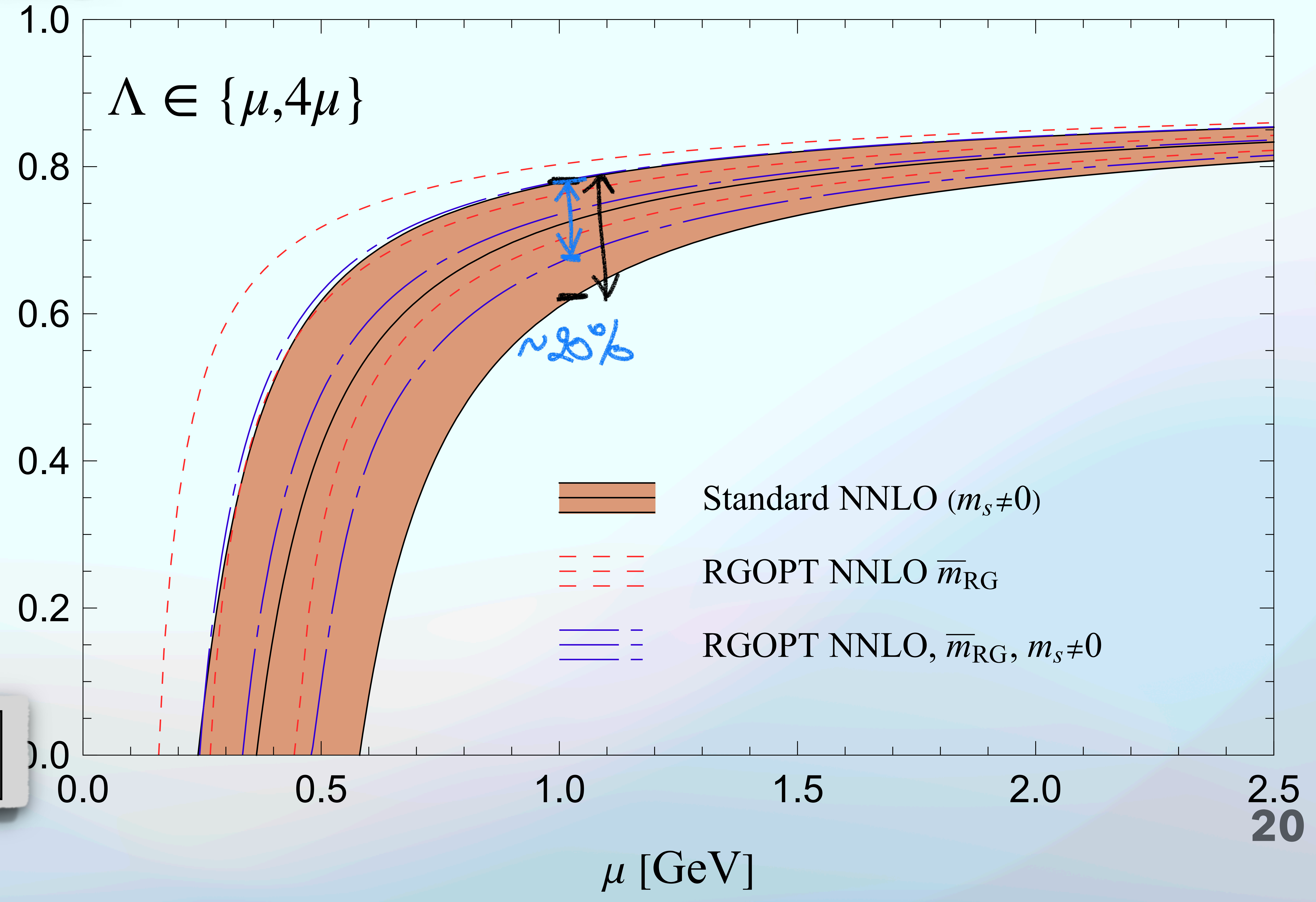


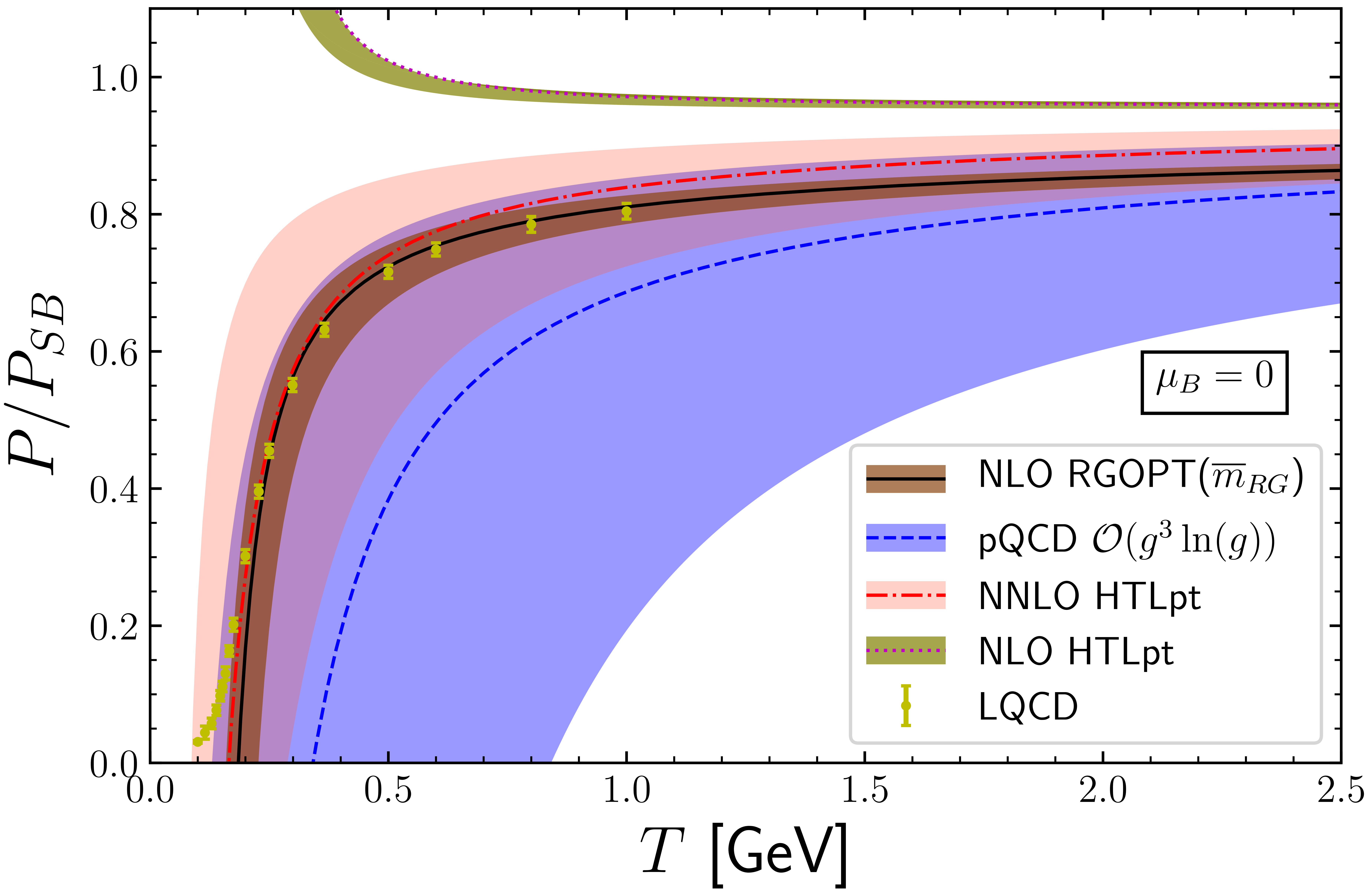
Summary

(1) RGOPT has been quite effective at zero temperature and at high temperature.

(2) Impressive reduction in the renormalization scale dependence (theoretical uncertainty) thanks to RGOPT

(3) The apparent "freedom" in choosing a prescription for "m" is in practice not one since only one prescription is physically relevant.

(4) Application of RGOPT at NNLO for hybrid stars and quark stars is soon to be published !


1.0
0.8
0.6
0.4
0.2
0.0
P/P_{SB}
0.0
0.5
1.0
1.5
2.0
2.5
T [GeV]
$\mu_B = 0$
NLO RGOPT($\overline{m}_{RG}$)
pQCD $\mathcal{O}(g^3 \ln(g))$
NNLO HTLpt
NLO HTLpt
LQCD

Optimized/Screened Perturbation Theory (OPT/SPT)

In hot/dense QCD, the fields develop medium-dressed masses.
It seems best to start from a massive action. OPT does so and optimize the parameter (mass).

$$\mathcal{L}(\alpha_s) \to \mathcal{L}(\delta\alpha_s, m(1-\delta))$$

variational parameter

Interpolation

and fix m **through local minima**

$$\left.\frac{\partial P}{\partial m}\right|_{m=\tilde{m}} = 0$$

OPT eq.

SPT

Karsch, Patkos, Petreczky (1997)

Parwani (1992)

HTLpt = OPT + HTL

Andersen, Braaten, Petitgirard, Strickland (2002)

Andersen, Petitgirard, Strickland (2004)

Andersen, Leganger, Strickland, Su, (2011)

Haque, Andersen, Mustafa, Strickland, Su,(2014)

Renormalization Group Optimized Perturbation Theory (RGOPT)

recipe

① Starting from the massless action, introduce a *fictitious* mass for the fields *(here, the quarks)*.

Keep the vacuum contributions !

RGOPT

Kneur, Neveu, QCD T=0, (2013)

Kneur, Neveu , QCD T=0, (2015)

LF, Kneur, $\lambda\phi^4, \mathrm{NNLO},$ (2021)

Kneur, Pinto, Restrepo Hot QCD(2021)

Very close to Lattice results for $T \gtrsim T_c$

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Vacuum energy anomalous mass dimension

② $$\mathcal{L}(\alpha_s, m) \rightarrow \mathcal{L}(\alpha_s, m) + m^4\,\mathbb{1}$$

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④ Evaluate your observable at $O(\alpha_s^k)$ then expand in δ at the same order and take $\delta \to 1$

RGOPT

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Solved once and for all !

Solve at leading order for any m

$a = \frac{\gamma_0}{2b_0}$, **Universal**

RGOPT

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LF, Kneur, $\lambda\phi^4$, NNLO, (2021)

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Very close to Lattice results for $T \gtrsim T_c$

Backup

Cut results

$$\mathcal{M}_3^{1c} = \frac{d_A \mu^4}{(2\pi)^2}\Bigg\{\Bigg(- C_A\Big[\Big(22\ln\frac{\Lambda}{m} + \frac{185}{3}\Big)\ln\frac{\Lambda}{m} + \frac{1111}{24} - \frac{4\pi^2}{3} + 4\pi^2\ln 2 - 6\zeta(3)\Big]$$
$$- C_F\Big[3\Big(12\ln\frac{\Lambda}{m} + 5\Big)\ln\frac{\Lambda}{m} + \frac{313}{8} + \frac{35\pi^2}{6} - 8\pi^2\ln 2 + 12\zeta(3)\Big] + N_f\Big[\frac{2}{3}\Big(6\ln\frac{\Lambda}{m} + 13\Big)\ln\frac{\Lambda}{m} + \frac{71}{12} + \frac{2\pi^2}{3}\Big]$$
$$+ 6 - 2\pi^2\Bigg)\hat{m}^2 z + 4\hat{m}^2 C_F\Big[3\Big(3\ln\frac{\Lambda}{m} + 4\Big)\ln\frac{\Lambda}{m} + 4\Big](\hat{u} - z)\Bigg\},$$

$$\frac{\mathcal{M}_3^{2c}}{(4\pi)^2} = d_A\Bigg\{C_A\Big(- \frac{16}{9}I_1^2 + \frac{62}{9}m^2 I_2 + \frac{5}{3}I_{1c} - \frac{10}{3}m^2 I_{2c} + I_{10} - \frac{22}{3}[I_1^2 - 2m^2 I_2]\ln\frac{\Lambda}{m}\Big)$$
$$+ C_F\Big(I_{11} + 24m^2[I_2 - I_{1b}I_1 + 2m^2 I_{2b} + 2m^2 I_8]\ln\frac{\Lambda}{m}\Big) + N_f\Big(\frac{10}{9}I_1^2 - \frac{20}{9}m^2 I_2 - \frac{2}{3}I_{1c} + \frac{4}{3}m^2 I_{2c}$$
$$+ [\frac{4}{3}I_1^2 - \frac{8}{3}m^2 I_2]\ln\frac{\Lambda}{m}\Big) - \frac{2}{3}I_{12}\Bigg\},$$

$$\frac{\mathcal{M}_3^{3c}}{(4\pi)^4} = - d_A\Bigg\{C_A\Big[2I_1 I_2 - 4(I_5 + I_7 - 2m^4 I_6)\Big] + C_F\Big[2I_1^2 I_{1b} - 4I_1 I_2 - 8m^2 I_1 I_{2b} + 8m^2 I_3 + 8m^4 I_{3b} - 2I_4$$
$$+ 8(I_5 + I_7 - 2m^4 I_6) - 8m^2 I_1 I_8 + 8m^4 I_9\Big]\Bigg\},$$

Backup

Renormalization group material

$$\Lambda_{\overline{\rm MS}} = \Lambda e^{-\frac{1}{2b_0 g^2}} \left(\frac{b_0 g^2}{1+\frac{b_1}{b_0}g^2}\right)^{-\frac{b_1}{2b_0^2}} \qquad m_s(\Lambda) = m_s(\Lambda_0)\left(\frac{g^2(\Lambda)}{g^2(\Lambda_0)}\right)^{\frac{\gamma_0}{2b_0}} \left(\frac{1+\frac{b_1}{b_0}g^2(\Lambda)}{1+\frac{b_1}{b_0}g^2(\Lambda_0)}\right)^{\frac{\gamma_1}{2b_1}-\frac{\gamma_0}{2b_0}} \qquad \Lambda\frac{d}{d\Lambda} = \Lambda\frac{\partial}{\partial\Lambda} + \beta(g^2)\frac{\partial}{\partial g^2} - \gamma_m(g^2)m\frac{\partial}{\partial m}\,,$$

$$\beta\left(g^2 \equiv 4\pi\alpha_s\right) = -2b_0 g^4 - 2b_1 g^6 + \cdots\,, \qquad \gamma_m\left(g^2\right) = \gamma_0 g^2 + \gamma_1 g^4 + \cdots\,, \qquad \Lambda\frac{d}{d\Lambda}\left[S(m,g)\right] \equiv \hat{\Gamma}^0(g)m^4 = m^4\sum_{k\geq 0}\Gamma_k^0 g^{2k}$$

$$s_0 = \frac{\Gamma_0^0}{2\left(b_0-2\gamma_0\right)} = \frac{3}{7}\,, \quad s_1 = \frac{1}{2\gamma_0}\left((b_1-2\gamma_1)s_0 - \frac{\Gamma_1^0}{2}\right) = -\frac{53}{224\pi^2}$$

$$s_2(N_h,N_l) = \frac{1}{b_0+2\gamma_0}\left((b_2-2\gamma_2)s_0 - 2\gamma_1 s_1 - \frac{\Gamma_2^0}{2}\right) \underset{N_h=3,N_l=0}{=} -0.00040082$$

$$s_3(N_h,N_l) = \frac{1}{2(b_0+\gamma_0)}\left((b_3-2\gamma_3)s_0 - 2\gamma_2 s_1 - (b_1+2\gamma_1)s_2 - \frac{\Gamma_3^0}{2}\right) \underset{N_h=3,N_l=0}{=} -0.00008304.$$

$$s_{2,1} = s_2(N_h=2,N_l=1) = -0.00024683, \quad s_{2,3} = s_2(N_h=1,N_l=2) = -0.00009284, \quad s_2^{nd} = \frac{9}{8\pi^4(2N_f-81)} \tag{A10}$$

Backup

Renormalization scheme change (RSC)

$$\gamma_2 \to \gamma_2' = \gamma_2 - 4b_0\, B_2$$
$$\gamma_3 \to \gamma_3' = \gamma_3 - 4b_1\, B_2$$

$$s_2 \to s_2' = s_2 + \frac{8b_0 s_0}{b_0 + 2\gamma_0} B_2 = -0.00040082 + \frac{216}{175} B_2,$$
$$s_{2,i} \to s_{2,i}' = s_{2,i} + \frac{216}{175} B_2,\ i = \{1, 3\}$$
$$(s_2^{nd})' = s_2^{nd}$$

$$s_3 \to s_3' = s_3 + \frac{4B_2}{b_0 + \gamma_0}\left(b_0 s_1 - \frac{b_0}{4}(a_{10} + 4a_{11}) + 2\frac{s_0}{b_0 + 2\gamma_0}(b_1\gamma_0 - b_0\gamma_1)\right) = -0.00008304 - \frac{111}{280\pi^2} B_2$$

where $a_{10} = -2(b_0 - 2\gamma_0)s_0,\ a_{11} = -(3/4)a_{10}$.

Backup

Fitting functions

$$G_1(\hat m) = 32\pi^4 \hat m^2 \Big(0.001715 - 0.000339\,\hat u + 0.002818\,\hat u^2 - 0.002282\,\hat u^3 + 0.005854 \ln \hat m - 0.018427\,\hat m^2 \ln^3 \hat m + 0.000444\,\hat m^2 \ln\left(\frac{1+\hat u}{\hat m}\right)\Big)$$

$$G_2(\hat m) = 32\pi^4 \hat m^2 \Big(-0.001363 + 0.000401\,\hat u + 0.003454\,\hat u^2 - 0.002983\hat u^3 + 0.021502\hat u^4 + 0.017914 \ln \hat m - 0.032789\,\hat m^2 \ln^2 \hat m + 0.002067\,\hat m^2 \ln\left(\frac{1+\hat u}{\hat m}\right)\Big)$$

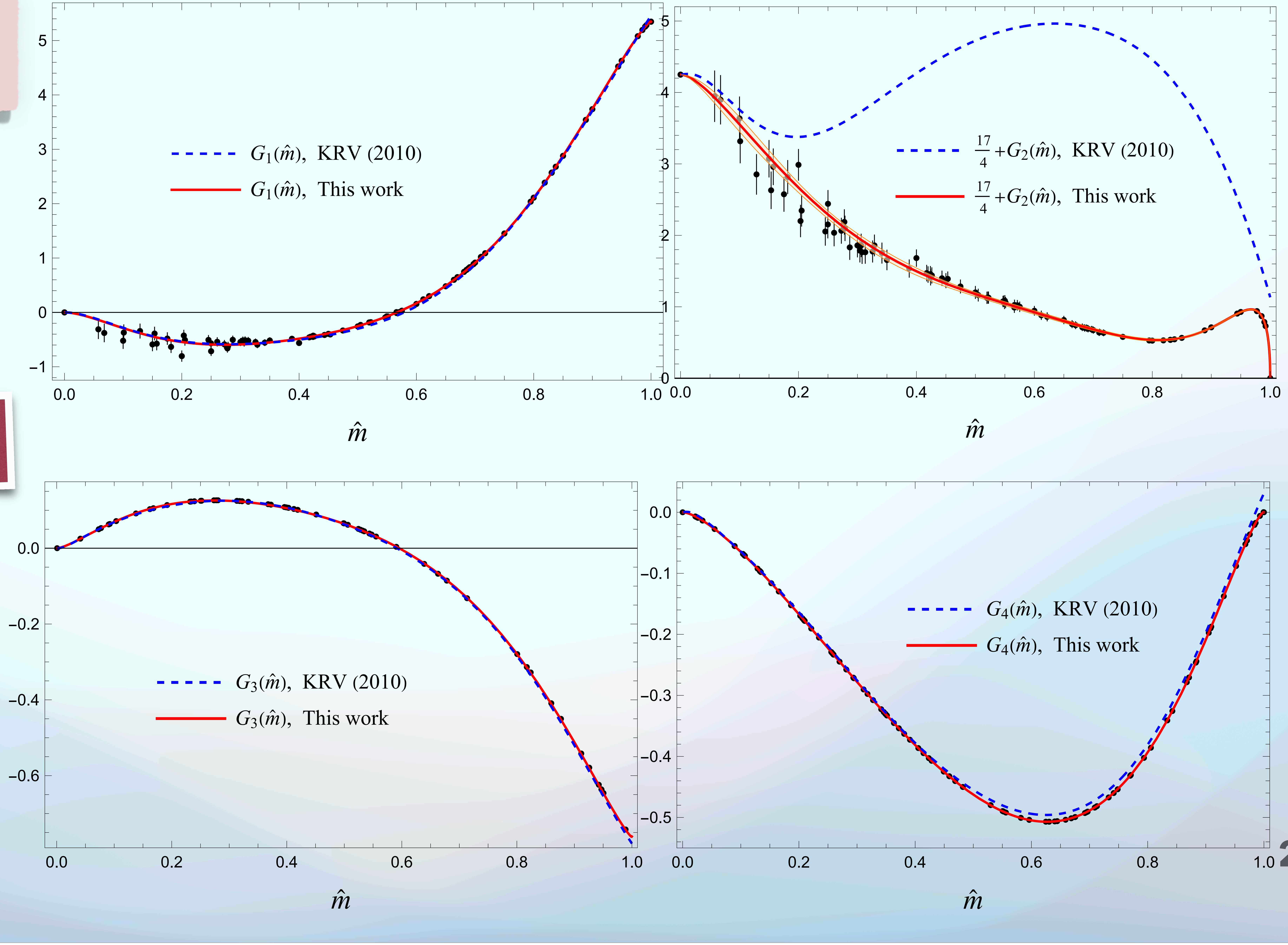
$$G_3(\hat m) = 32\pi^4 \hat m^2 \Big(-0.000244 - 0.002192\,\hat u + 0.000086\,\hat u^2 + 0.001895\,\hat u^3 + 0.000054 \ln \hat m + 0.000521 \ln^2 \hat m + 0.002176\,\hat m^2 \ln\left(\frac{1+\hat u}{\hat m}\right)\Big)$$

$$G_4(\hat m) = 32\pi^4 \hat m^2 \Big(-0.0020405\,\hat u + 0.0003254\,\hat u^3 + 0.0001777\,\hat u^4 + 0.0002580\hat u^5 - 0.0003811 \ln \hat m - 0.0003289 \ln^2 \hat m + 0.0005292\hat m^2 \ln \hat m + 0.0004012\hat m^2 \ln^2 \hat m + 0.0020462\,\hat m^2 \ln\left(\frac{1+\hat u}{\hat m}\right)\Big)$$

$$G_3^{\rm bis}(\hat m) = 32\pi^4 \hat m^2 \Big(-0.000244 - 0.003777\hat u + 0.000319\hat u^2 + 0.001263\hat u^3 + 0.000322 \ln \hat m + 0.000572 \ln^2 \hat m + 0.003743\hat m^2 \ln\left(\frac{1+\hat u}{\hat m}\right)\Big).$$

Backup

G_i comparisons



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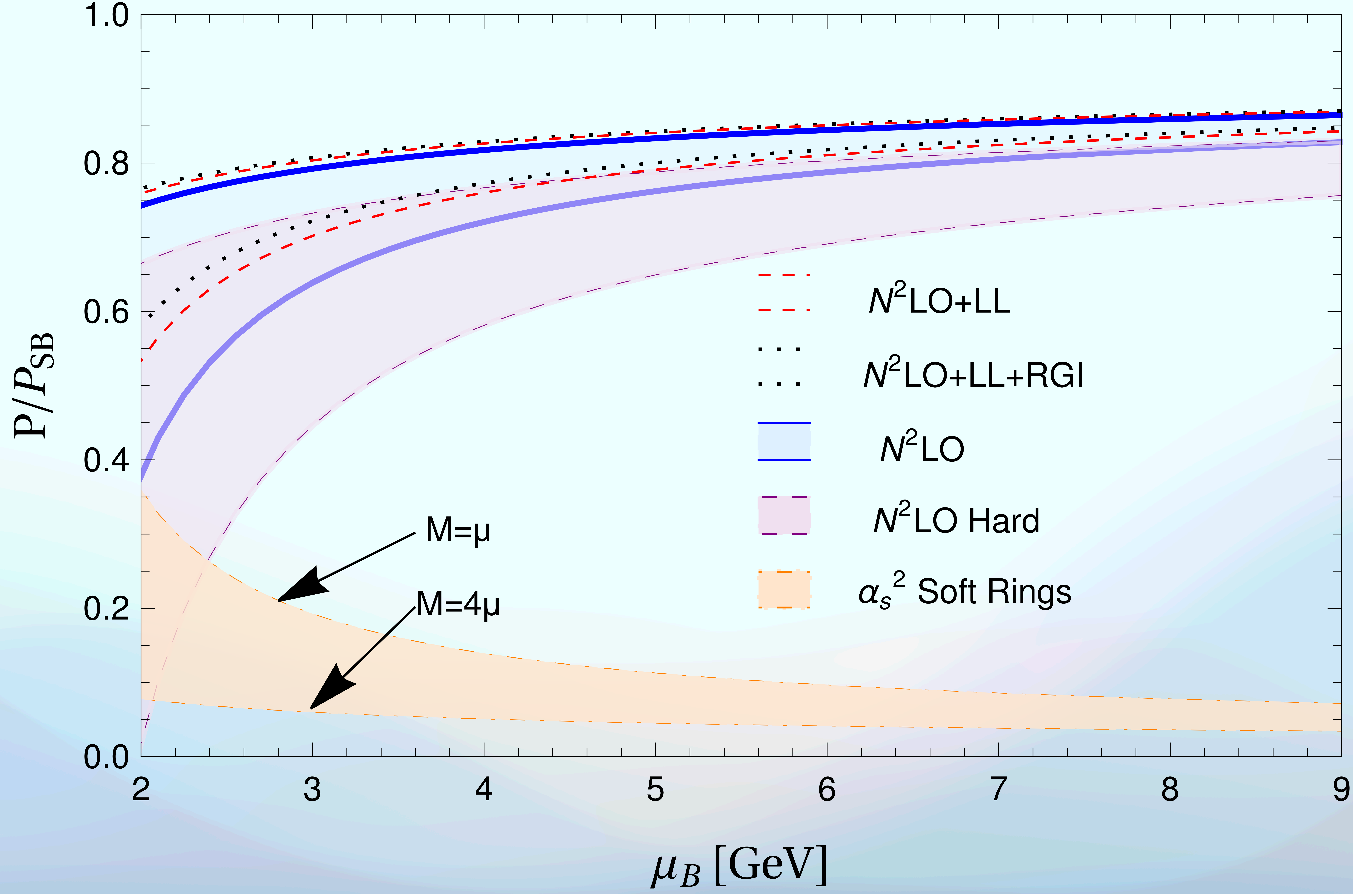

1.0
0.8
0.6
0.4
0.2
0.0
2
3
4
5
6
7
8
9
P/P_SB
μ_B [GeV]
N2LO+LL
N2LO+LL+RGI
N2LO
N2LO Hard
αs2 Soft Rings
M=μ
M=4μ

$$\left[\Lambda\frac{\partial}{\partial\Lambda}\mathcal{P}+\beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right]_{\overline{m}_{RG}}=0$$

$$\left.\frac{\partial P}{\partial m}\right|_{m=\tilde{m}}=0$$

Det. eq. $$\frac{\partial f_{\mathrm{RG}}}{\partial g^2}\frac{\partial f_{\mathrm{OPT}}}{\partial m}-\frac{\partial f_{\mathrm{RG}}}{\partial m}\frac{\partial f_{\mathrm{OPT}}}{\partial g^2}=0$$

Solving RG & Det. : $(\overline{m}_{\mathrm{RG}}, \tilde{B}_2)$

For Cold Quark Matter , the prescription is unique !