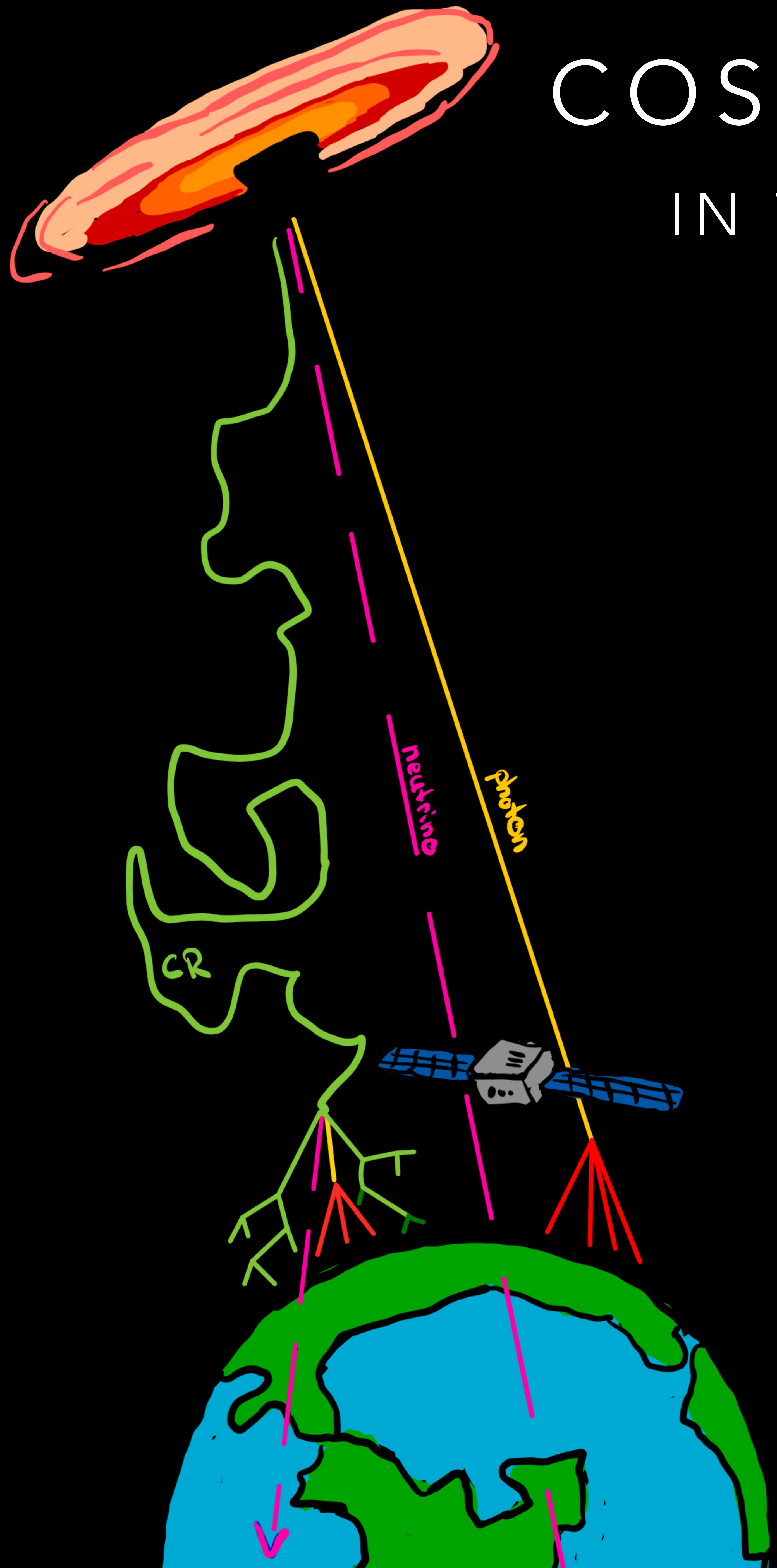
A GENERALIZED FERMI MODEL OF SHEAR ACCELERATION

SOPHIE AERDKER - APC, PARIS & RUHR UNIVERSITY BOCHUM

WITH LUKAS MERTEN, JULIA TJUS, MARTIN LEMOINE, FRANK RIEGER

COSMIC RAY TRANSPORT

IN TURBULENT MAGNETIC FIELDS



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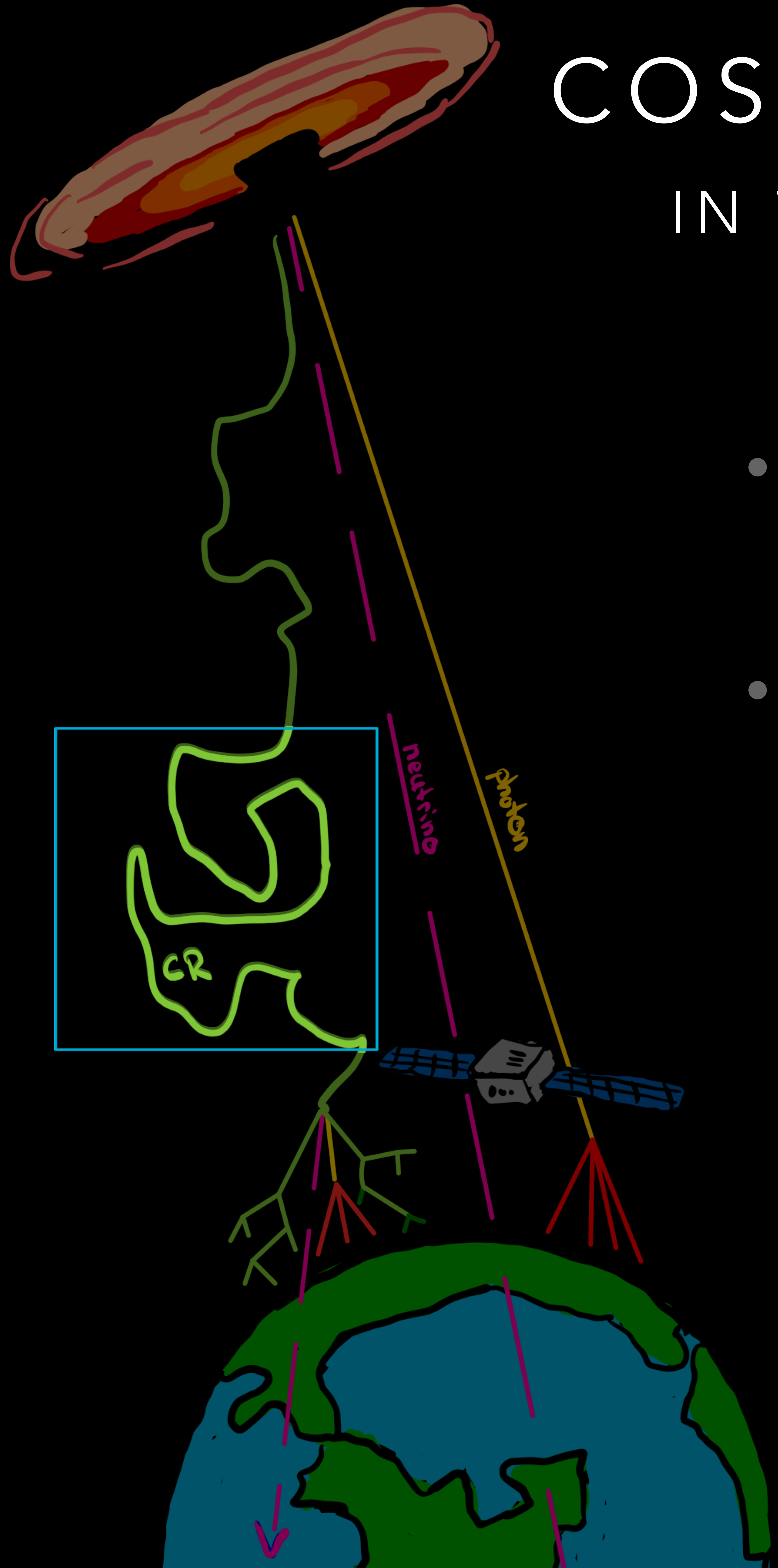
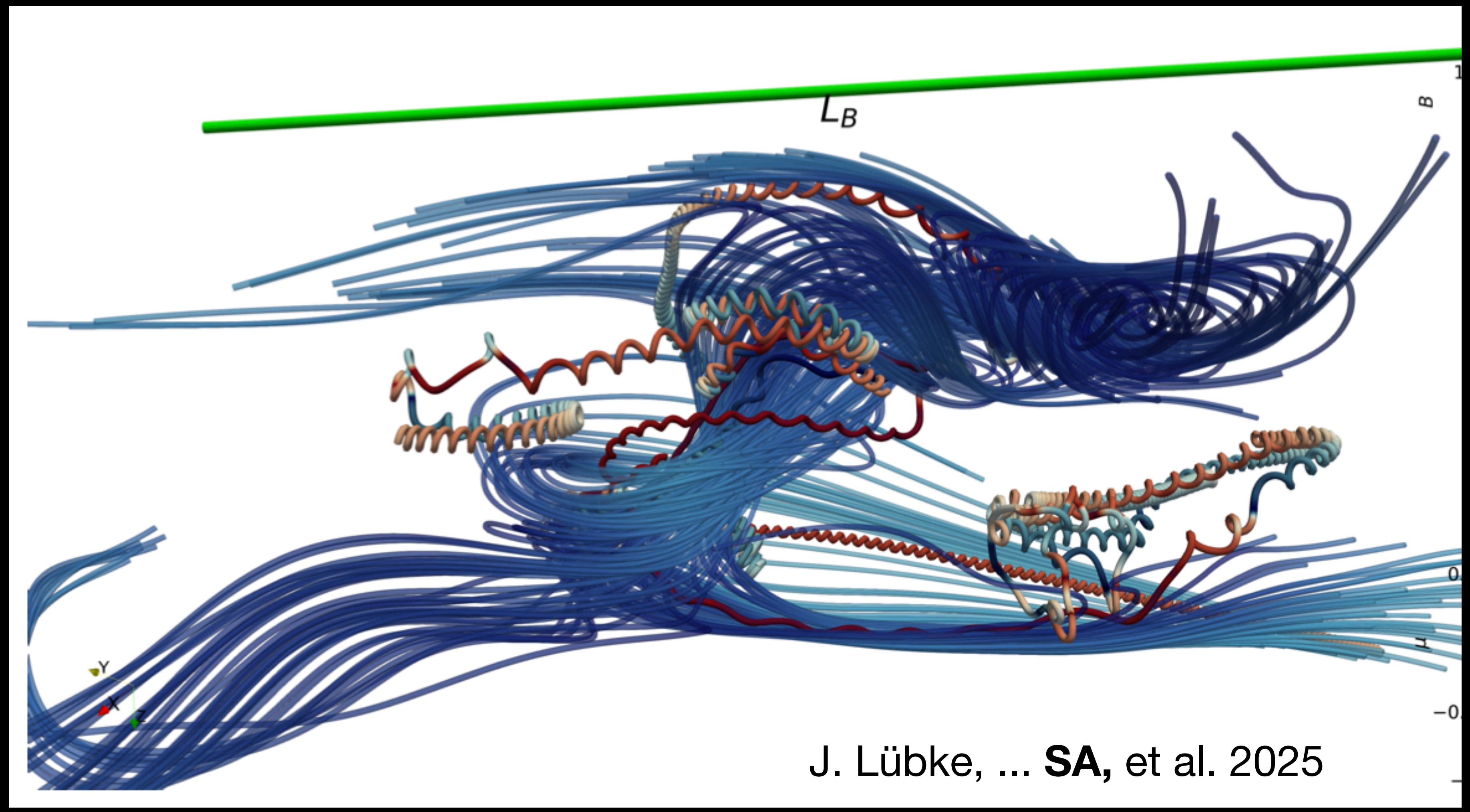


- Charged particles gyrate around magnetic field lines & scatter on inhomogeneties

- Good knowledge of the magnetic field & turbulence



J. Lübke, ... **SA,** et al. 2025

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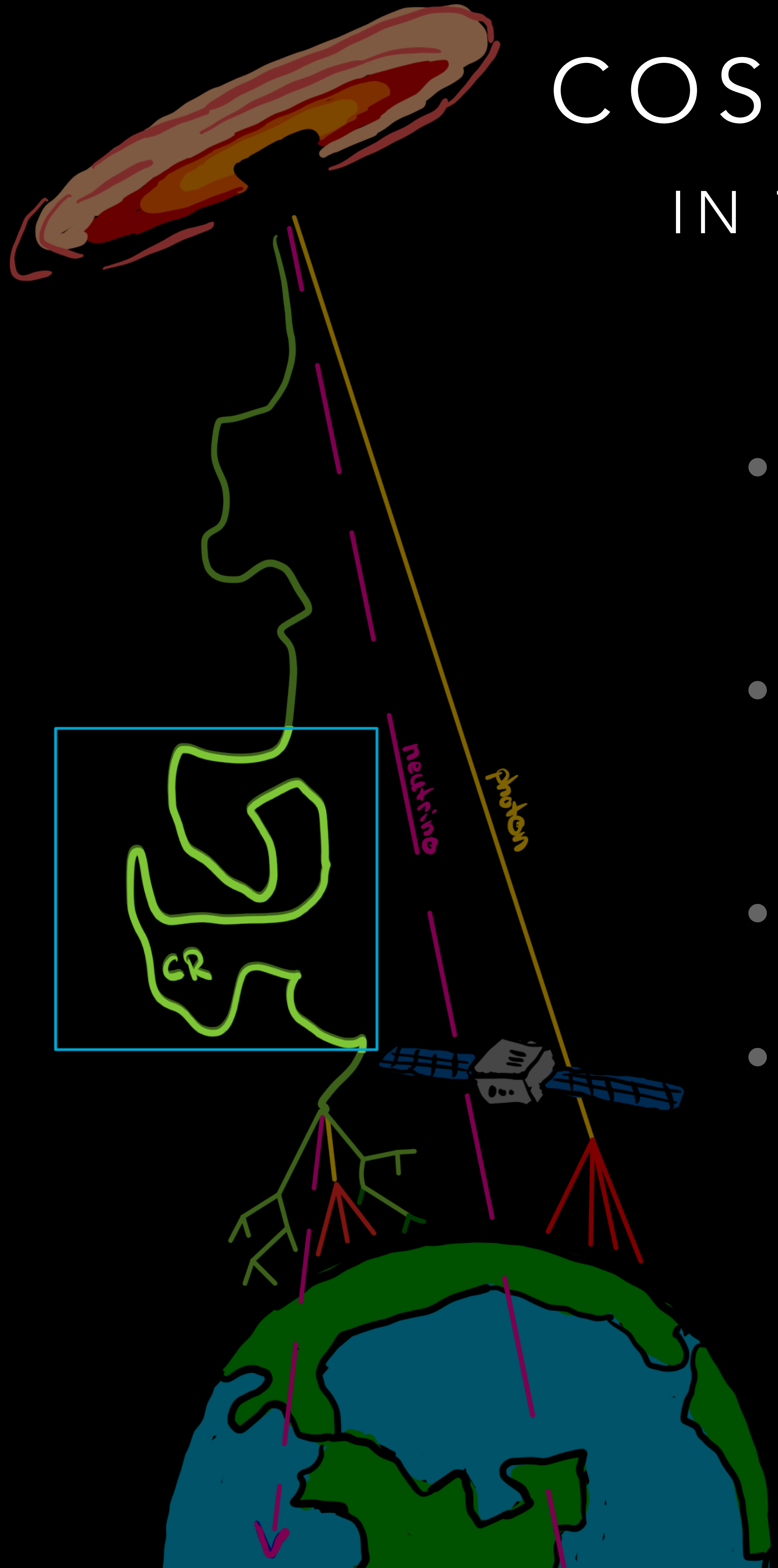
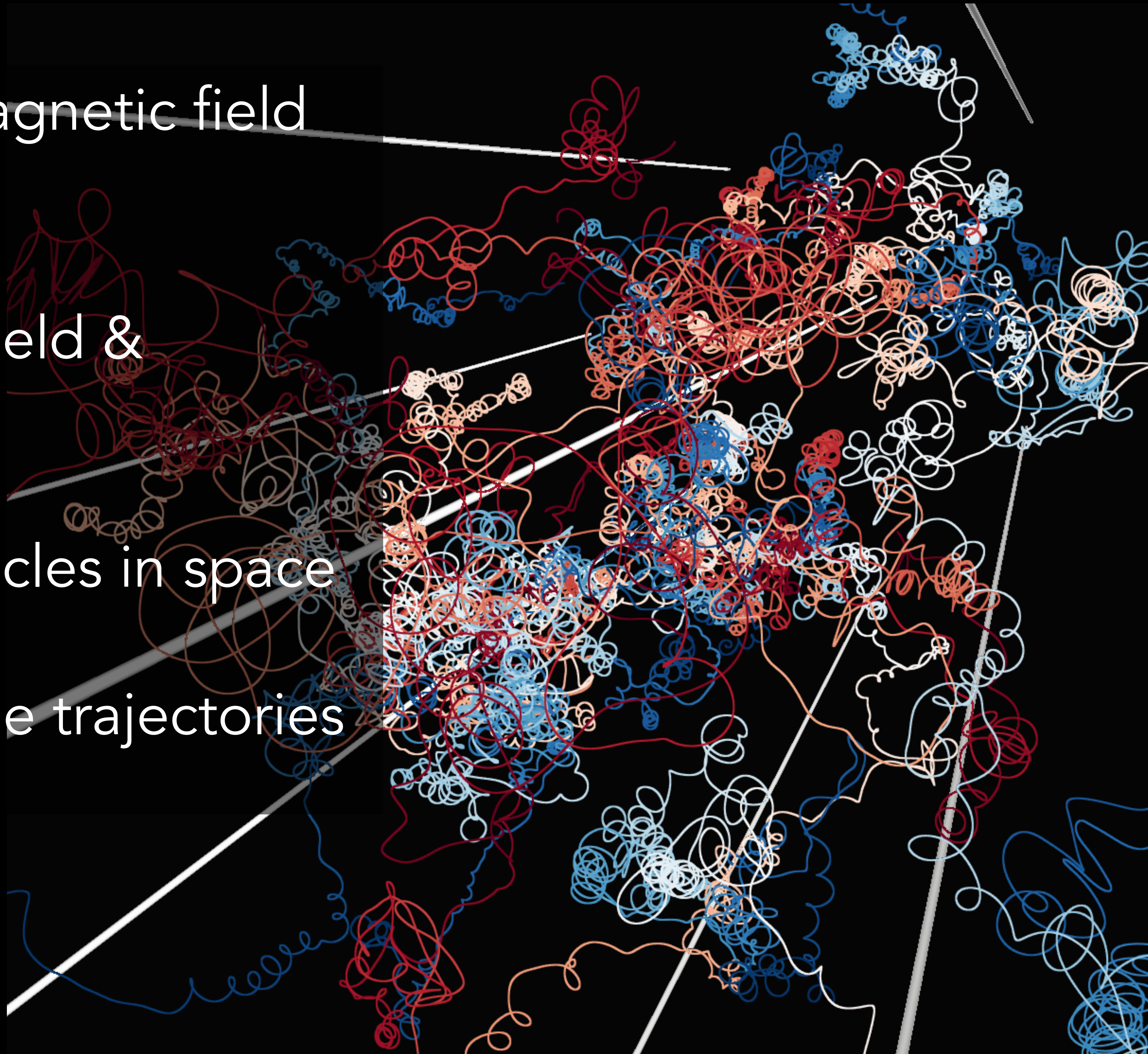


- Charged particles gyrate around magnetic field lines & scatter on inhomogeneties
- Good knowledge of the magnetic field & turbulence
- Scattering leads to diffusion of particles in space
- Unfeasible to track individual particle trajectories



Credit: Lukas Merten

ENSEMBLE AVERAGED APPROACH

TEST PARTICLE TRANSPORT EQUATION

$$\frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f = \nabla \cdot \left(\hat{\kappa} \nabla f\right) + \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D \frac{\partial f}{\partial p} \right) + \frac{p}{3} \left(\nabla \cdot \vec{u} \right) \frac{\partial f}{\partial p} + S(\vec{x}, p, t)$$

advection | spatial diffusion | momentum diffusion | adiabatic energy change

ENSEMBLE AVERAGED APPROACH

TRANSPORT EQUATION - FOKKER-PLANCK EQUATION

$$\frac{\partial n}{\partial t} = \frac{1}{2}\nabla^2(2\hat{\kappa}n) - \nabla\left[\left(\nabla\hat{\kappa} + \vec{u}\right)n\right] + \frac{1}{2}\frac{\partial^2}{\partial p^2}(2Dn) - \frac{\partial}{\partial p}\left[\left(\frac{\partial D}{\partial p} + \frac{2D}{p} - \frac{p}{3}\nabla\cdot\vec{u}\right)n\right] + S_n$$

ENSEMBLE AVERAGED APPROACH

TRANSPORT EQUATION & STOCHASTIC DIFFERENTIAL EQUATIONS

$$\frac{\partial n}{\partial t} = \frac{1}{2}\nabla^2(2\hat{\kappa}n) - \nabla\left[\left(\nabla\hat{\kappa} + \vec{u}\right)n\right] + \frac{1}{2}\frac{\partial^2}{\partial p^2}(2Dn) - \frac{\partial}{\partial p}\left[\left(\frac{\partial D}{\partial p} + \frac{2D}{p} - \frac{p}{3}\nabla\cdot\vec{u}\right)n\right] + S_n$$

FROM FOKKER-PLANCK TO STOCHASTIC DIFFERENTIAL EQUATIONS:

$$\mathrm{d}\vec{x} = \left(\nabla\cdot\hat{\kappa} + \vec{u}\right)\mathrm{d}t + \sqrt{2\hat{\kappa}}\,\mathrm{d}\overrightarrow{\omega}_{x,t}$$

$$\mathrm{d}p = \left(\frac{2D}{p} + \frac{\partial D}{\partial p} - \frac{p}{3}\nabla\cdot\vec{u}\right)\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\omega_{p,t}$$

ENSEMBLE AVERAGED APPROACH

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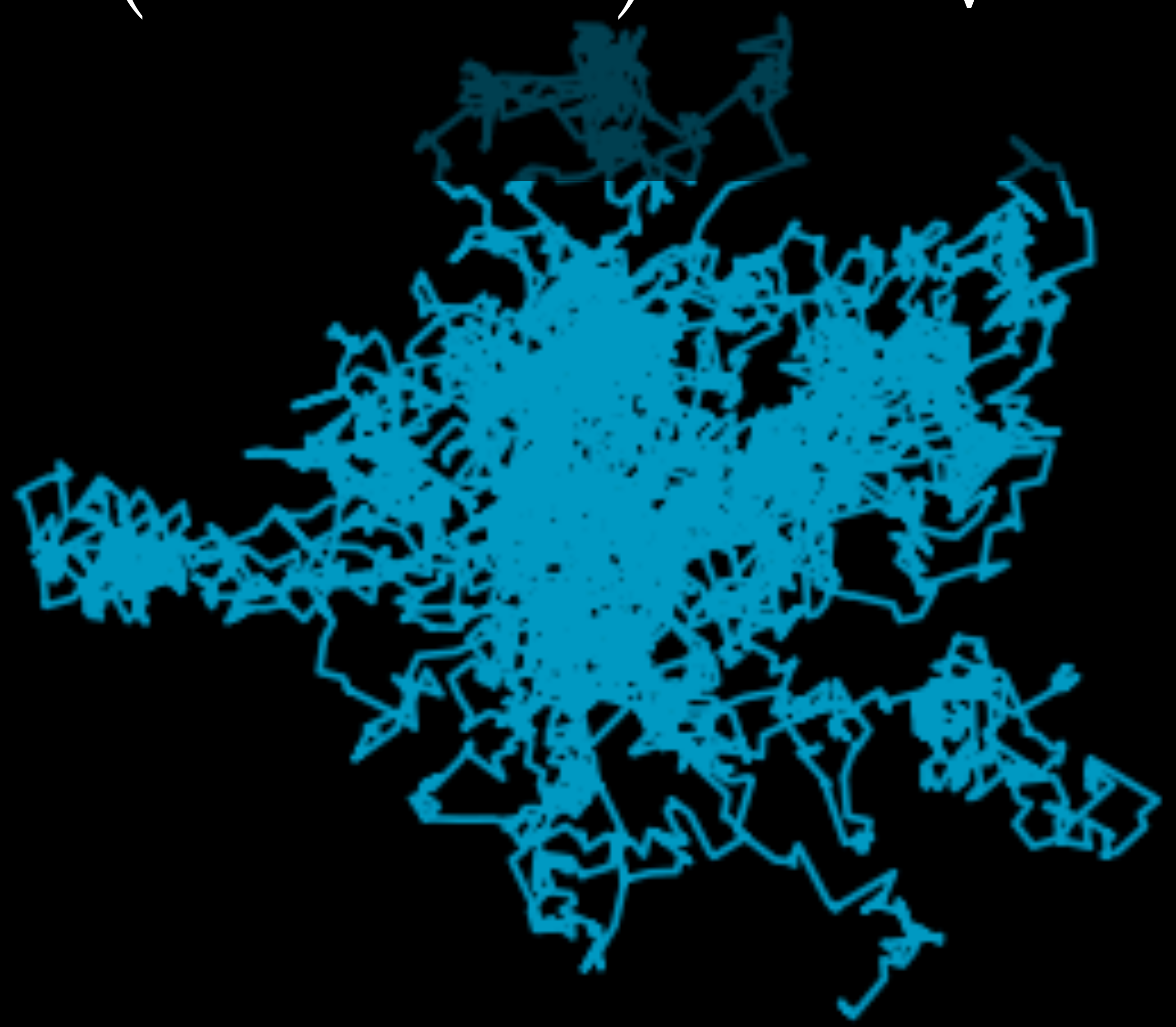


Pseudo-particles propagated with Stochastic Differential Equations

CR/Propa

Cosmic Ray Propagation Framework

L. Merten & SA, CPC, 2025

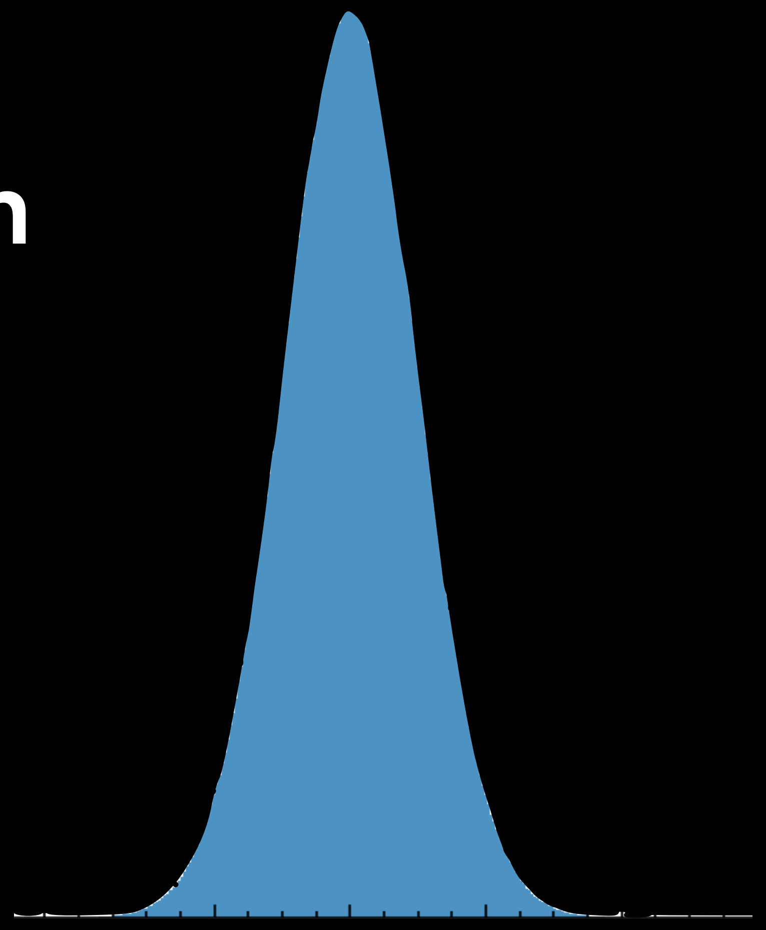
BROWNIAN MOTION

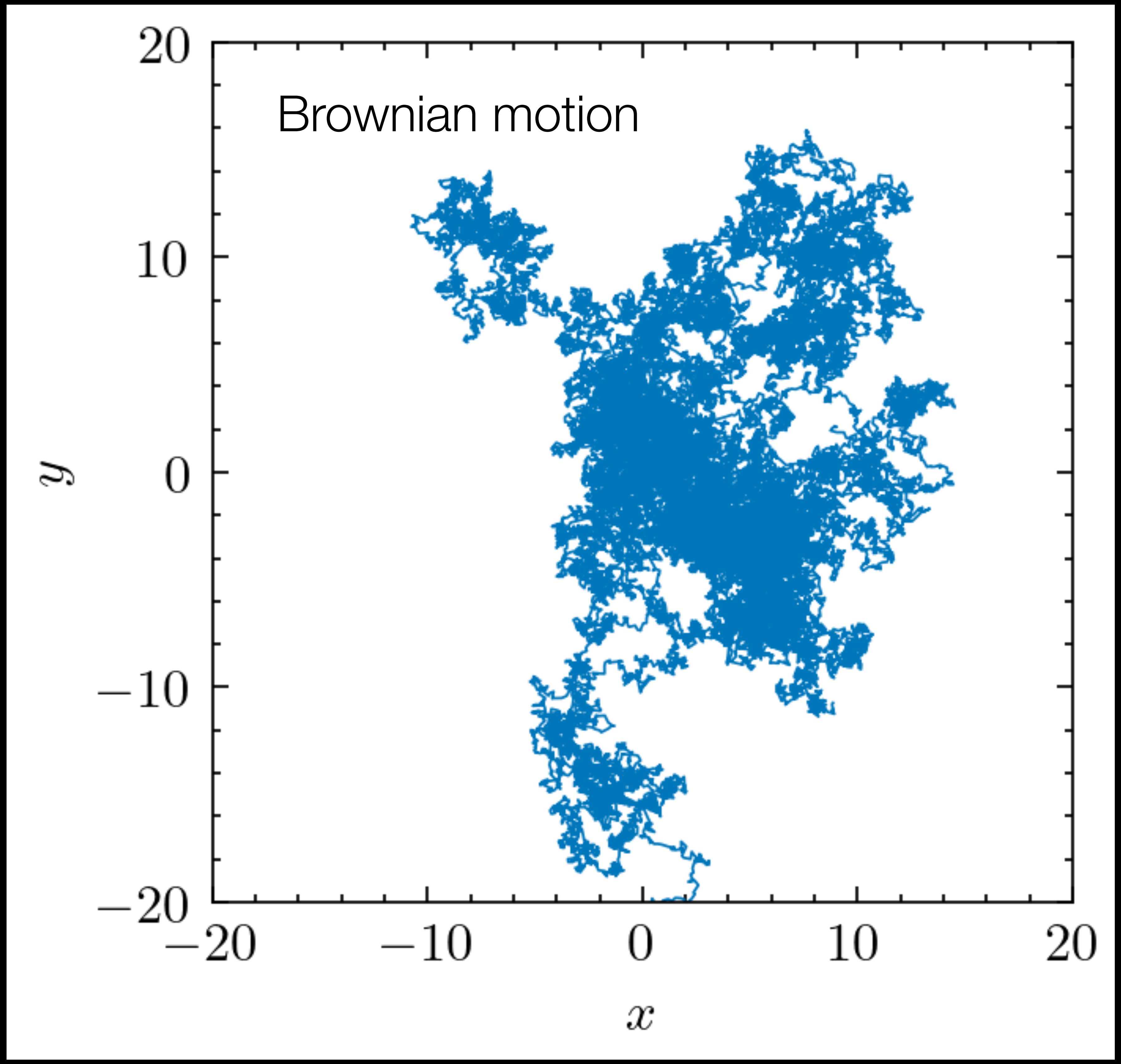
INTEGRATING STOCHASTIC DIFFERENTIAL EQUATIONS

- Pure diffusion, integrated with **Euler-Maruyama** scheme:

$$\vec{x}_{t+1} = \vec{x}_t + \sqrt{2\hat{\kappa}}\sqrt{\Delta t}\vec{\eta}_{x,t}$$

- Random numbers η drawn from a **Gaussian distribution**





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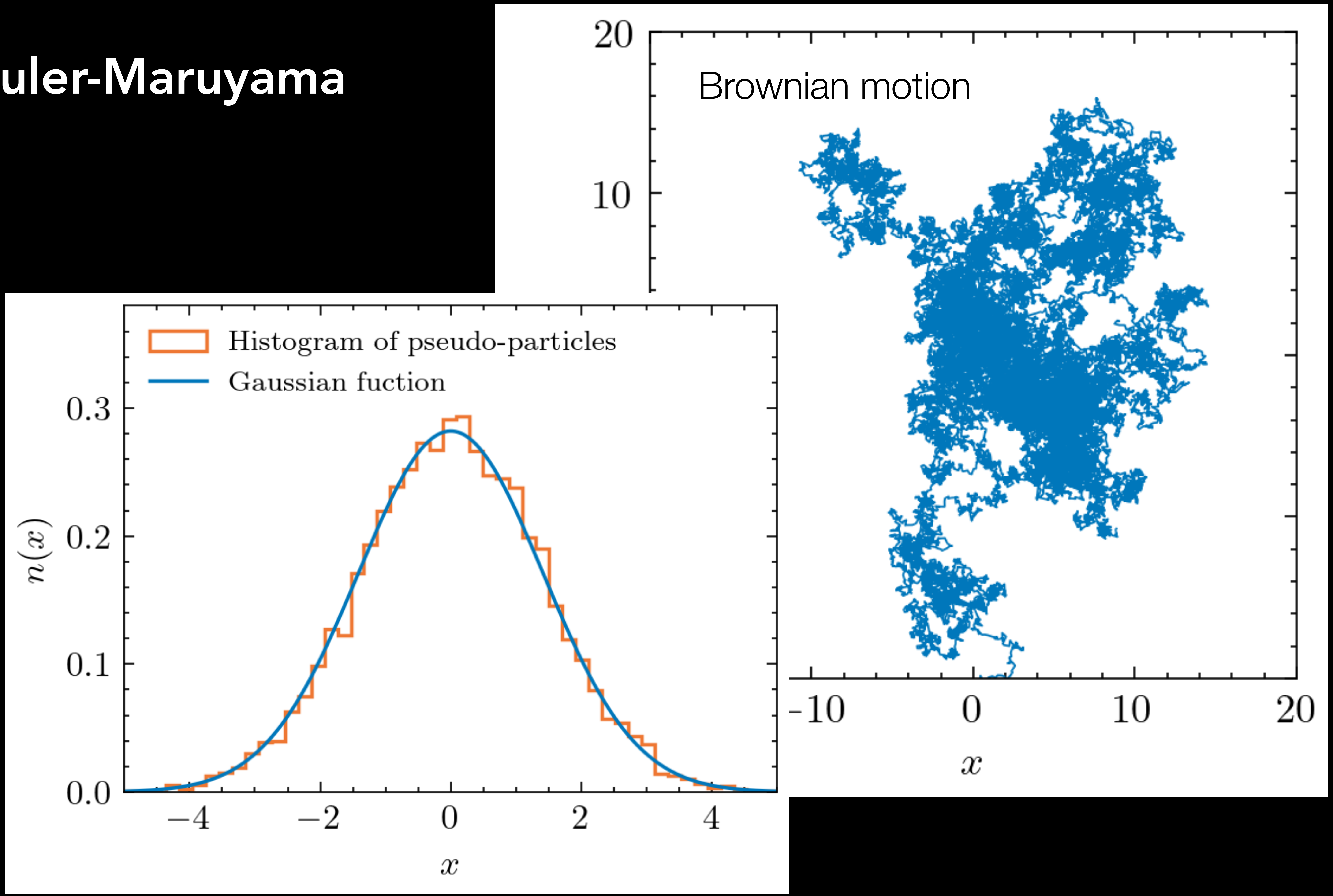
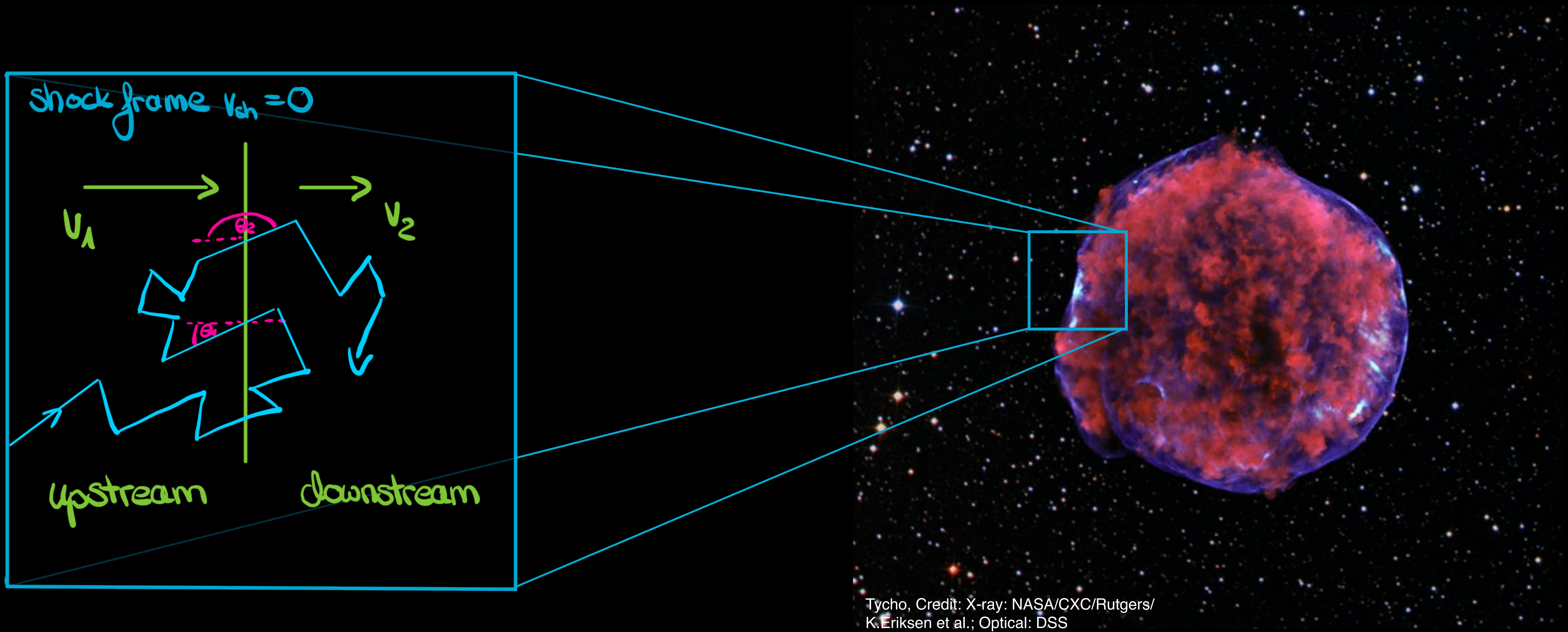
- Recover macroscopic quantities



DIFFUSIVE SHOCK ACCELERATION

Tycho, Credit: X-ray: NASA/CXC/Rutgers/ K.Eriksen et al.; Optical: DSS



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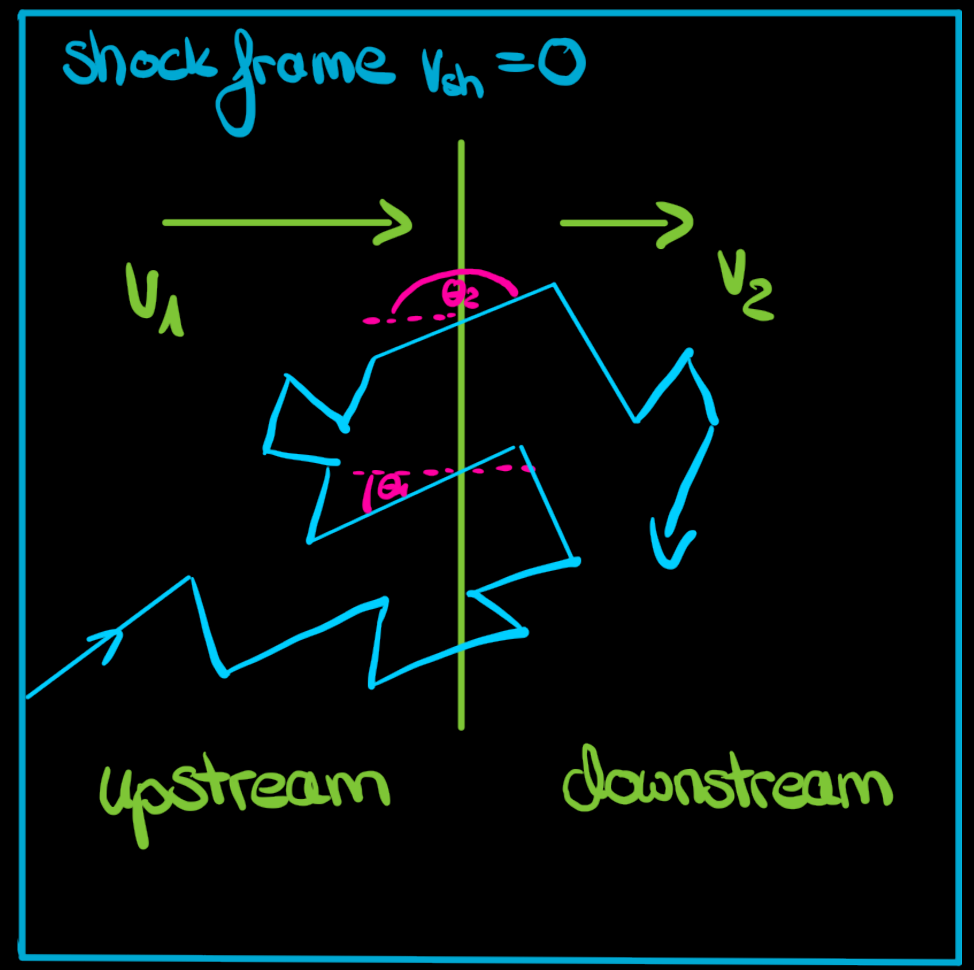
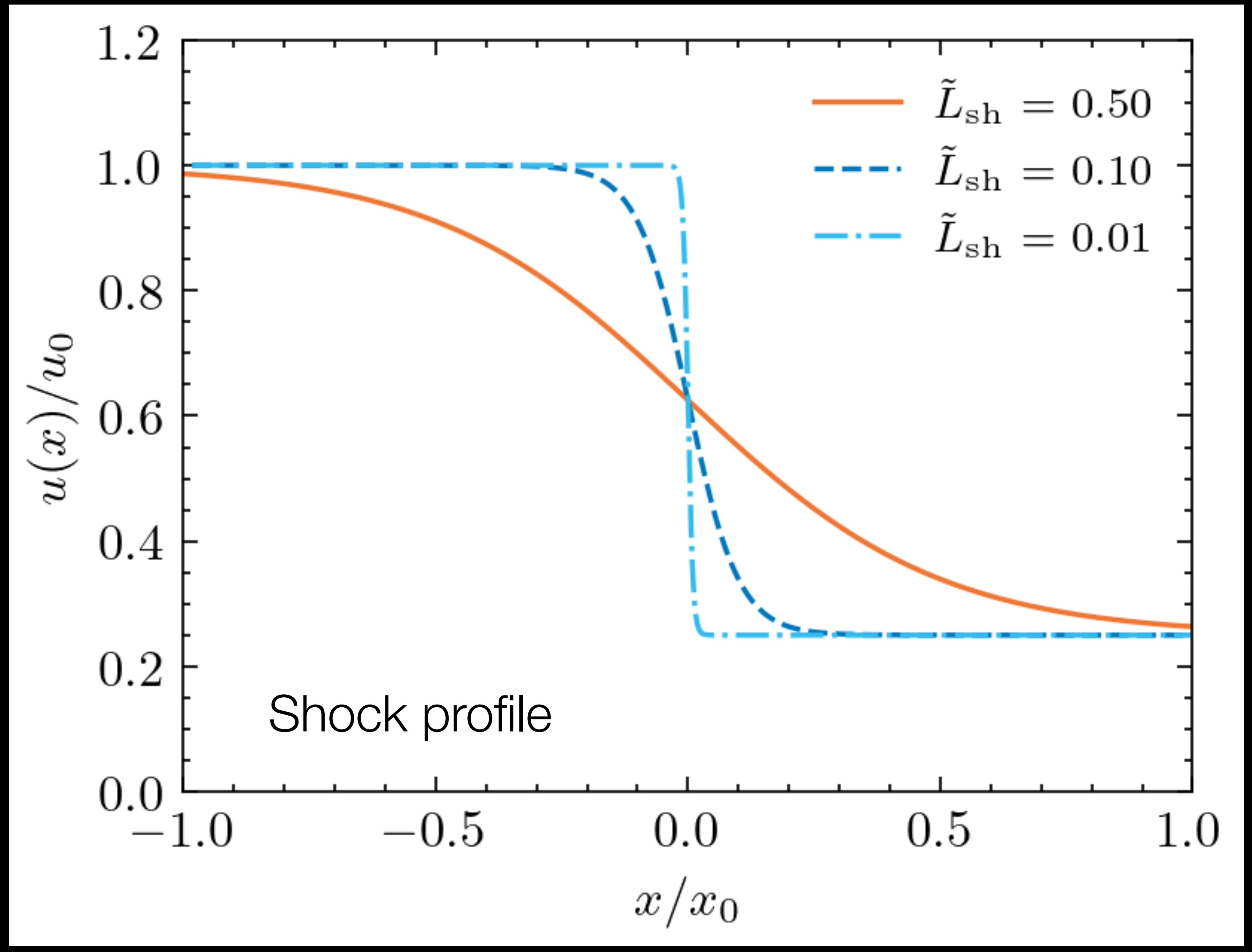
PARTICLE ACCELERATION

DIFFUSIVE SHOCK ACCELERATION



$$\mathrm{d}x = u(x)\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_{x,t}$$

$$\mathrm{d}p = -\frac{p}{3}\frac{\partial u}{\partial x}\mathrm{d}t$$



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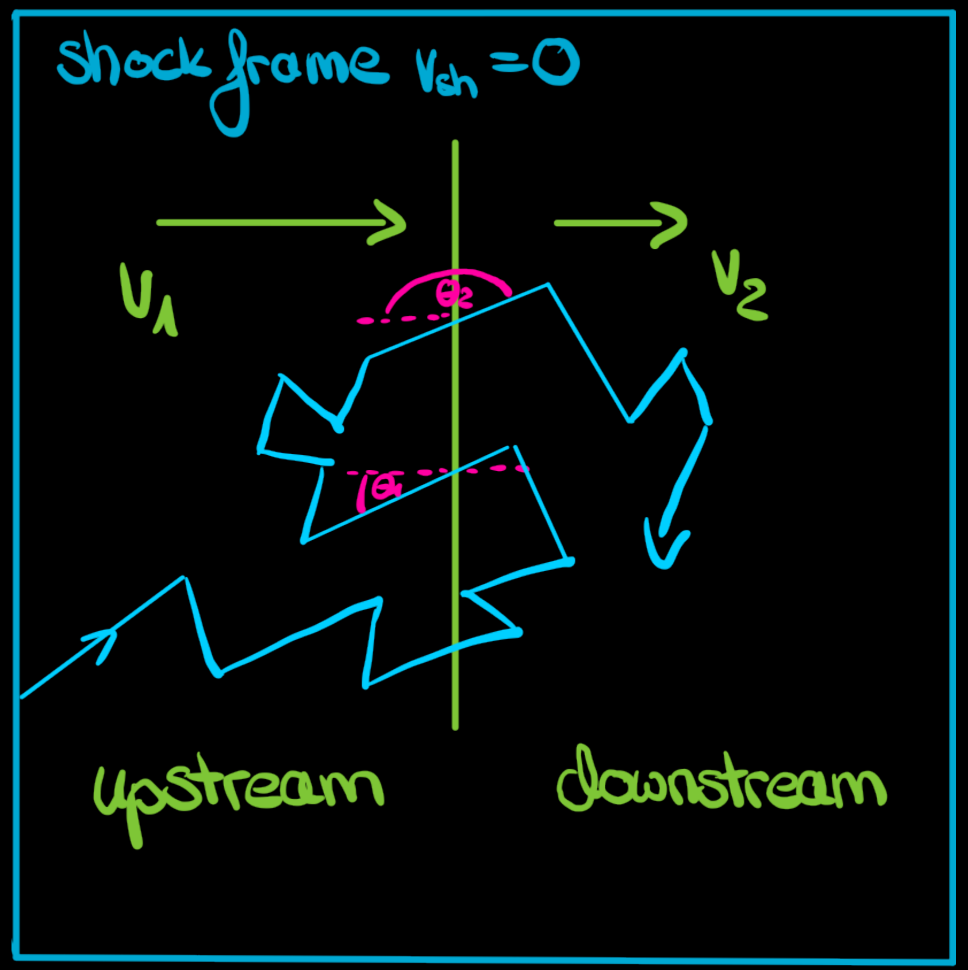

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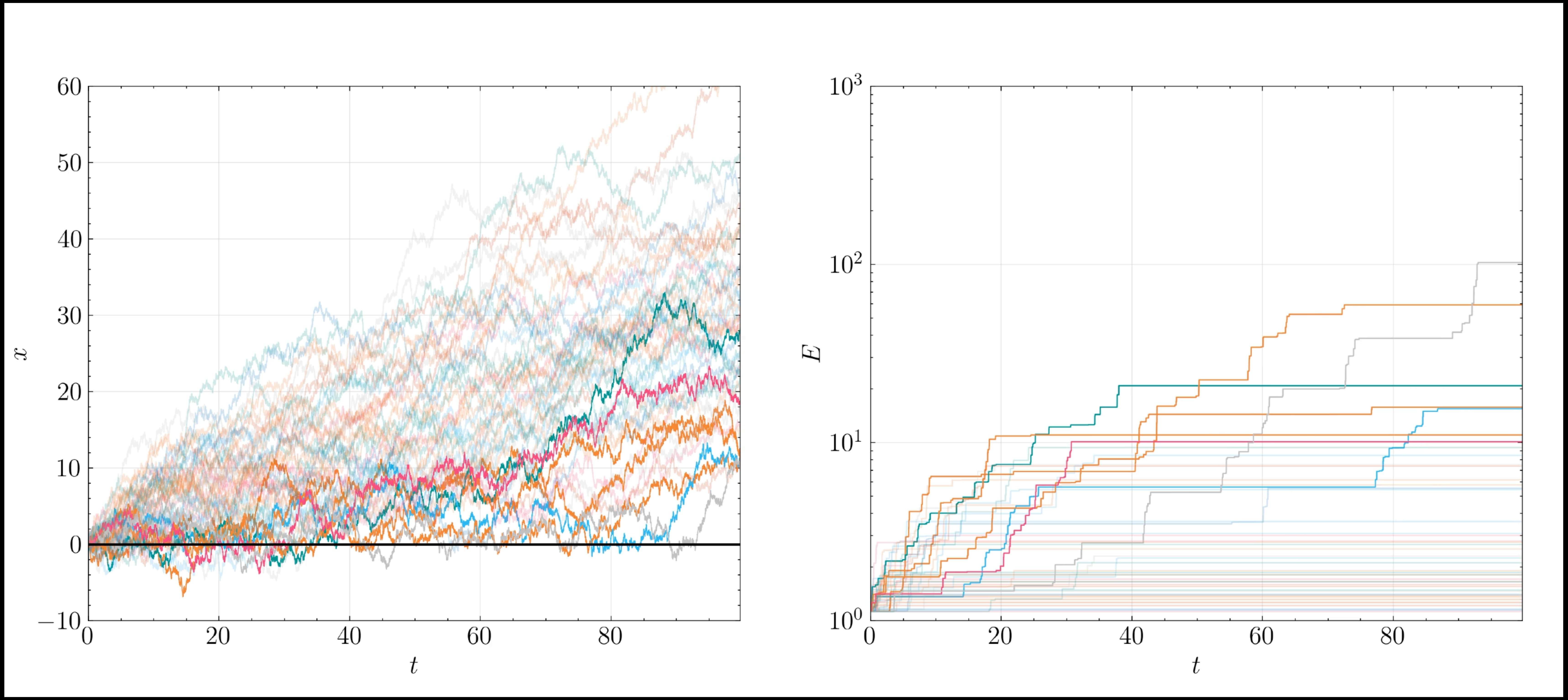
$$u\Delta t < L_{\mathrm{sh}} \lesssim \sqrt{2\kappa\Delta t}$$



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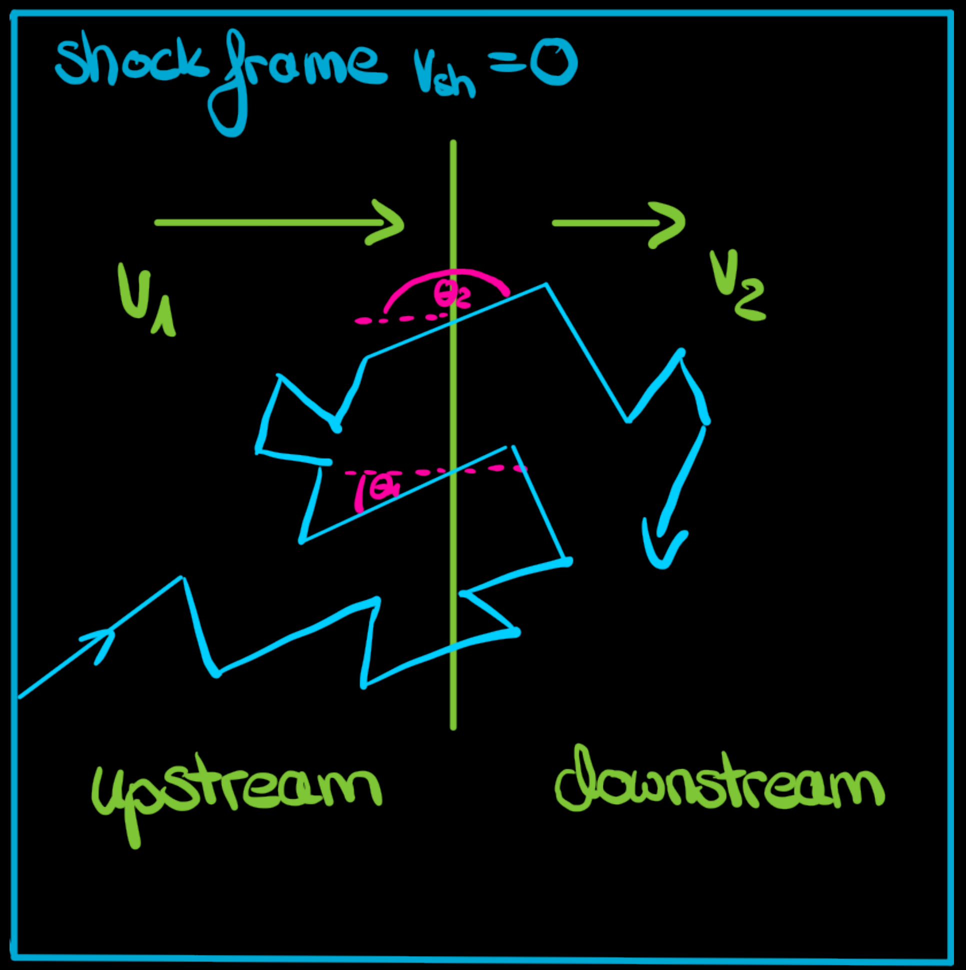

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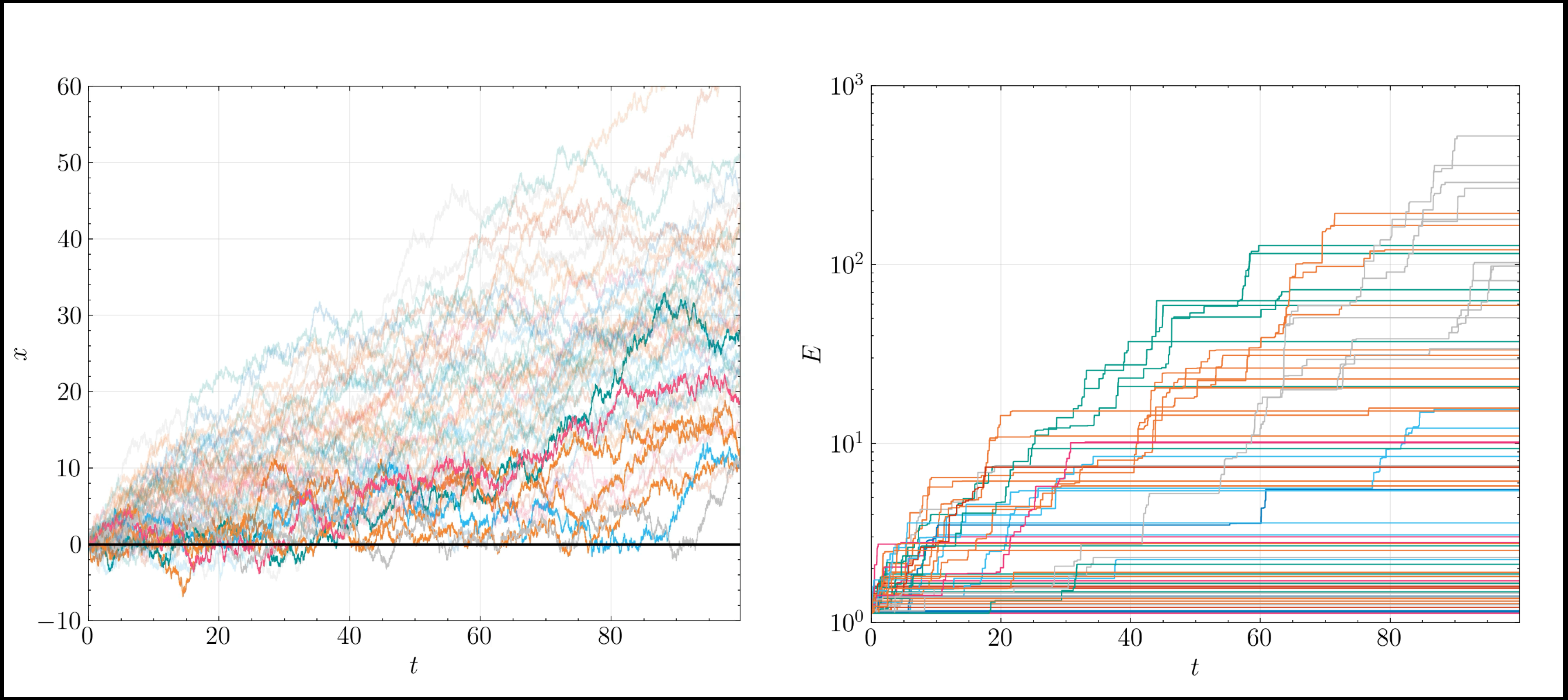
$$w_i = 1/n_{\mathrm{split}}$$

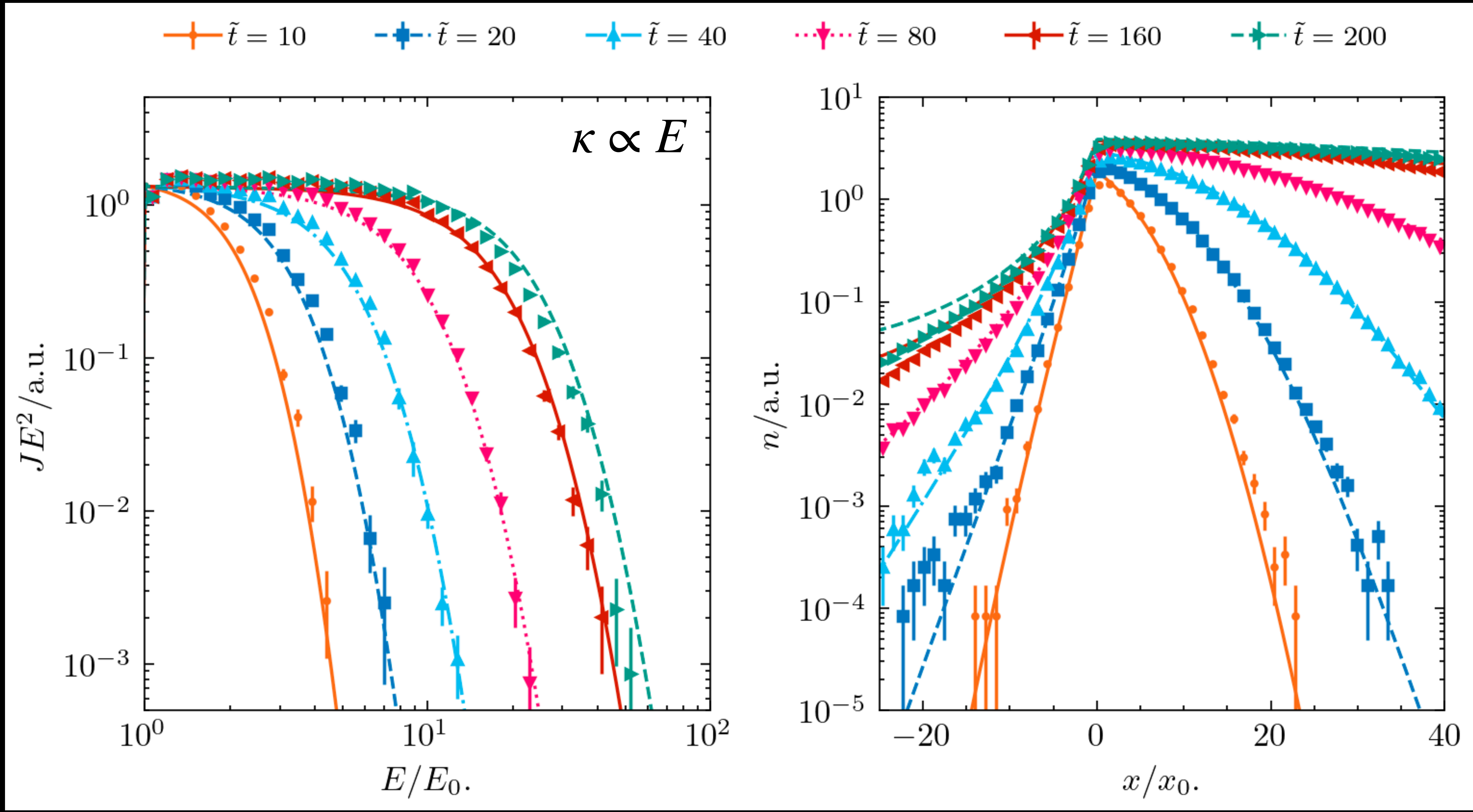
$$\mathrm{d}p = -\frac{p}{3}\frac{\partial u}{\partial x}\,\mathrm{d}t$$

- Importance splitting



TIME-DEPENDENT SPECTRA

DSA AT 1D PLANAR SHOCK



Aerdker+, 2024

- Validation: Comparison against grid code VLUGR3 solving the transport equation
- Moving and colliding shocks: SA, Habegger+, 2025
- Superdiffusive shock acceleration: Effenberger, SA+, 2024 & SA, Merten+, 2025

EXCURSION: SUPERDIFFUSION

ANOMALOUS DIFFUSION

LEVY FLIGHT MODEL

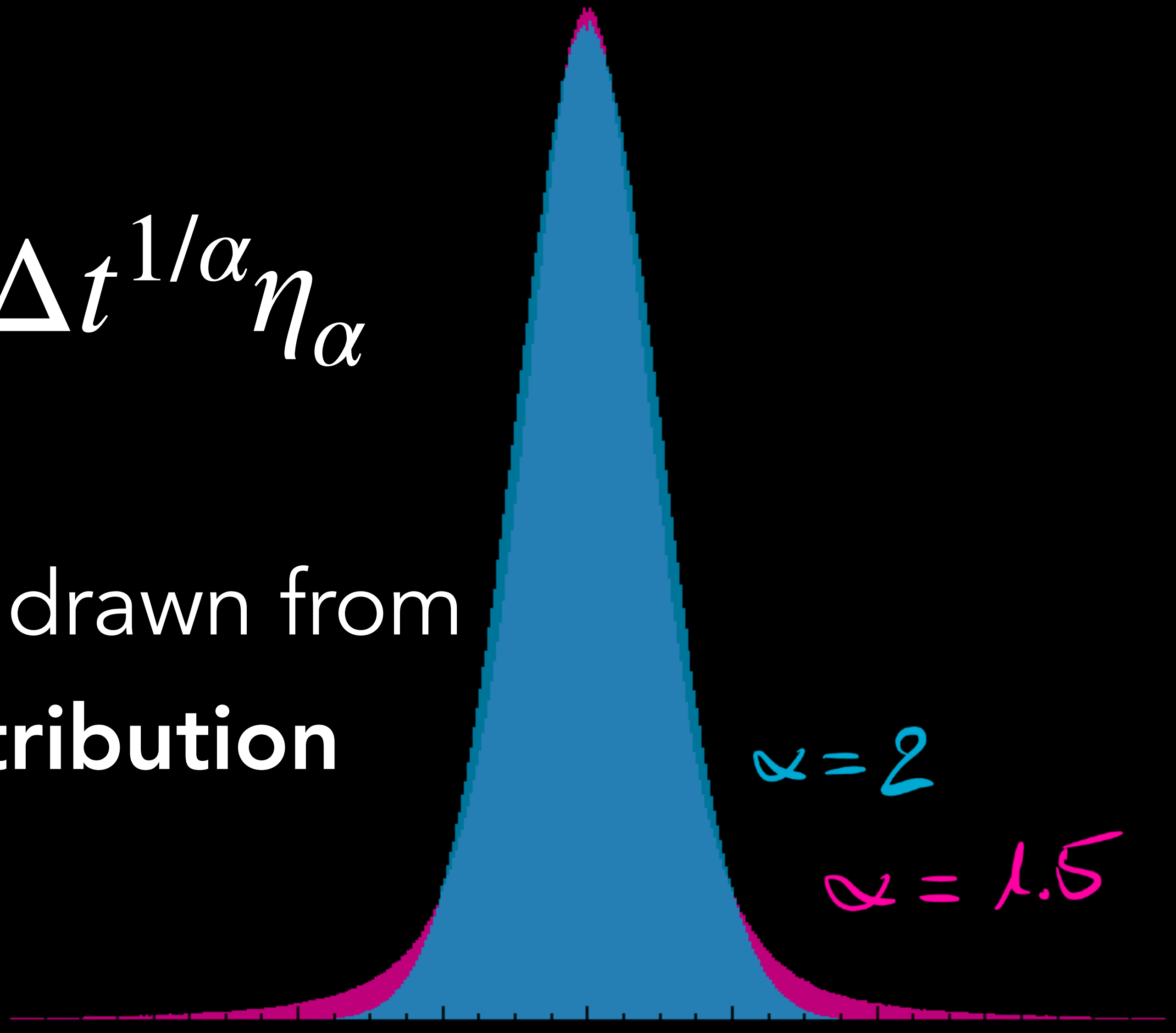
$$\mathrm{d}x = u\,\mathrm{d}t + \sqrt{2}\,\left(\kappa_\alpha\right)^{1/\alpha}\mathrm{d}L_\alpha(t)$$

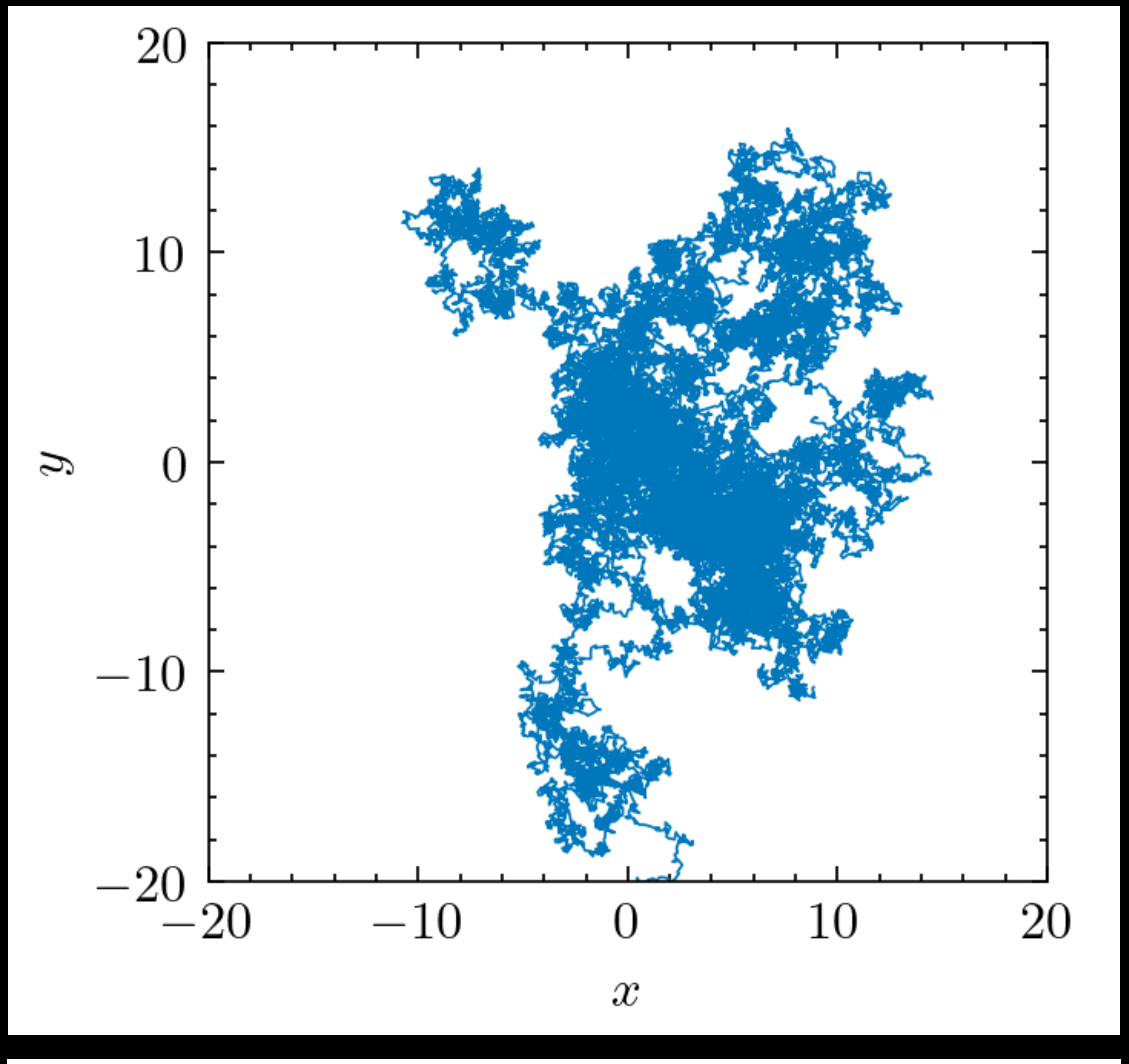
$$\mathrm{d}p = -\frac{p}{3}\frac{\partial u}{\partial x}\,\mathrm{d}t$$

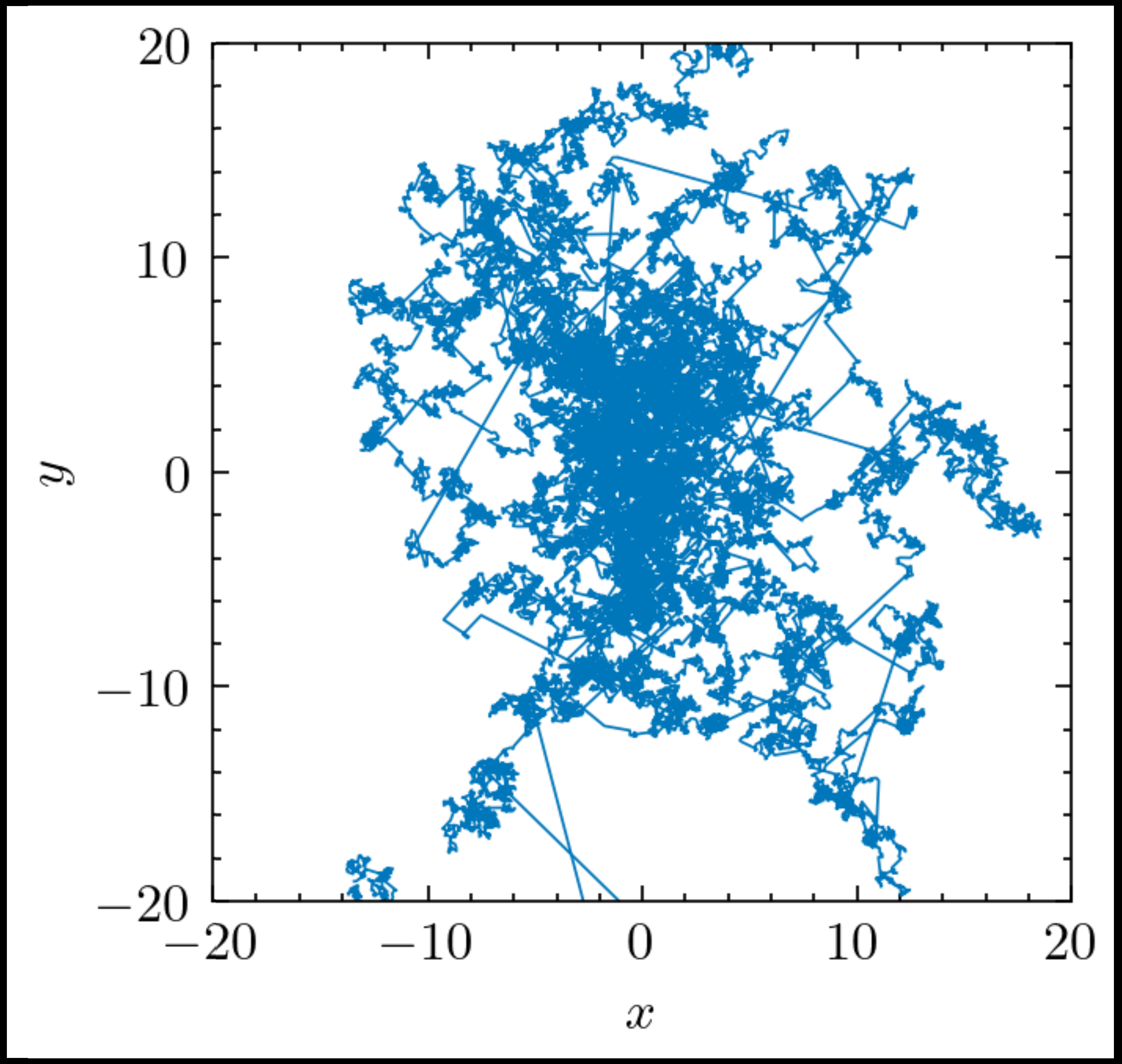
$$\Delta L_\alpha = \Delta t^{1/\alpha}\eta_\alpha$$

- Random numbers η drawn from a **α-stable Lévy distribution**

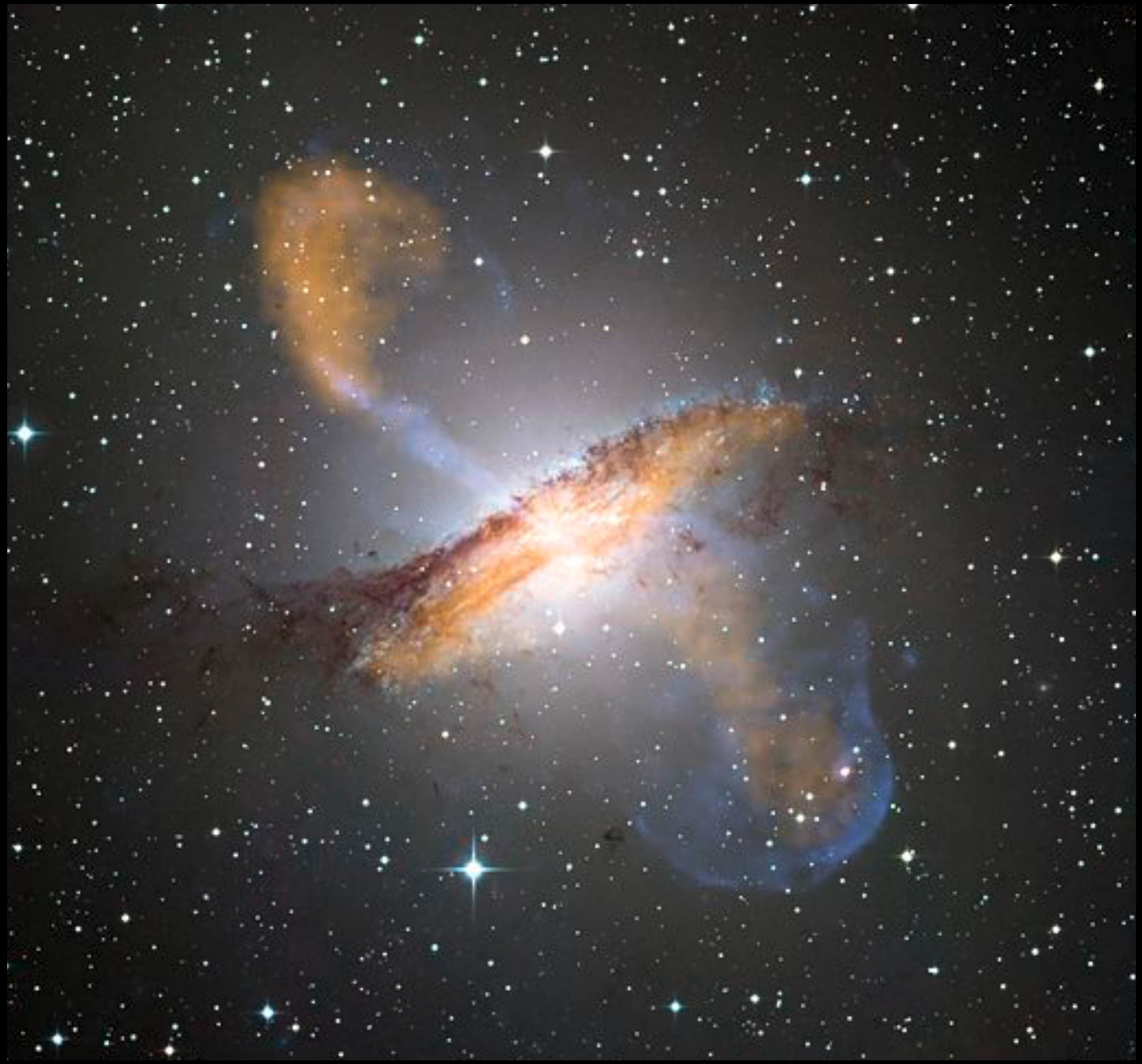
(e.g. Janicki&Weron,







SHEAR ACCELERATION



FOKKER-PLANCK PICTURE

- Assuming isotropic spatial diffusion, non-relativistic gradual shear
- Momentum change expressed as diffusive process:

$$D_{pp} = \frac{1}{15}\left(\partial_x u_z\right)^2 p^2 \tau(p)$$

$$A_p = \frac{6-q}{15}\left(\partial_x u_z\right)^2 p\tau(p)$$

e.g. Rieger, 2019;
Liu, Rieger, Aharonian 2017

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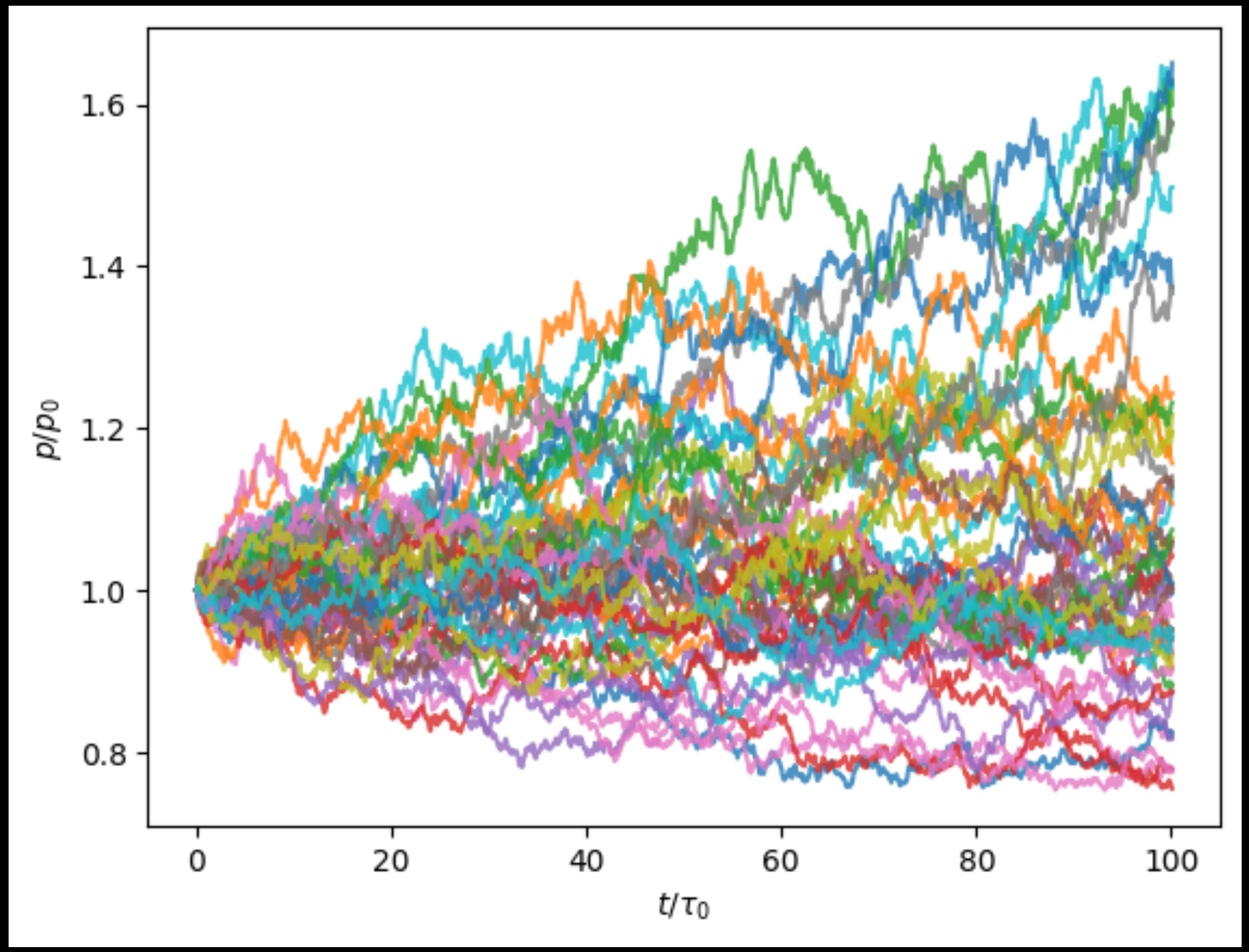
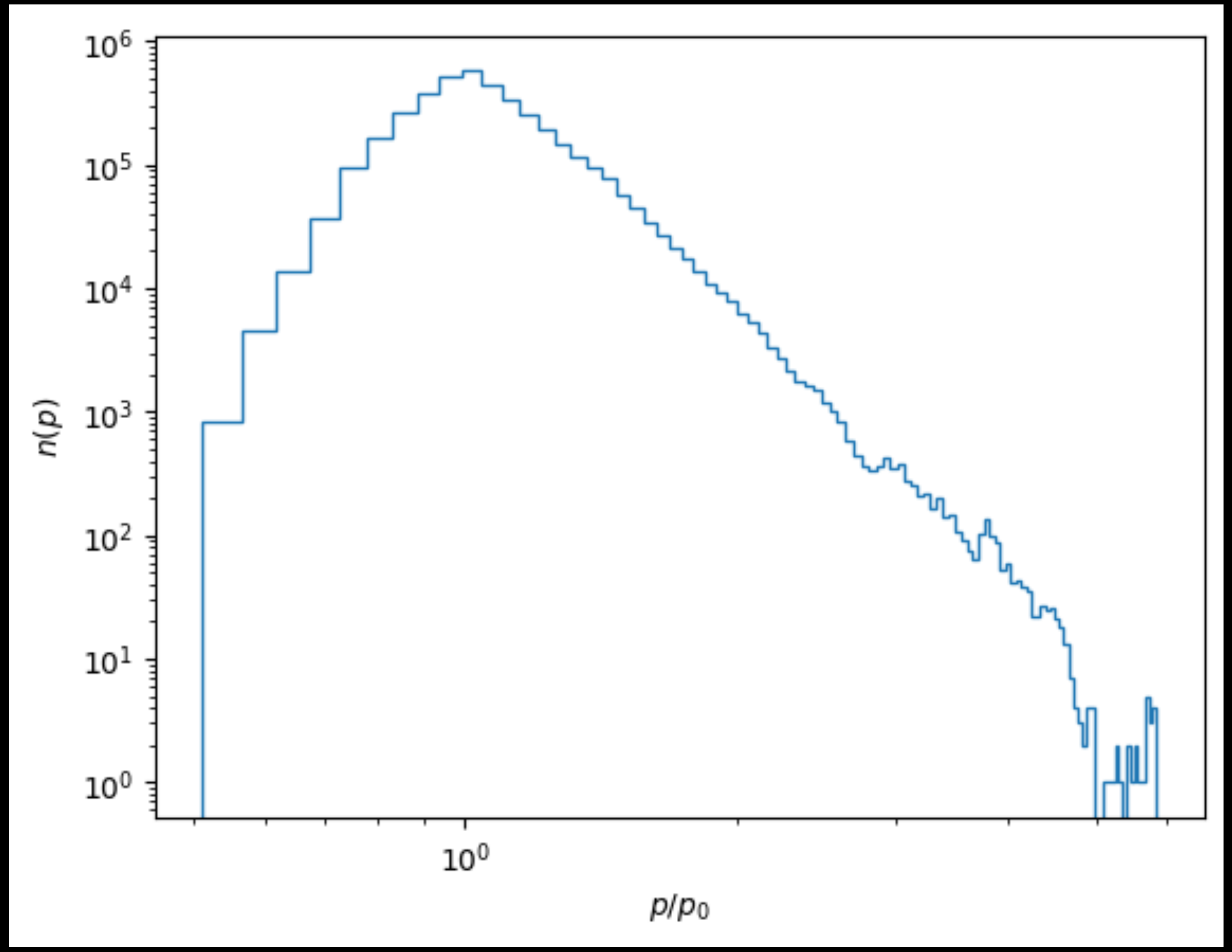
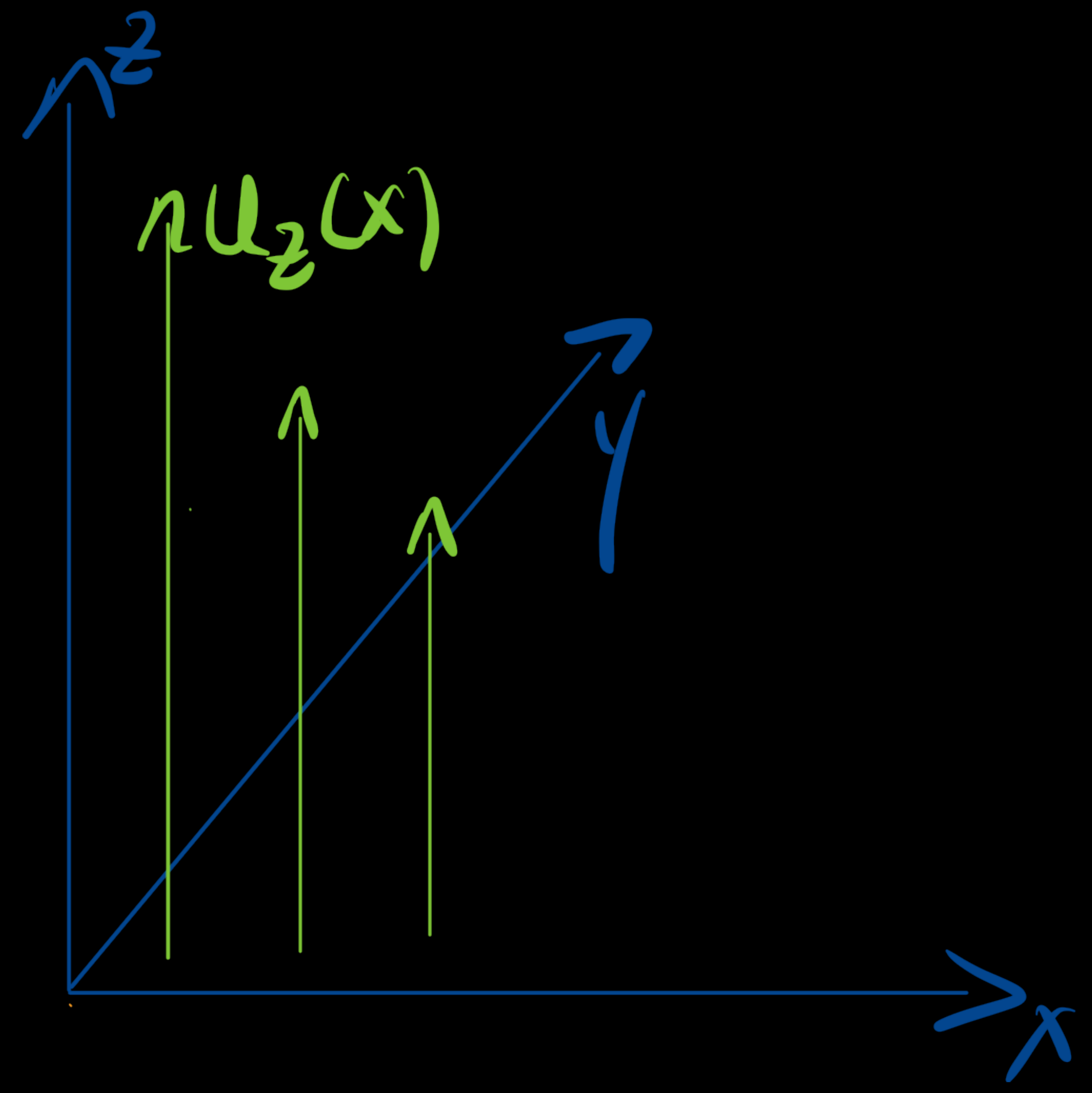




GENERALIZED FERMI MODEL

COMOVING FRAME

- Track particle momentum in the frame where the electric field vanishes
- Velocity perpendicular to magnetic field: $\mathbf{v}_{\mathrm{E}} = c\mathbf{E} \times \mathbf{B}/B^2$
- Velocity parallel to magnetic field unspecified
- Energy changes given by acceleration, compression and shear of the frame velocity



SHEAR ACCELERATION

IN THE COMOVING FRAME

$$\dot{p}' = -\,p'\left(\mu_2'\mu_3'\sigma_{zx}\right) \qquad \sigma_{zx} = \partial_x u_z/\gamma_E$$

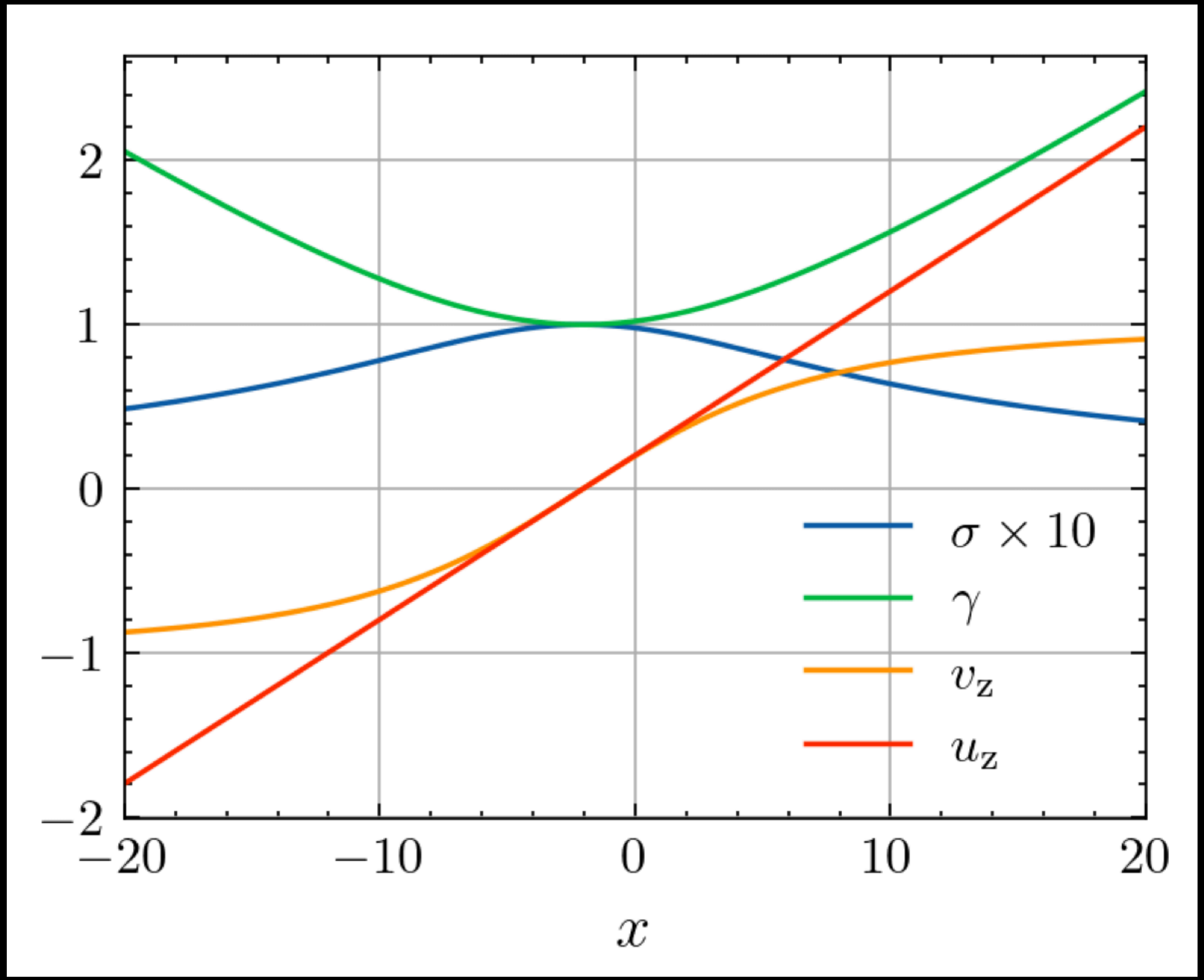
$$\dot{\mu}_1' = \mu_1'\mu_2'\mu_3'\sigma_{zx} + \xi_1$$

$$\dot{\mu}_2' = \mu_2'^2\mu_3'\sigma_{zx} + \xi_2$$

$$\dot{\mu}_3' = -\,\mu_2'(1-\mu_3'^2)\sigma_{zx} + \xi_3$$

Lemoine, 2025

- Need to track evolution of angular variables μ_i', not only position in space!



Infinite linear shear profile in $u_z(x)$: σ_{zx} maximal when $\gamma(x) \gtrsim 1$

Aerdker+, in prep.

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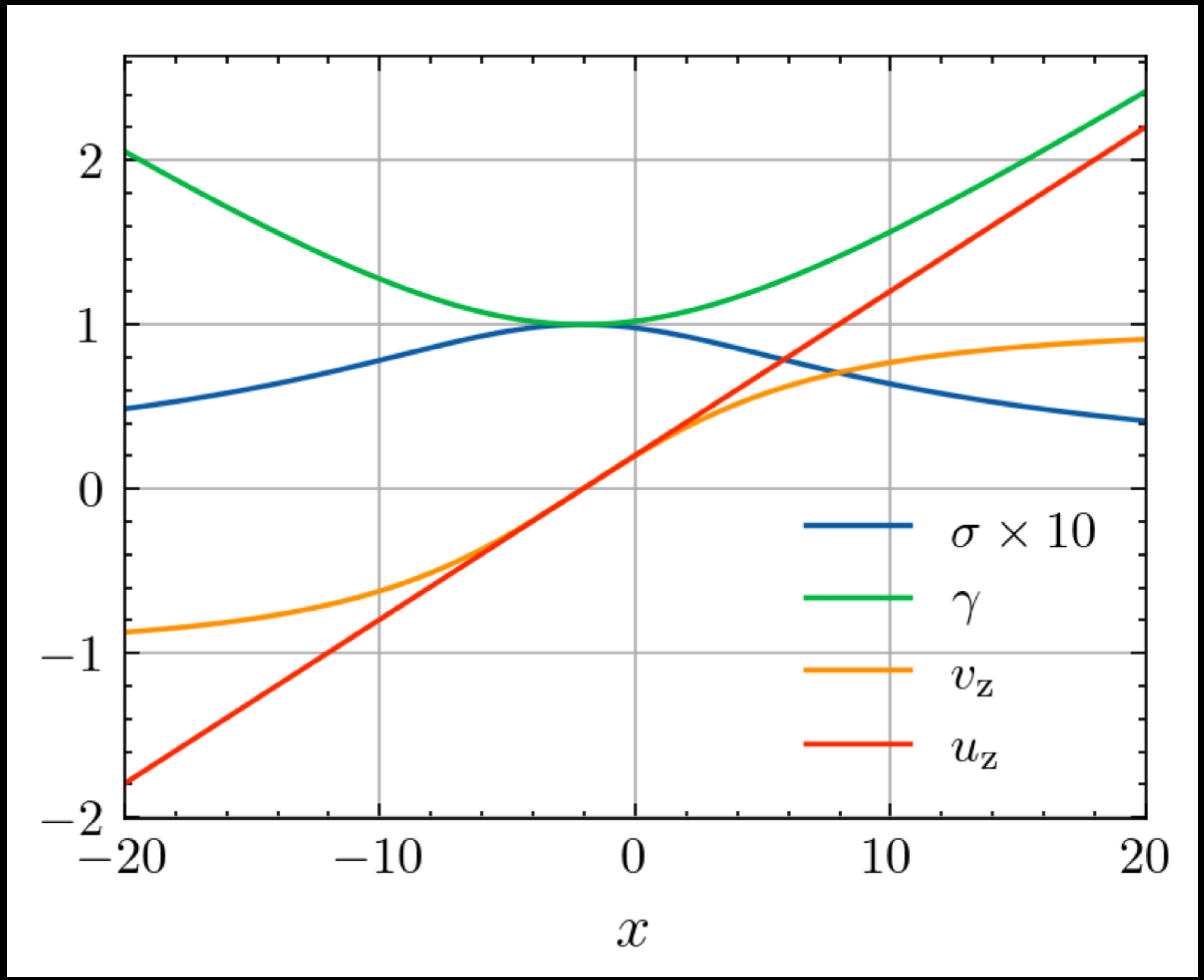

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EXCURSION: PARTICLE SCATTERING

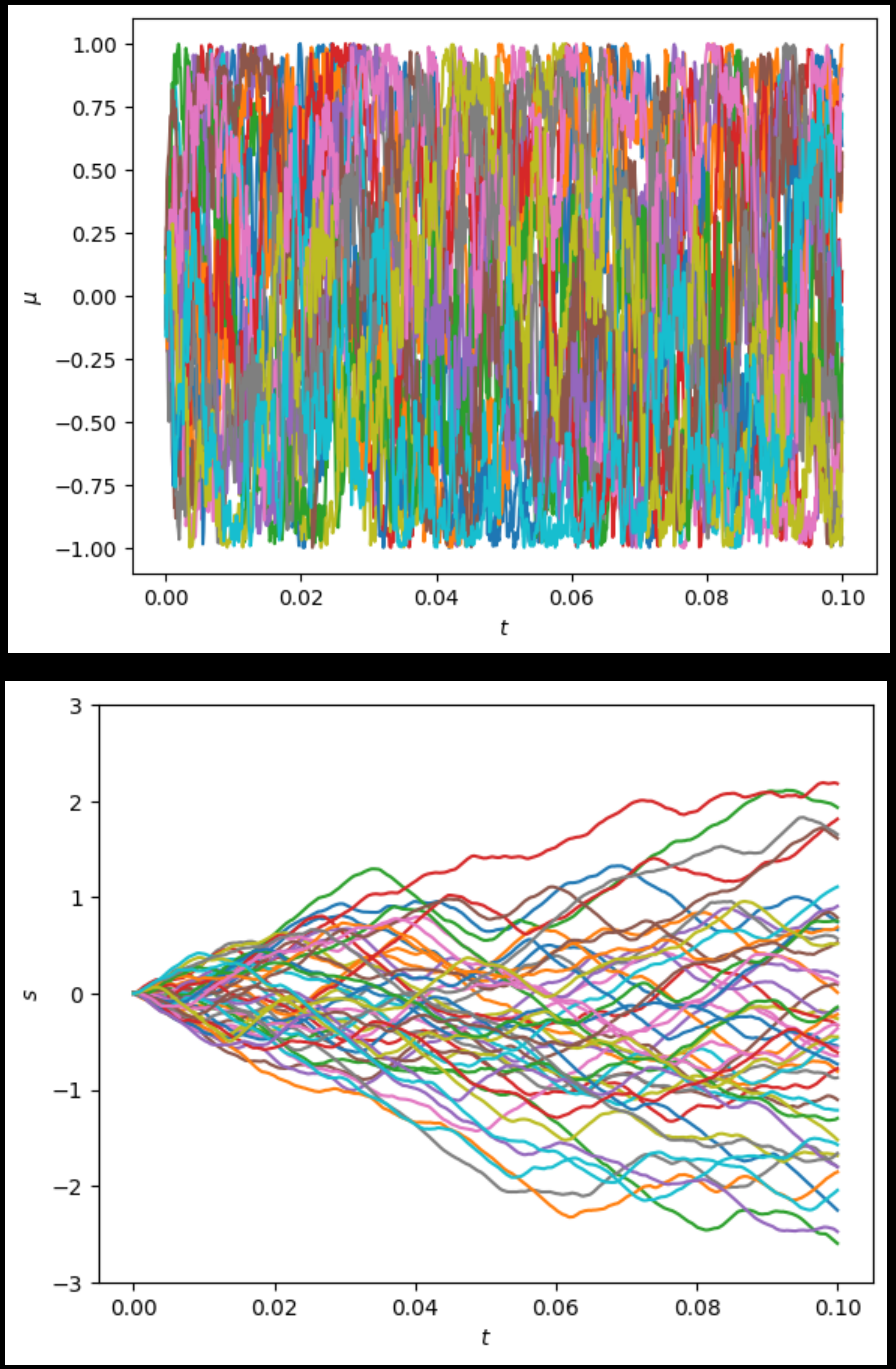
PITCH-ANGLE SCATTERING

ORNSTEIN-UHLENBECK PROCESS

- Scattering in pitch-angle leads to diffusion along the magnetic field

$$d\mu_1 = -2\mu_1 D_0 \mathrm{d}t + \sqrt{2D_0(1-\mu_1^2)}\mathrm{d}W_\mu$$

$$ds = v\mu \mathrm{d}t$$



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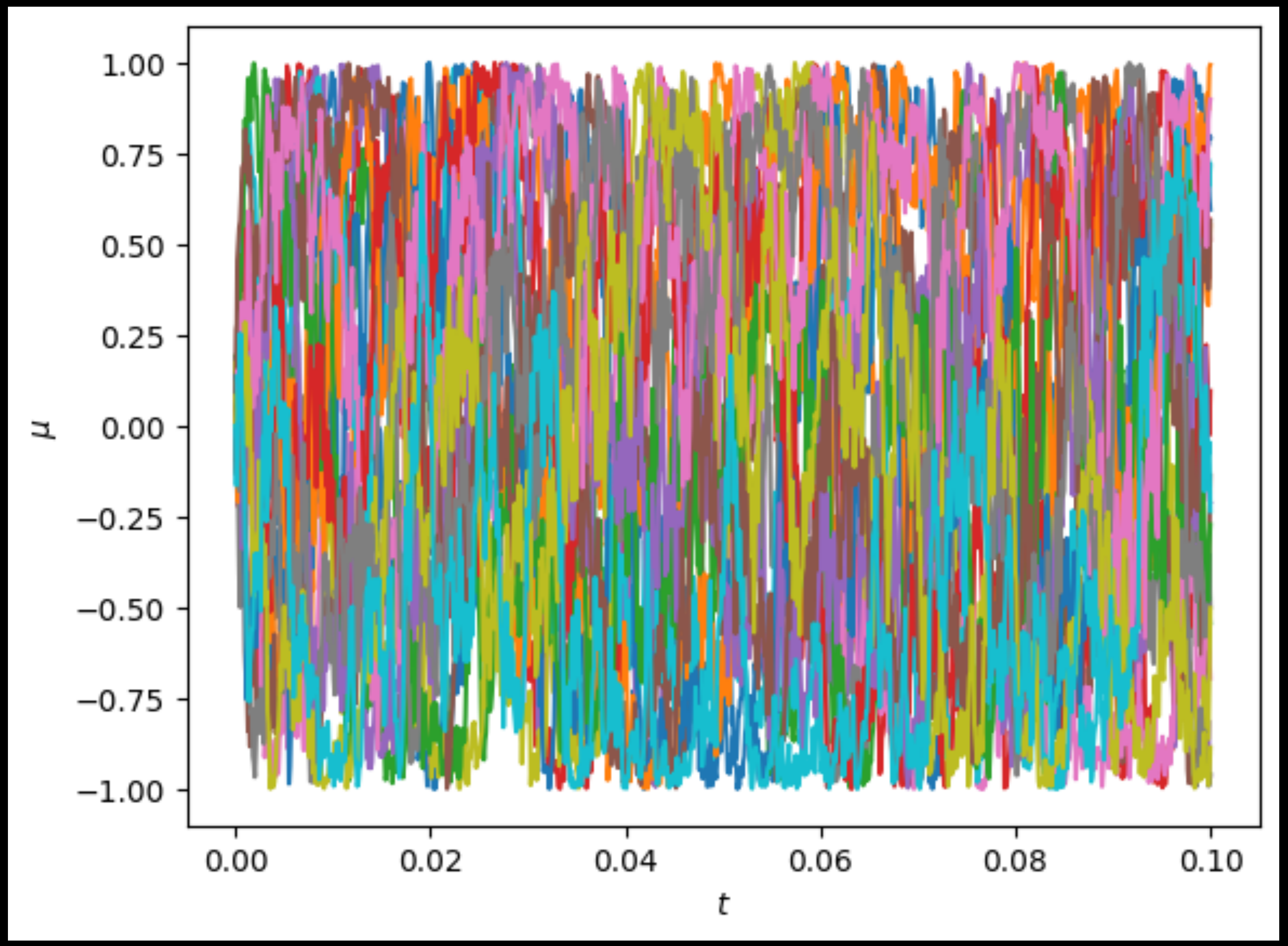
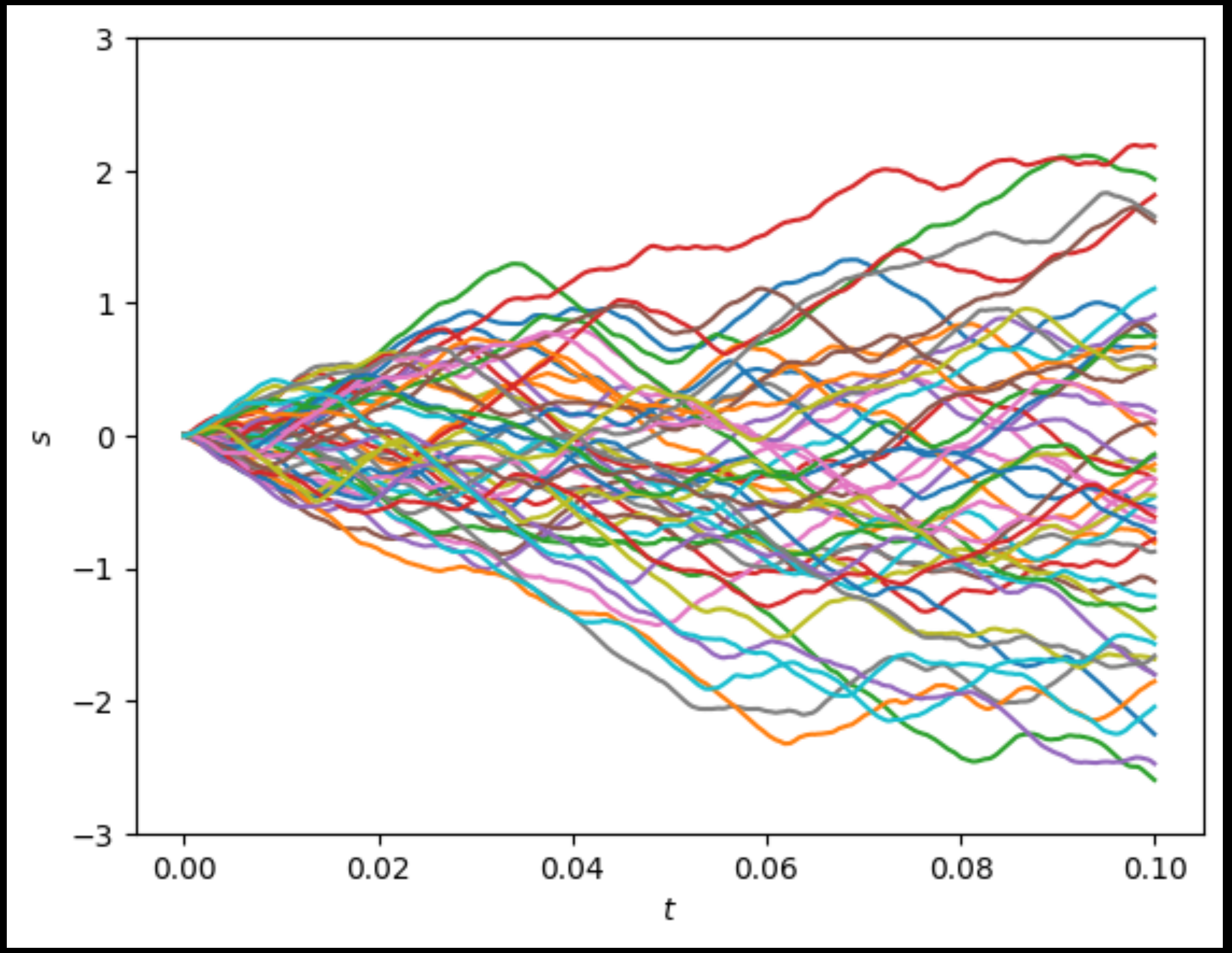
$$ds = v\mu \mathrm{d}t$$

- Scattering in the gyrophase leads to (an-)isotropic diffusion in space

$$\mathrm{d}\phi = \Omega \mathrm{d}t + \sqrt{2D_\phi}\mathrm{d}W_{\phi,t}$$

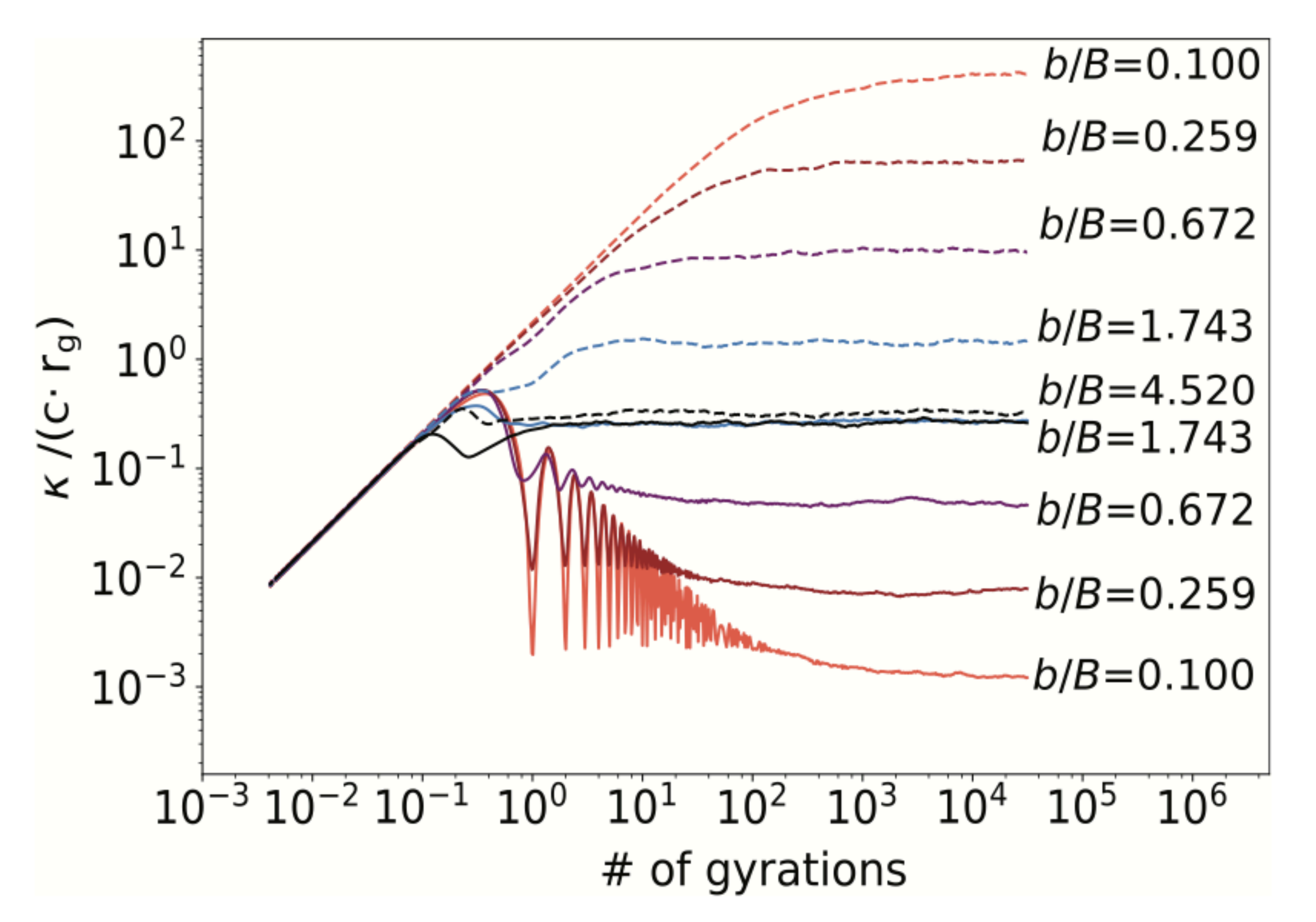




PARTICLE SCATTERING

(AN-)ISOTROPIC DIFFUSION IN SPACE

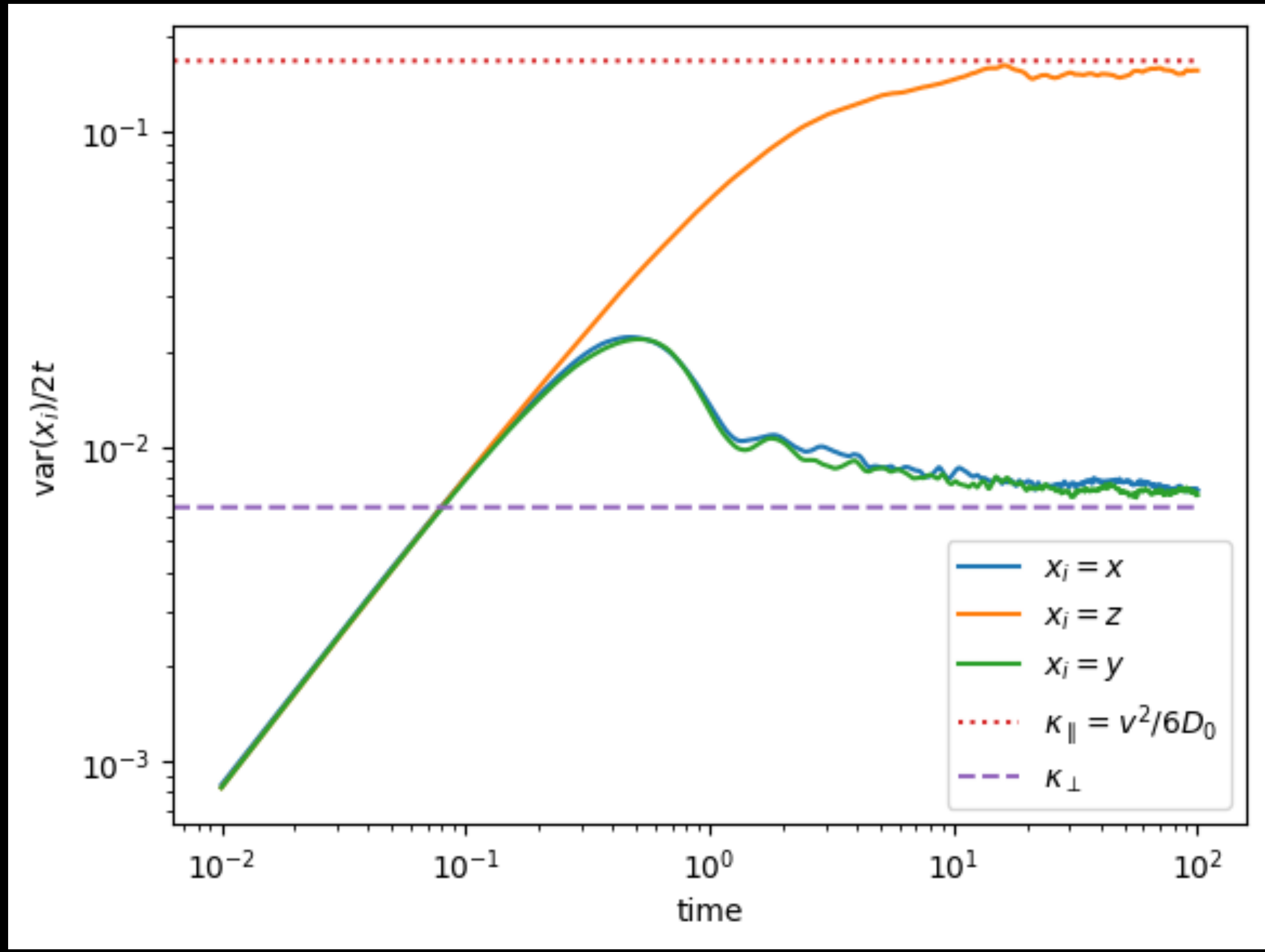
- Running diffusion coefficient: $< (\Delta x)^2 > /2t$



Particle transport in synthetic turbulence: Reichherzer, et al. 2020

$$\kappa_{\parallel} = \frac{v^2 \tau_{\parallel}}{3}$$

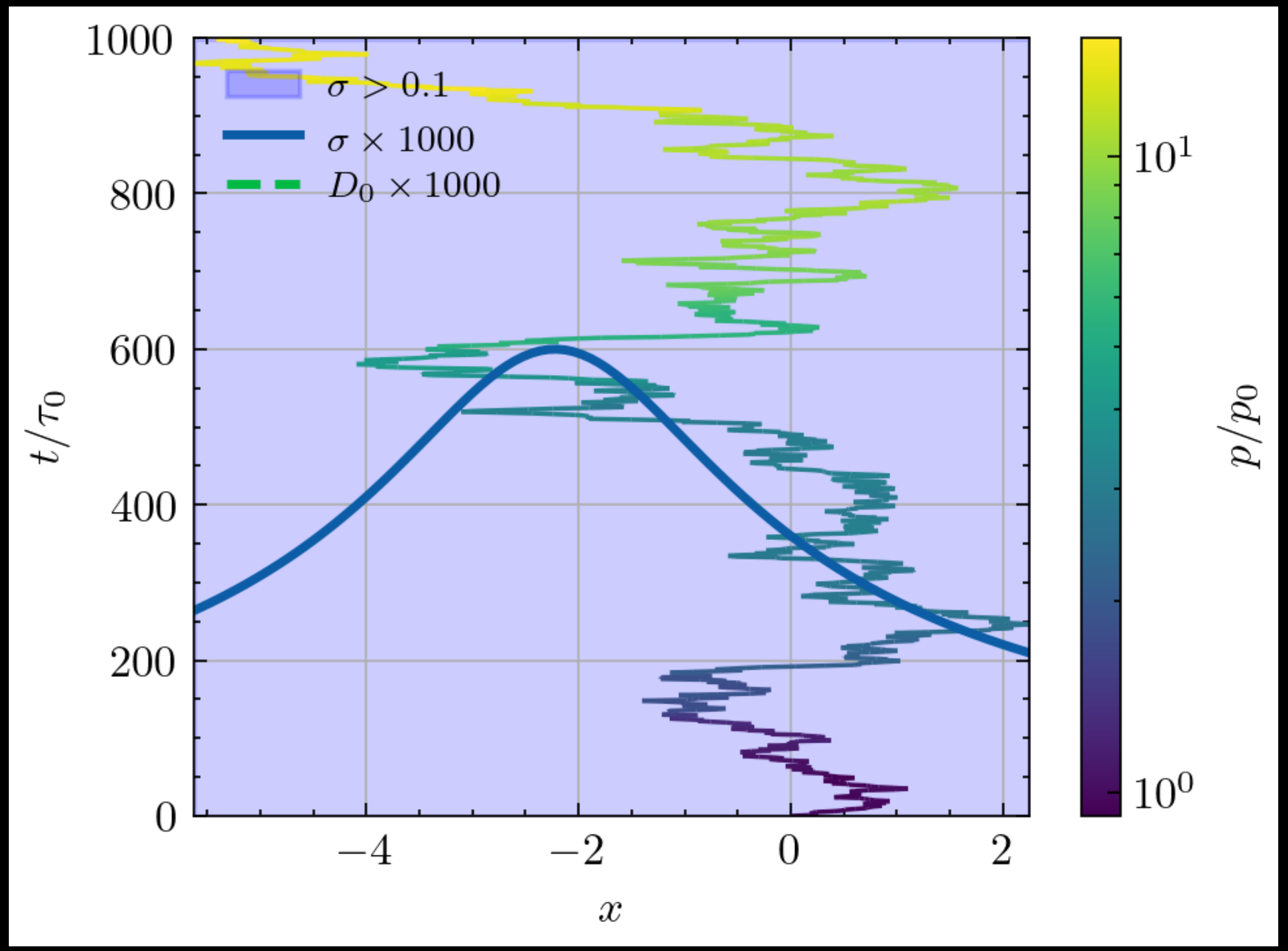
$$\kappa_{\perp} = \frac{v r_g}{3} \frac{\Omega \tau_{\perp}}{1 + \Omega^2 \tau_{\perp}^2}$$

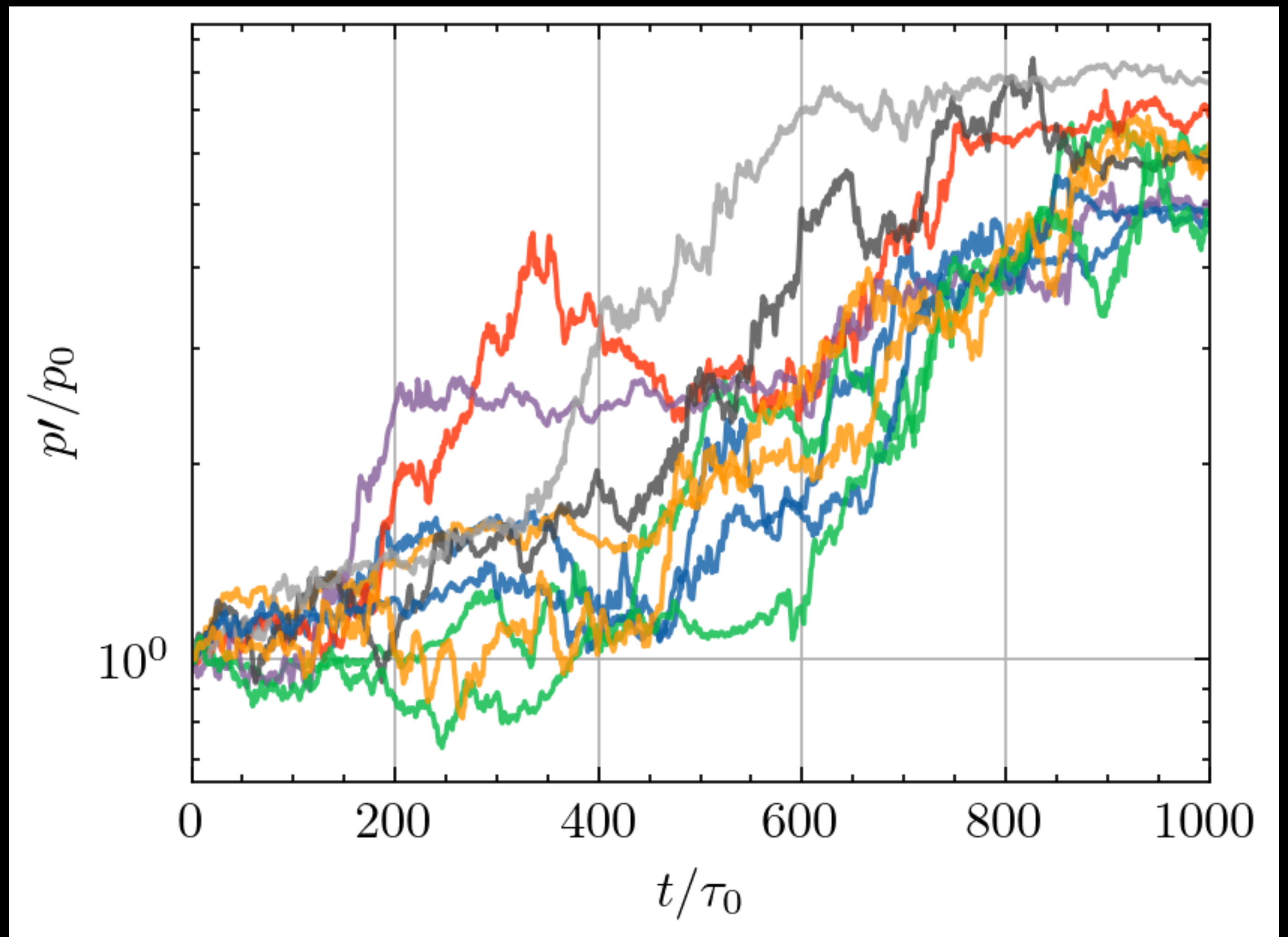


Pitch-angle & gyrophase scattering with SDEs, Aerdker+, in prep.

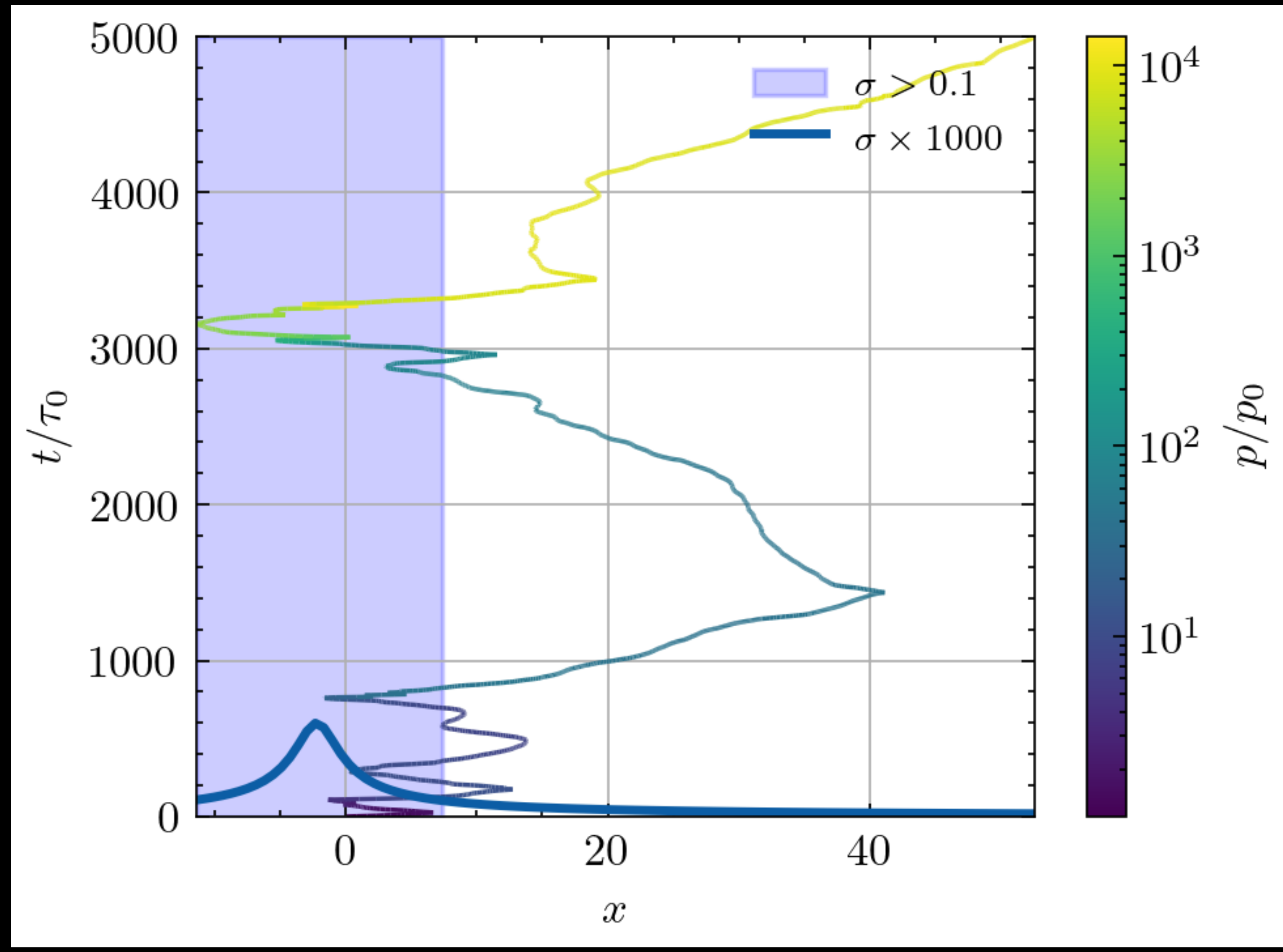
SUBDIFFUSION IN MOMENTUM

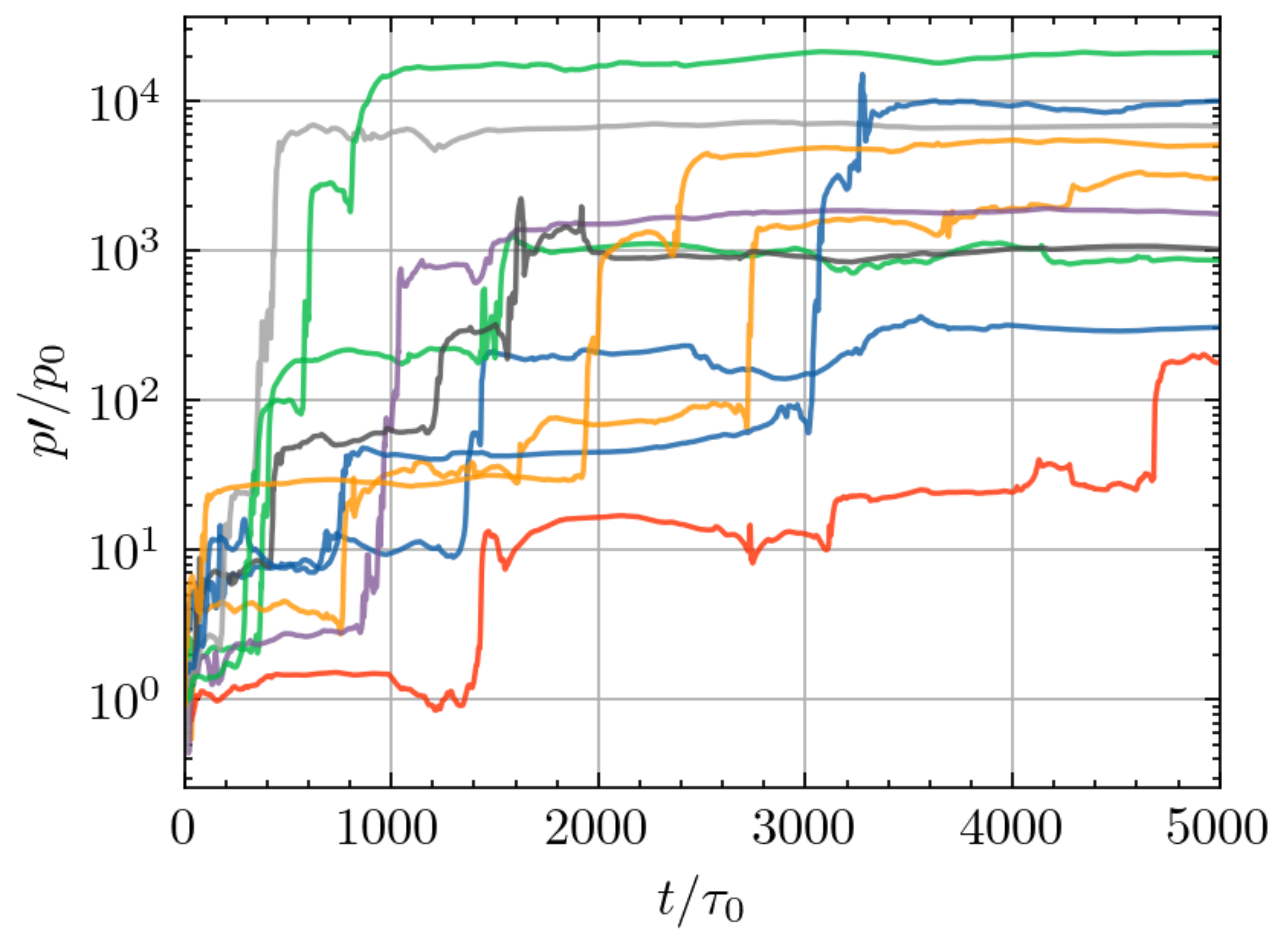
NON-GRADUAL SHEAR ACCELERATION





- Reproducing Fokker-Planck expectations for $\tau'\sigma < 1$ at early times





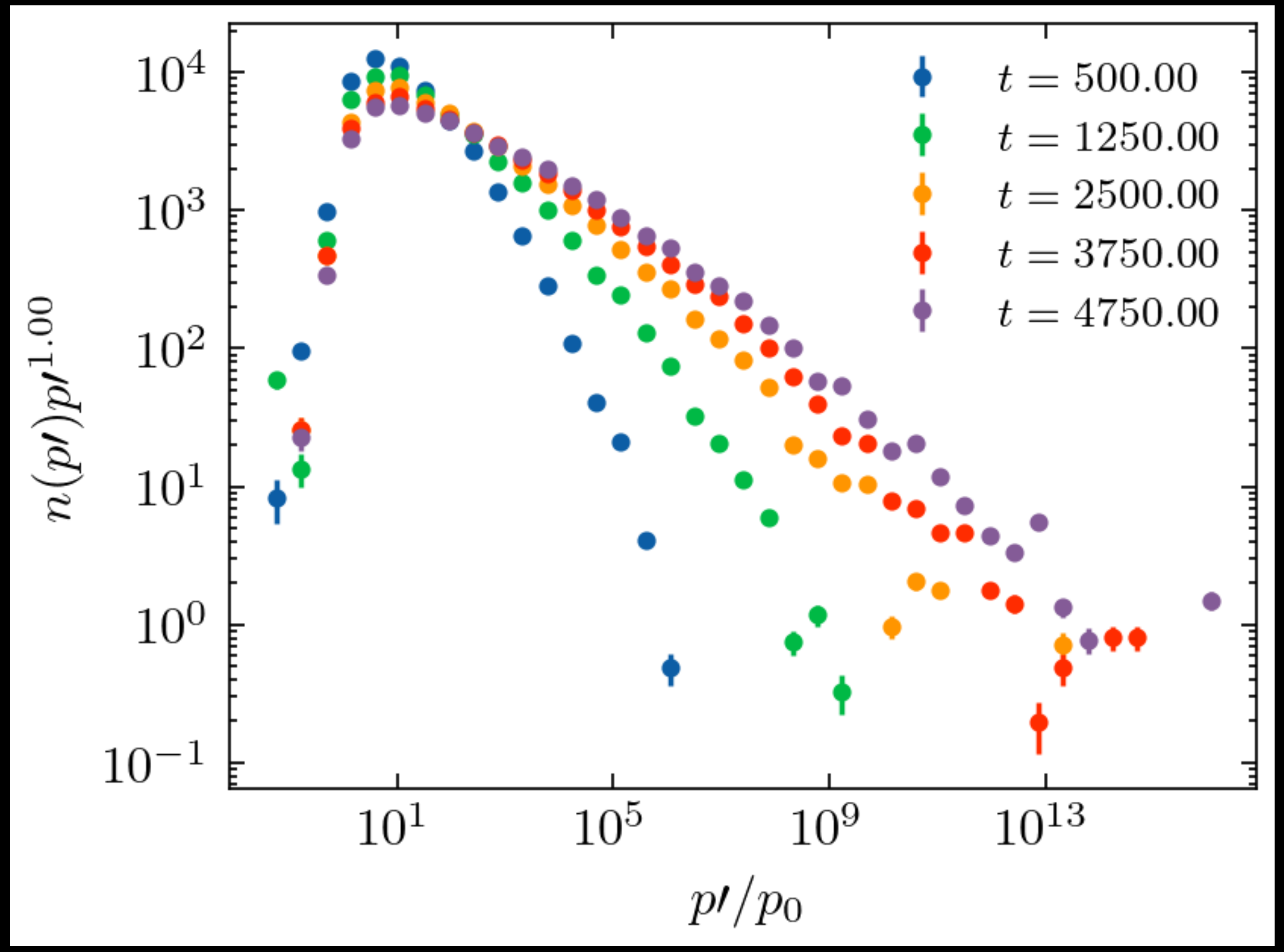
- Non-gradual shear acceleration ($\tau'\sigma > 1$) shows subdiffusion in momentum

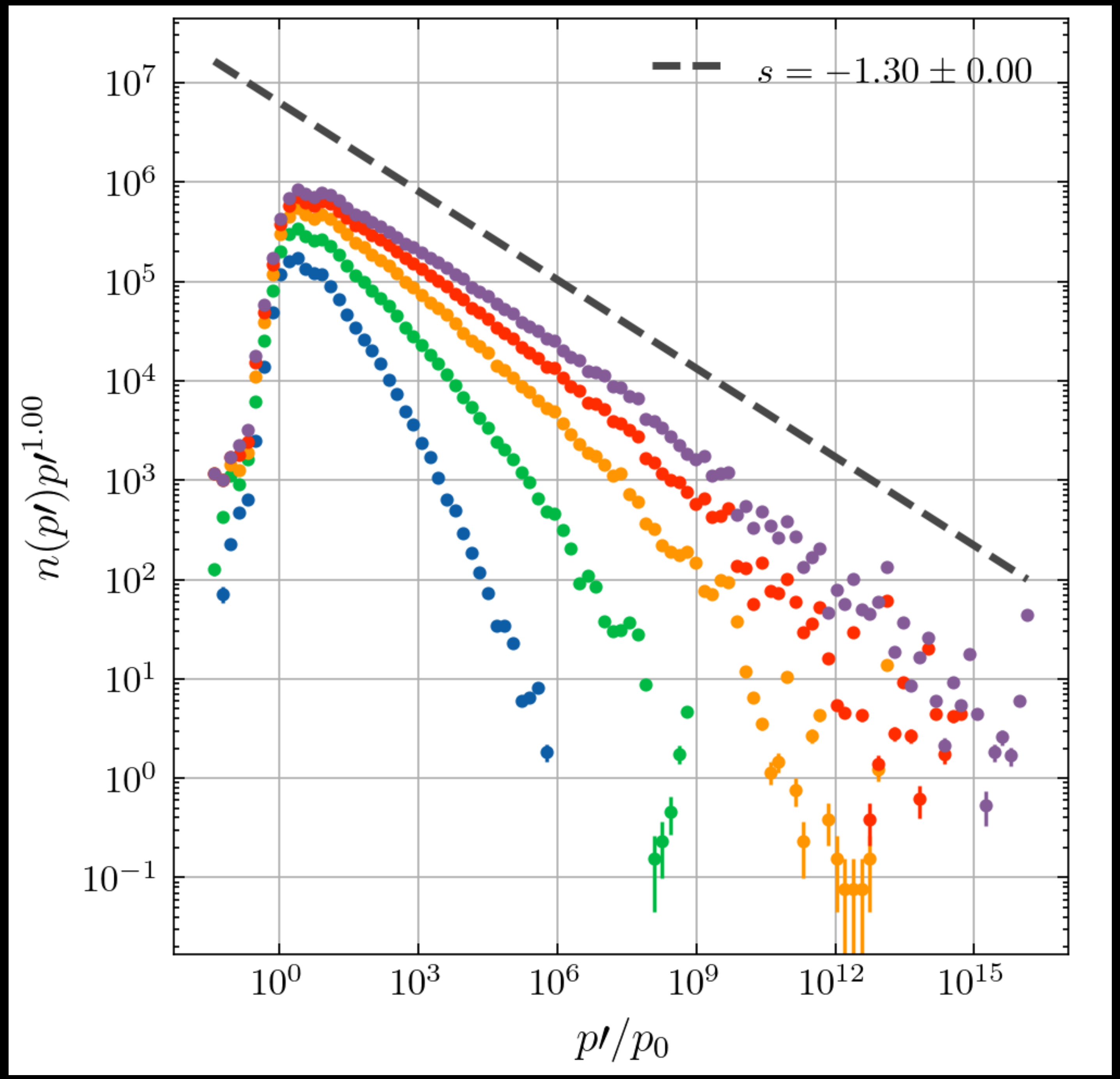
Aerdker+, in prep.

PARTICLE SPECTRA

NON-GRADUAL SHEAR ACCELERATION

- Time-dependent spectra show power-laws - explained by power-law waiting time distributions
- Integrated spectra are steeper compared to simple Fokker-Planck models - correction for the flow profile (Rieger & Duffy, 2019) points to soft spectra
- Highest energy particles in the region of strongest shear - where jet is mildly-relativistic
- Addition of loss processes, escape, ... to come!





STOCHASTIC PARTICLE TRANSPORT

- Fast & flexible approach:
  - Not dependent on source: Re-weighting and recycling of simulation data
  - Extension to anomalous diffusion
- Part of the CRPropa framework:
  - Parallelization, adaptive time step & importance splitting
  - Magnetic field line integration
- Applications:
  - (Super-)diffusive shock acceleration, momentum diffusion, Galactic transport
  - Gradual & non-gradual shear acceleration

BACKUP

SHEAR ACCELERATION

IN THE COMOVING FRAME

$$\mathrm{d}p' = -\,p'\left(\mu_2'\mu_3'\sigma_{32}\right)\mathrm{d}t$$

$$\mathrm{d}\mu_1' = \left(\mu_1'\mu_2'\mu_3'\sigma_{32} - 2\mu_1' D_0\right)\mathrm{d}t + \sqrt{2D_0(1-\mu_1'^2)}\mathrm{d}W_{\mu_1'}$$

$$\mathrm{d}\phi' = \sqrt{2D_{\phi'}}\mathrm{d}W_{\phi'}$$

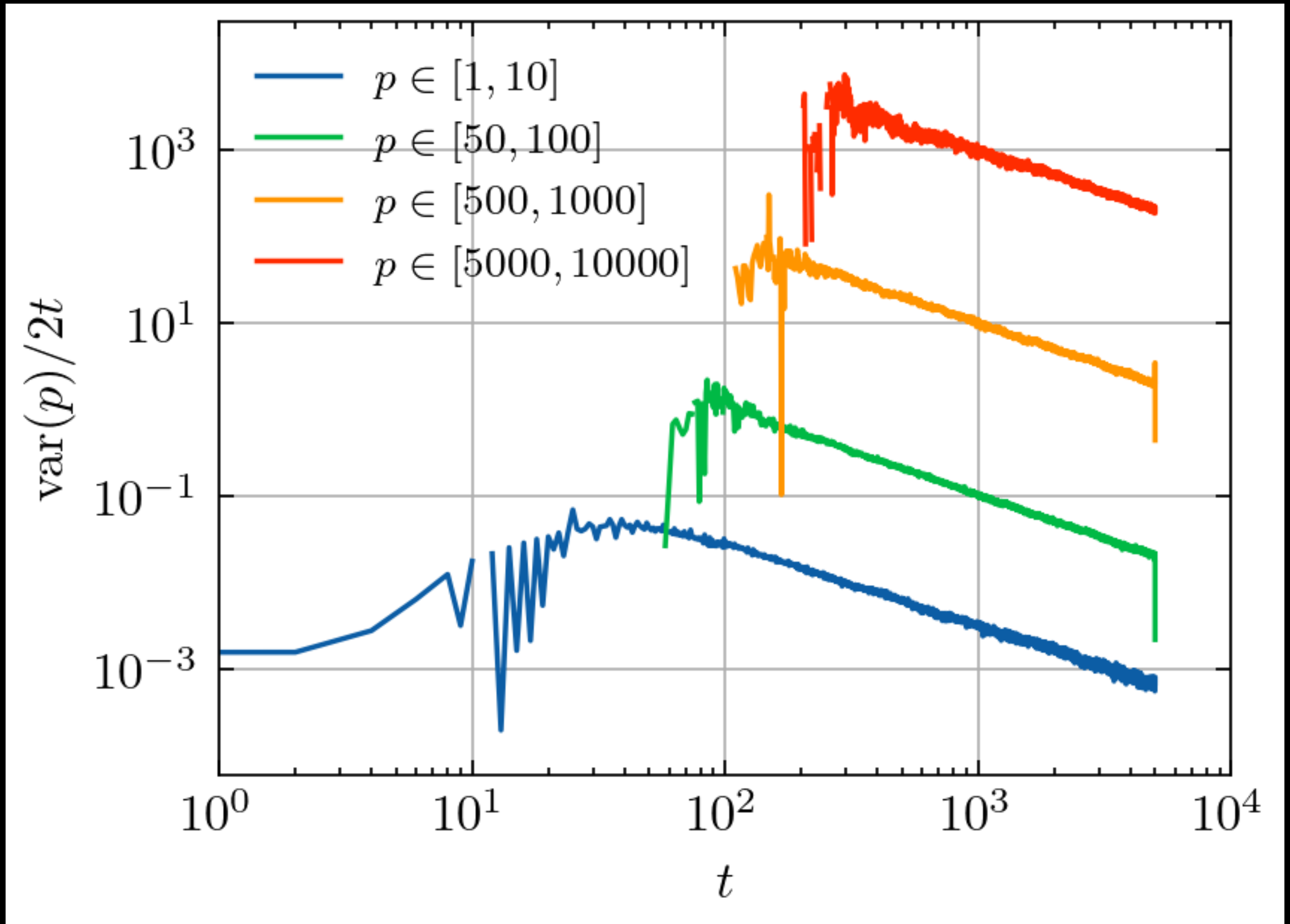
$$\mathrm{d}\mu_2' = \mu_2'^2\mu_3'\sigma_{32}\mathrm{d}t + f(\mu_1', \phi')$$

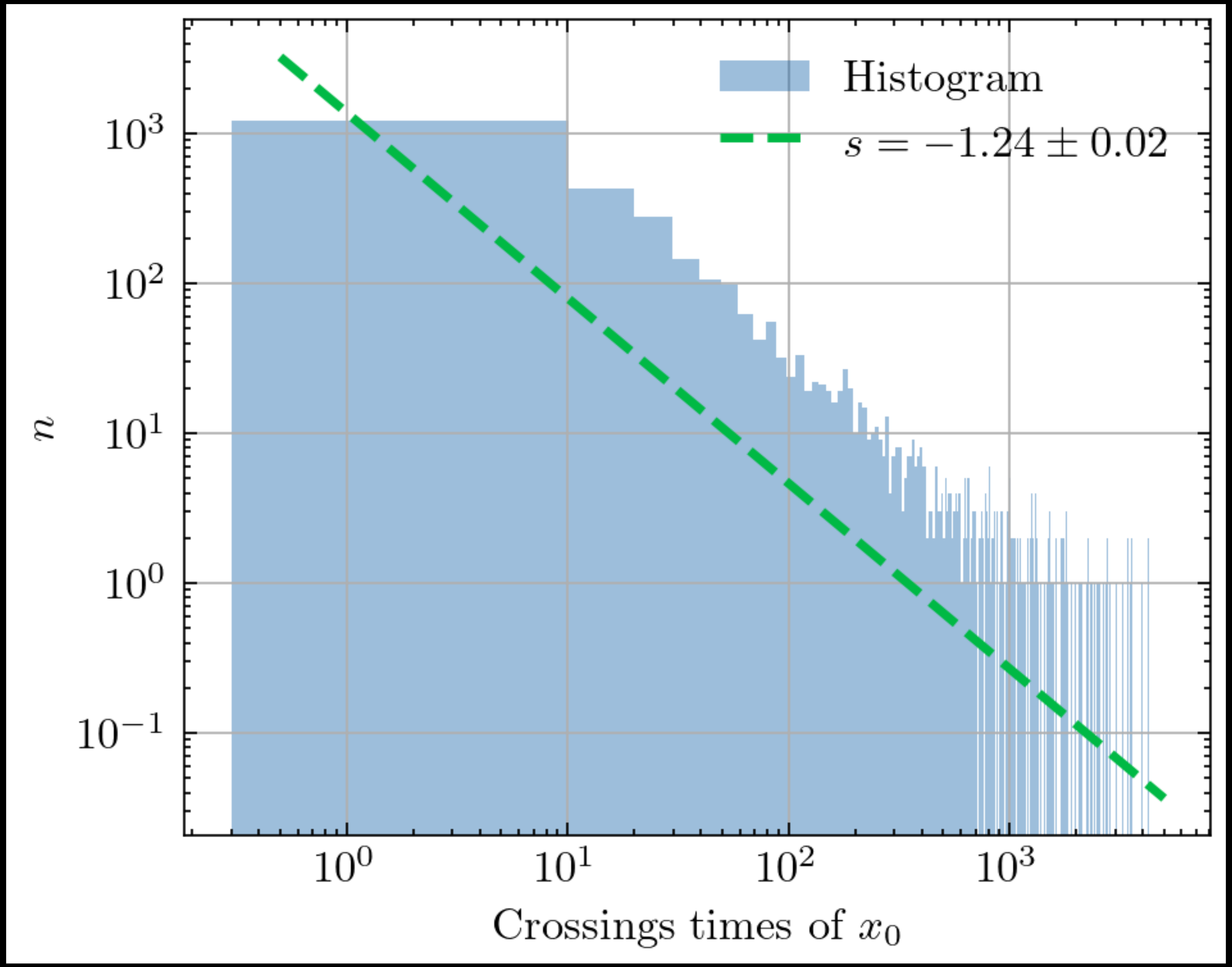
$$\mathrm{d}\mu_3' = -\,\mu_2'(1-\mu_3'^2)\sigma_{32}\mathrm{d}t + g(\mu_1', \phi')$$

- Every time step $\Delta t'$:
  - Boost time and momentum (t', p') to lab frame (t, p), depending on particle position & momentum

SUBDIFFUSION IN MOMENTUM

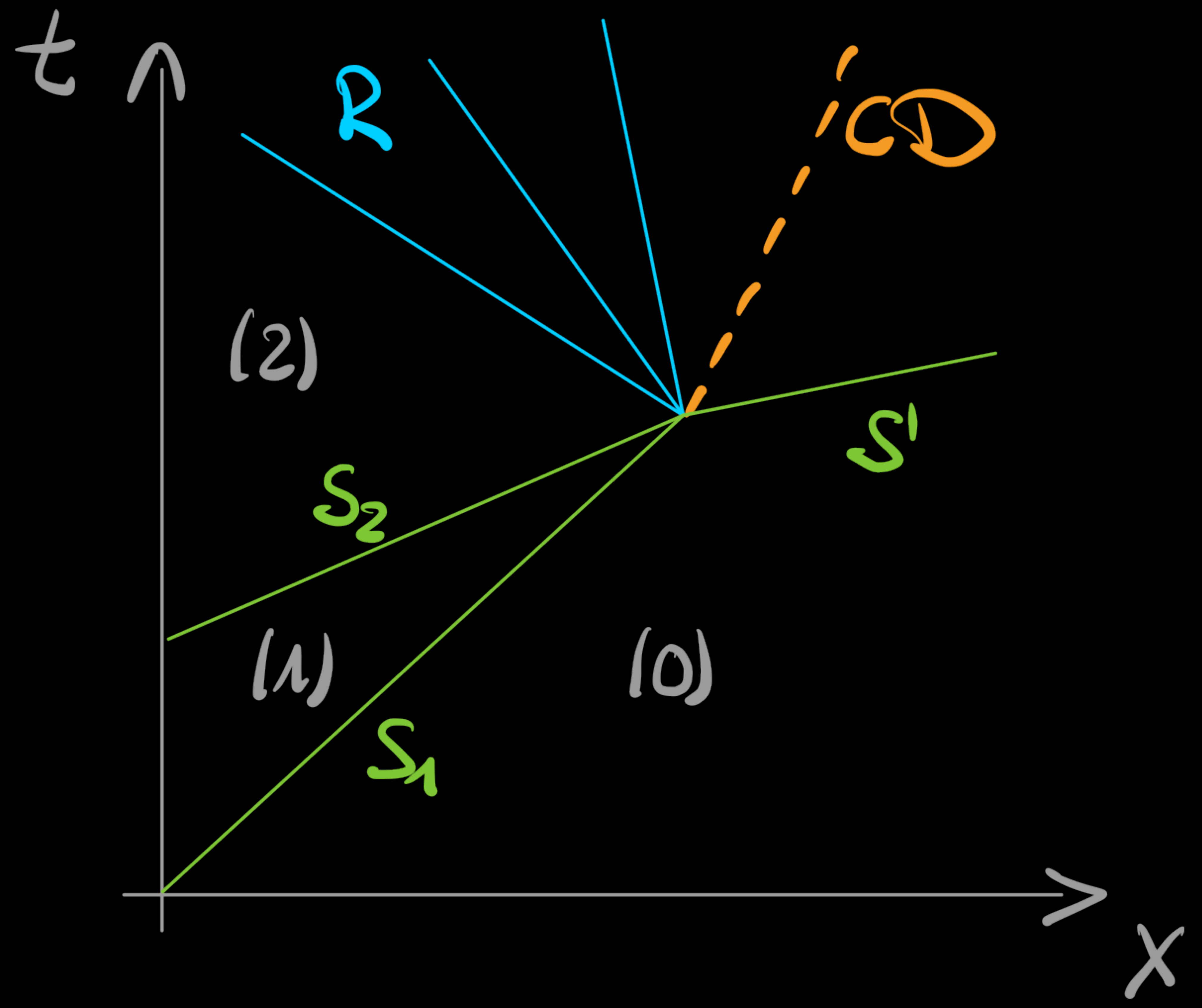
NON-GRADUAL SHEAR ACCELERATION





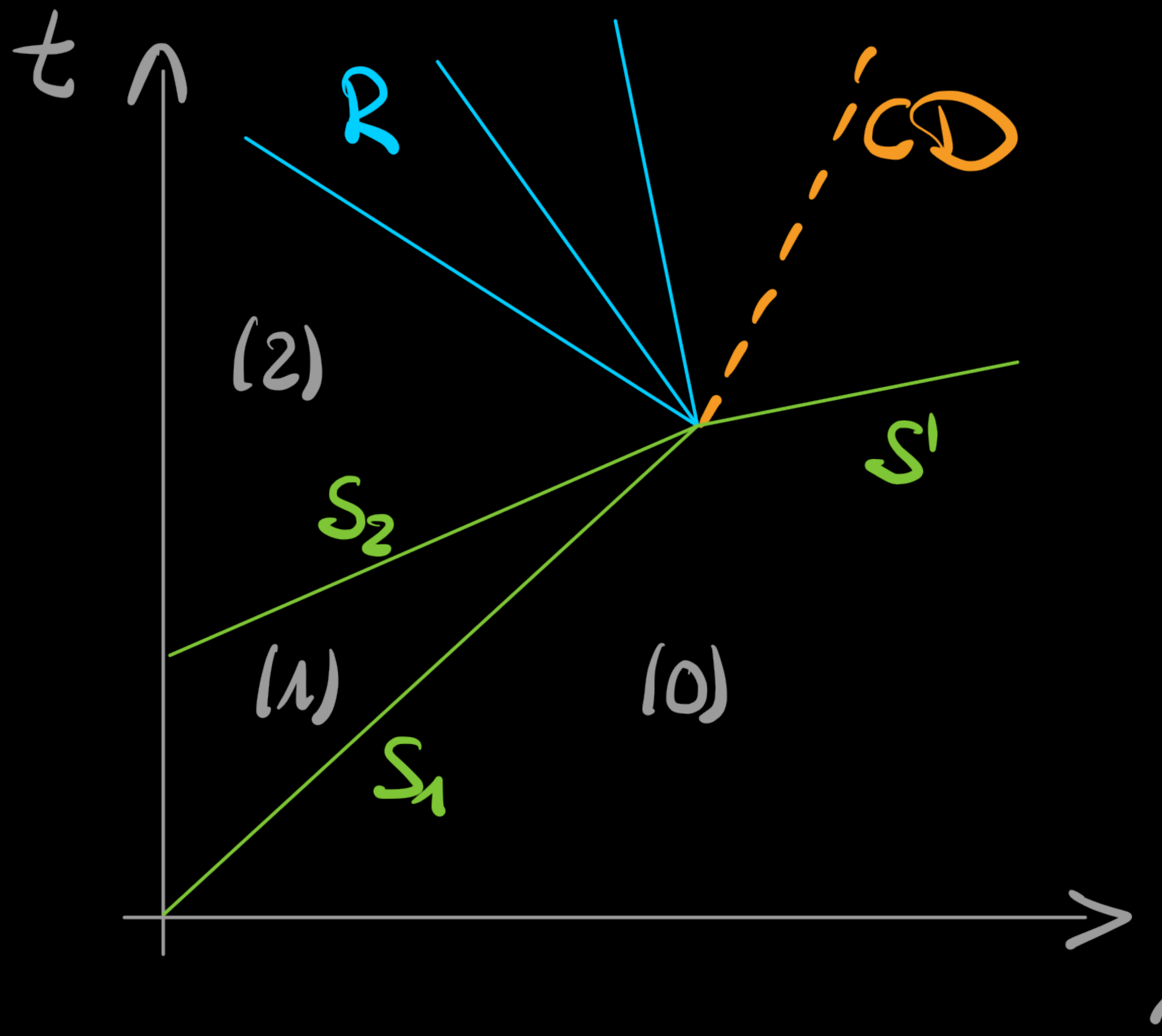
Aerdker+, in prep.

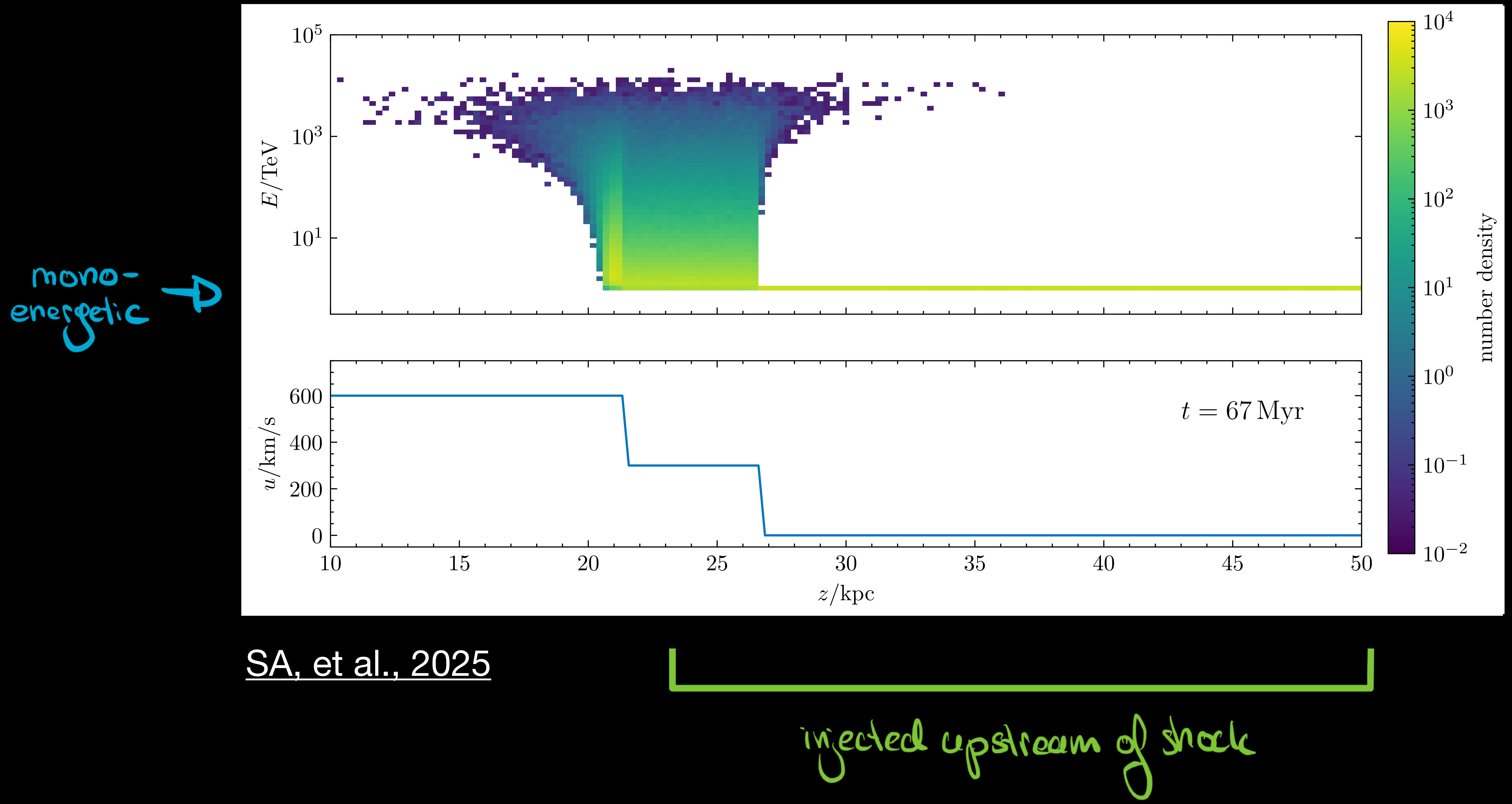
COLLIDING SHOCKS



COLLIDING SHOCKS

CATCHING UP WITH EACH OTHER

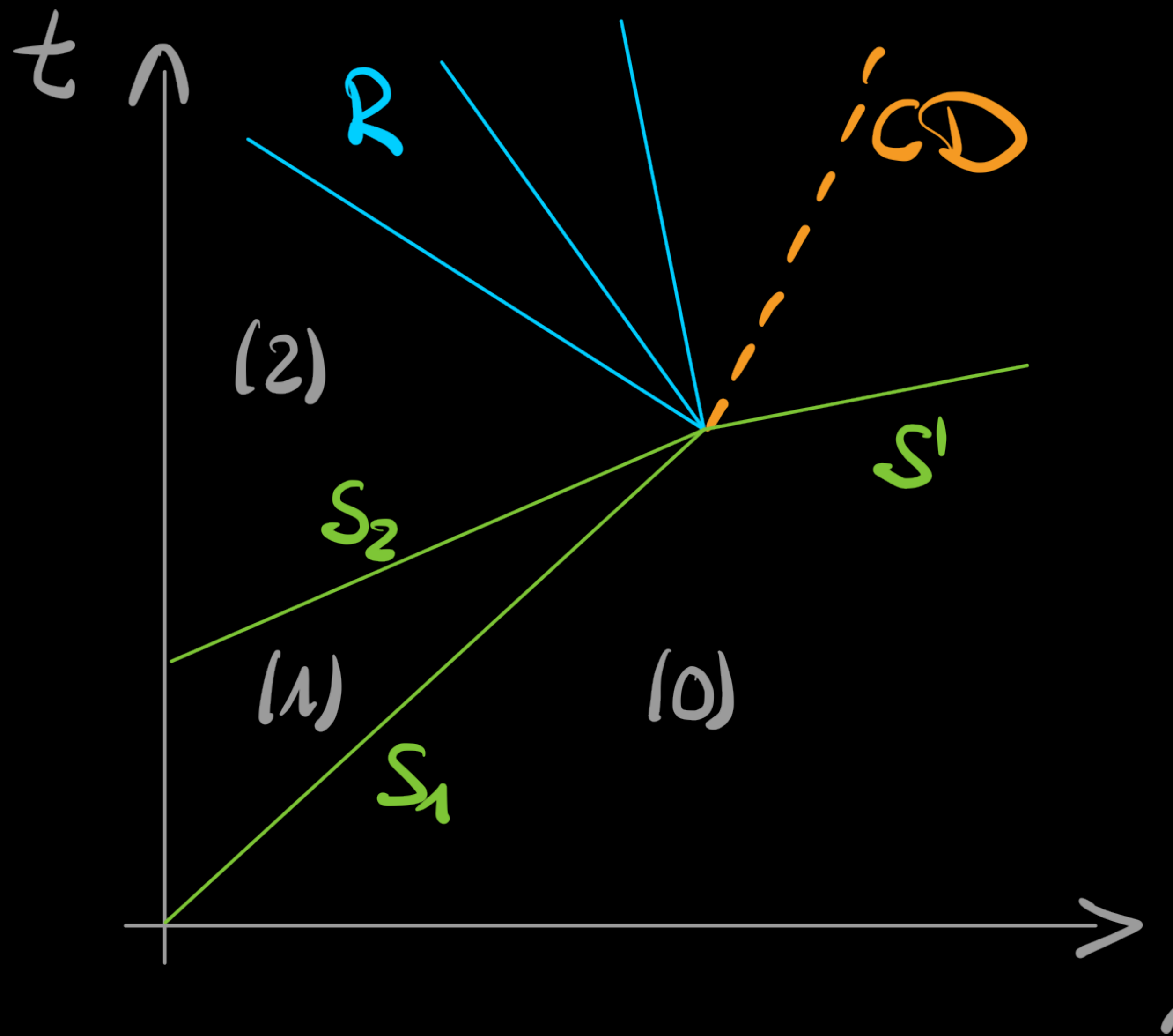


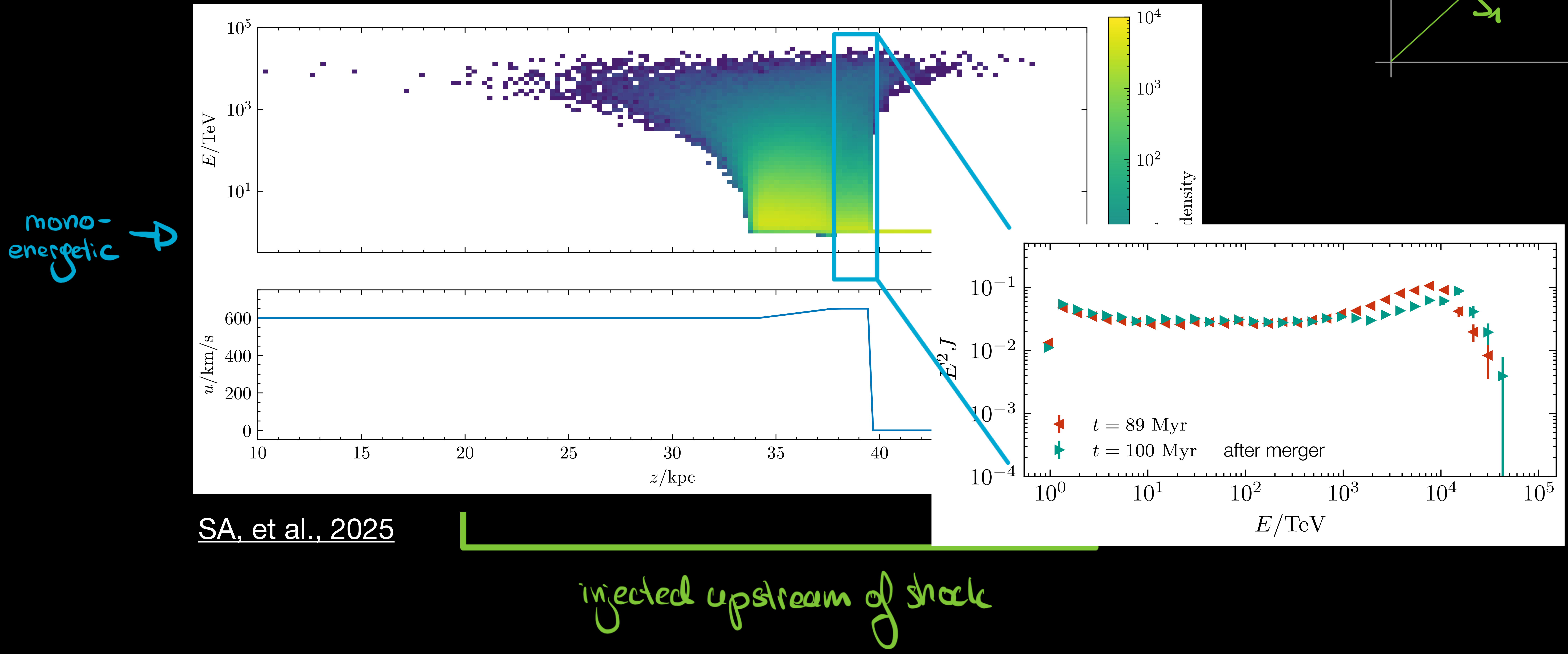


SA, et al., 2025

COLLIDING SHOCKS

CATCHING UP WITH EACH OTHER

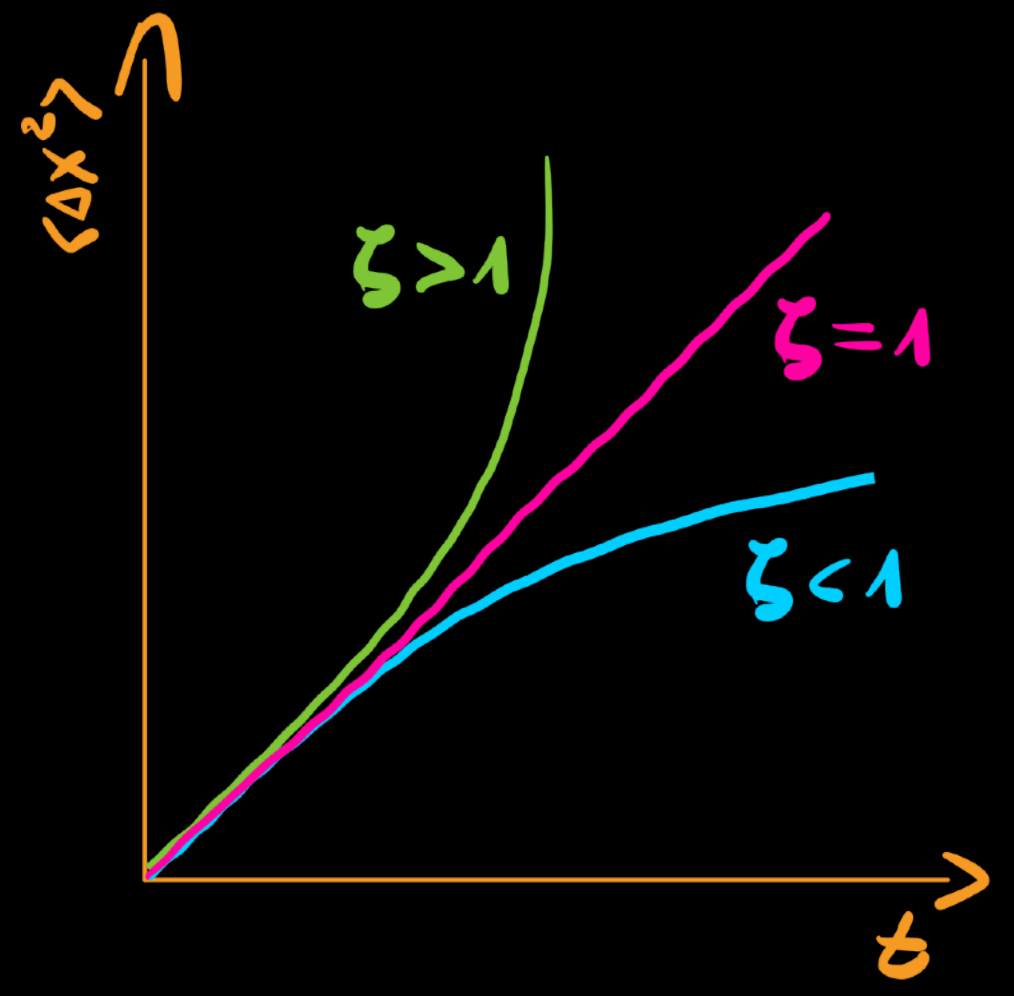




SA, et al., 2025

SUPERDIFFUSION

SOLVING FRACTIONAL TRANSPORT EQUATIONS



$$\frac{\partial f(x,t)}{\partial t} = {}_0D_t^{1-\beta}\left(\frac{\partial}{\partial x}V'(x) + \kappa\nabla^\alpha\right)f(x,t)$$

Riemann-Liouville fractional derivative

$K = K_\xi^{1/\xi}$

Riesz derivative

$${}_0D_t^{1-\beta}f(t) = \frac{1}{\Gamma(\beta)}\frac{d}{dt}\int_0^t (t-s)^{\beta-1}f(s)ds \qquad \nabla^\alpha f(x) = -\frac{1}{2\cos(\alpha\pi/2)}\left({}_{-\infty}D_x^\alpha + {}_xD_{+\infty}^\alpha\right)f(x)$$

SUPERDIFFUSION

SPACE-FRACTIONAL TRANSPORT EQUATION

$$\frac{\partial f}{\partial t} = \kappa_\alpha(p)\frac{\partial^\alpha f}{\partial |x|^\alpha} - u\frac{\partial f}{\partial x} + \frac{p}{3}\frac{\partial u}{\partial x}\frac{\partial f}{\partial p} + S$$

generalized Itô lemma

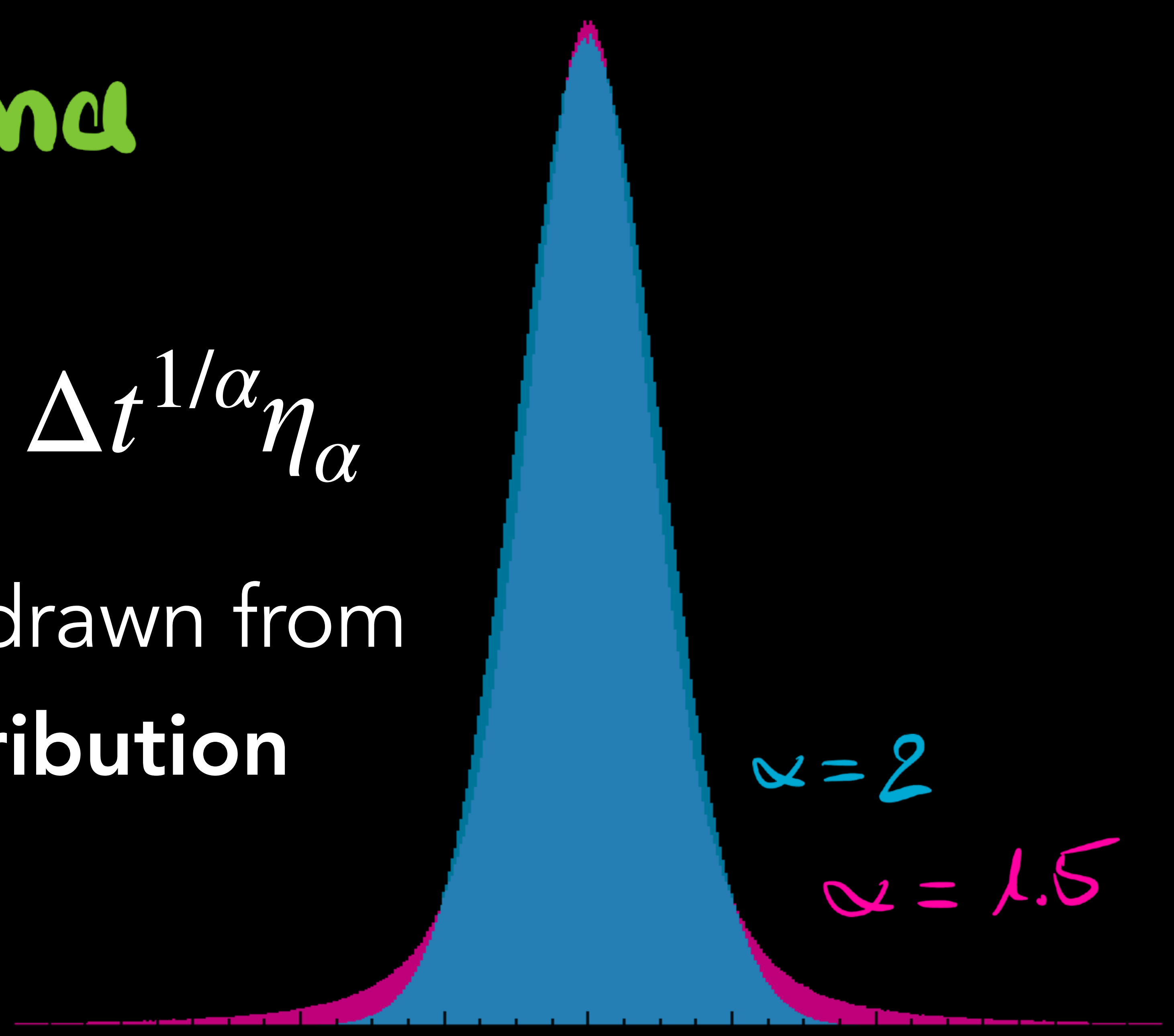
$$\mathrm{d}x = u\,\mathrm{d}t + \sqrt{2}\left(\kappa_\alpha\right)^{1/\alpha}\mathrm{d}L_\alpha(t)$$

$$\mathrm{d}p = -\frac{p}{3}\frac{\partial u}{\partial x}\mathrm{d}t$$

$$\Delta L_\alpha = \Delta t^{1/\alpha}\eta_\alpha$$

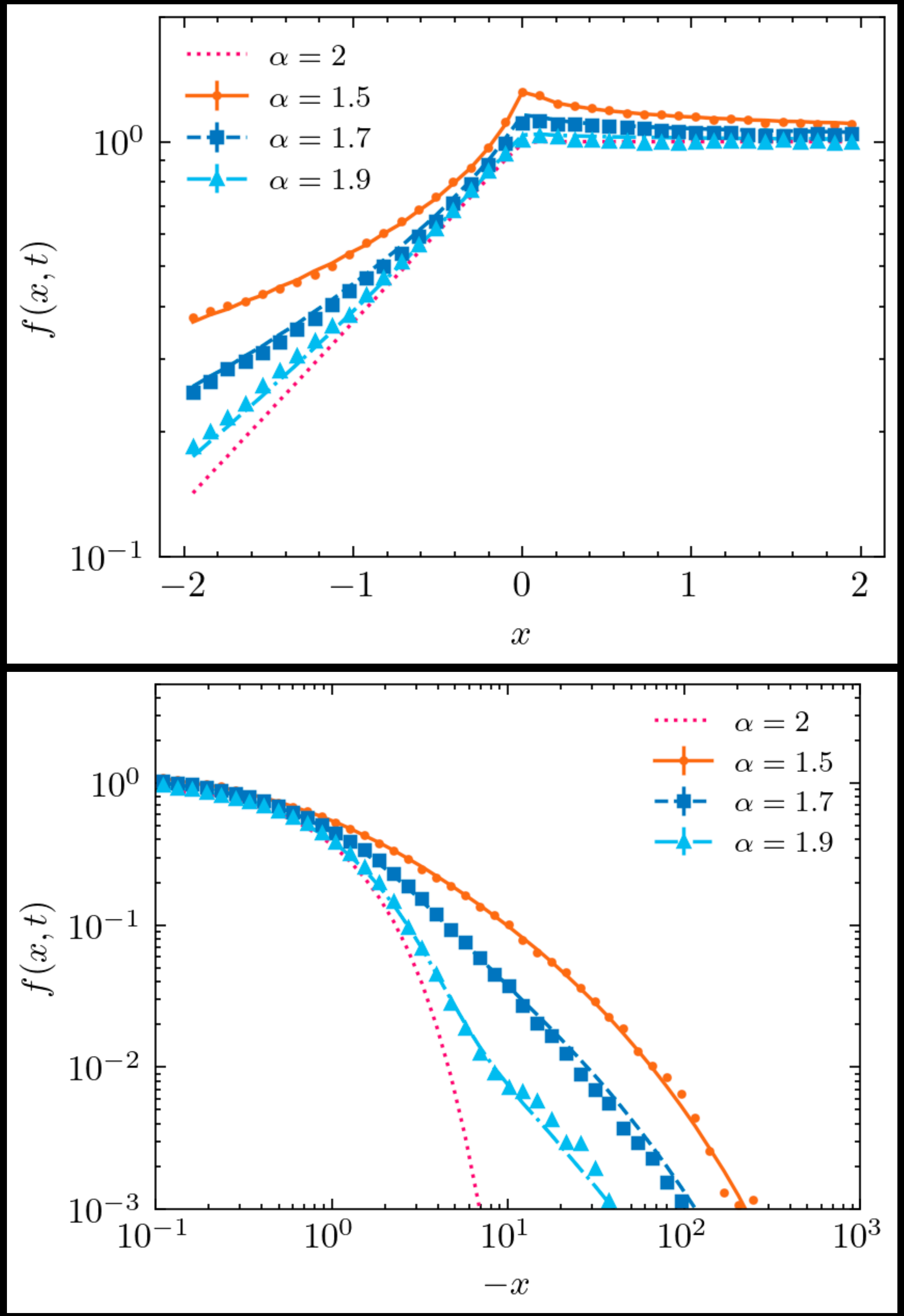
- Random numbers η drawn from a α**-stable Lévy distribution**

(e.g. Janicki&Weron, 94)

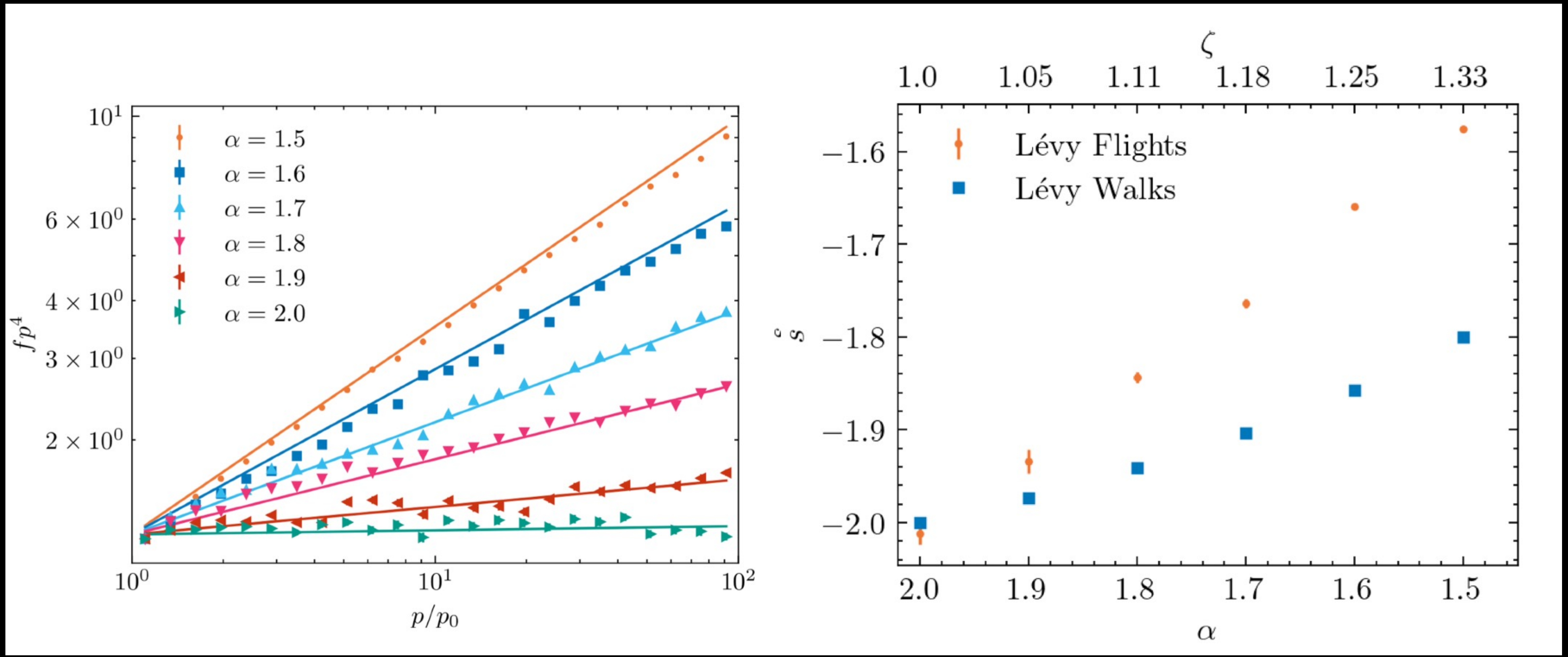


SUPERDIFFUSION

TRANSPORT AND ACCELERATION AT SHOCKS



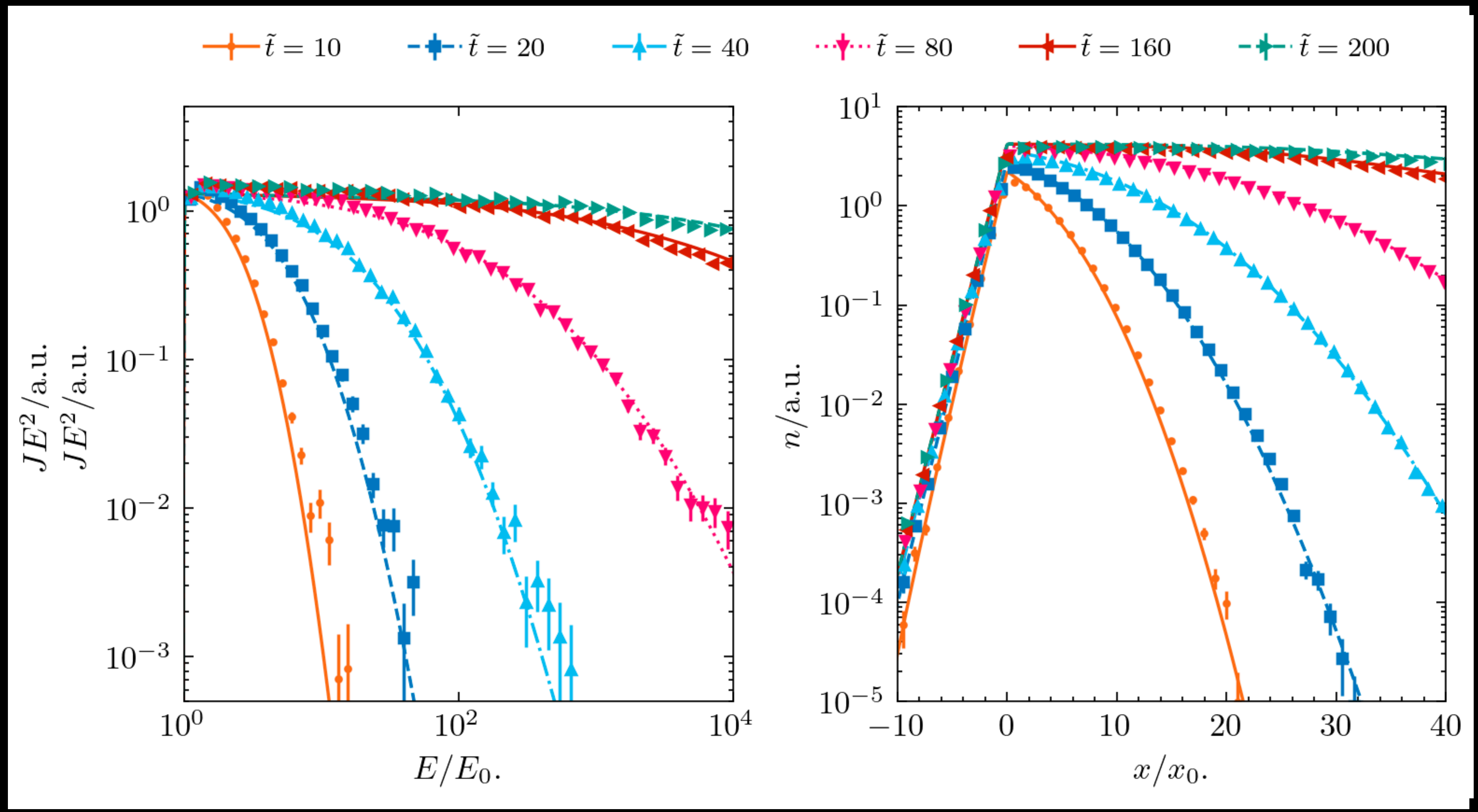
Effenberger, SA, et al., A&A (2024)

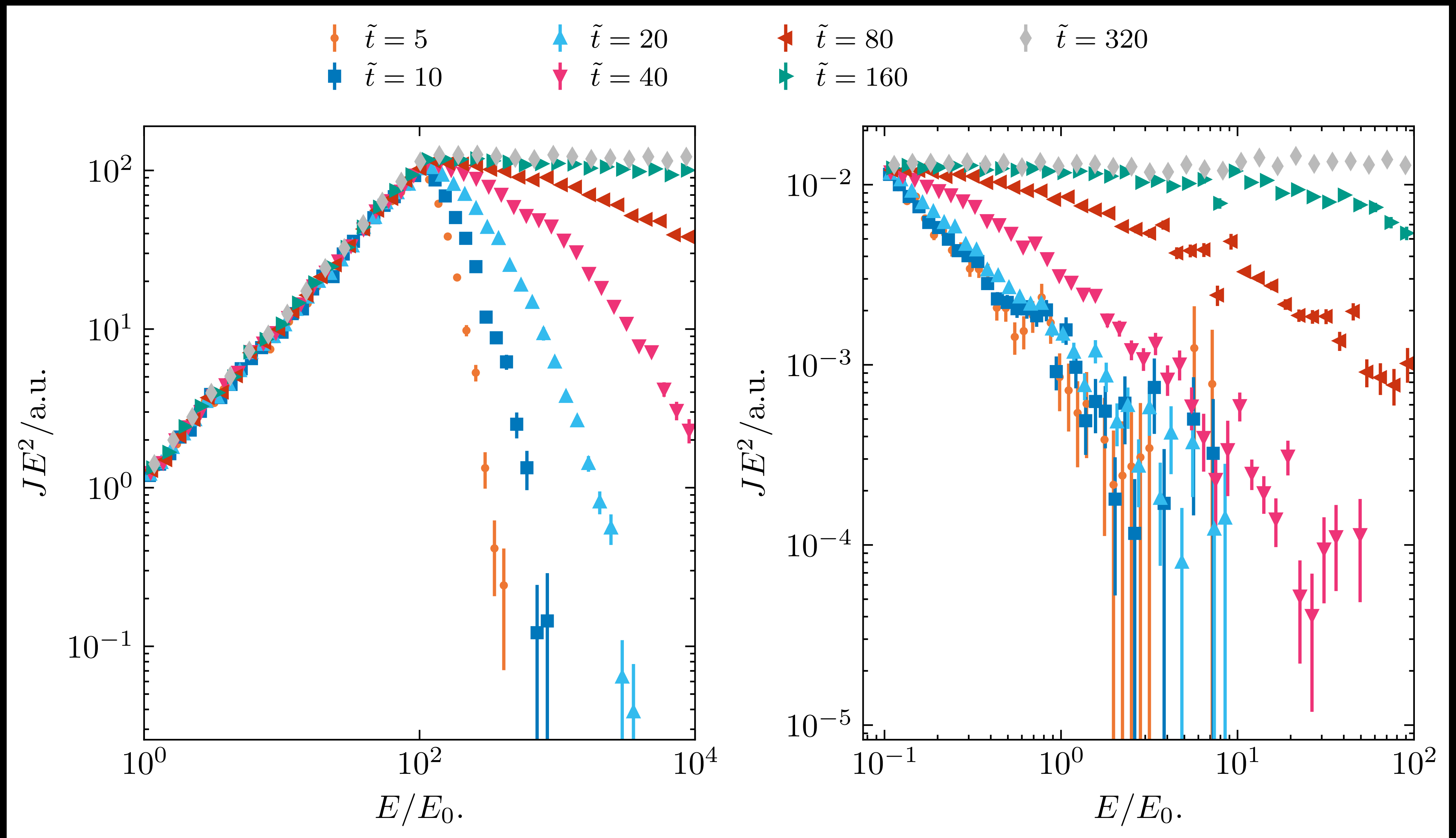


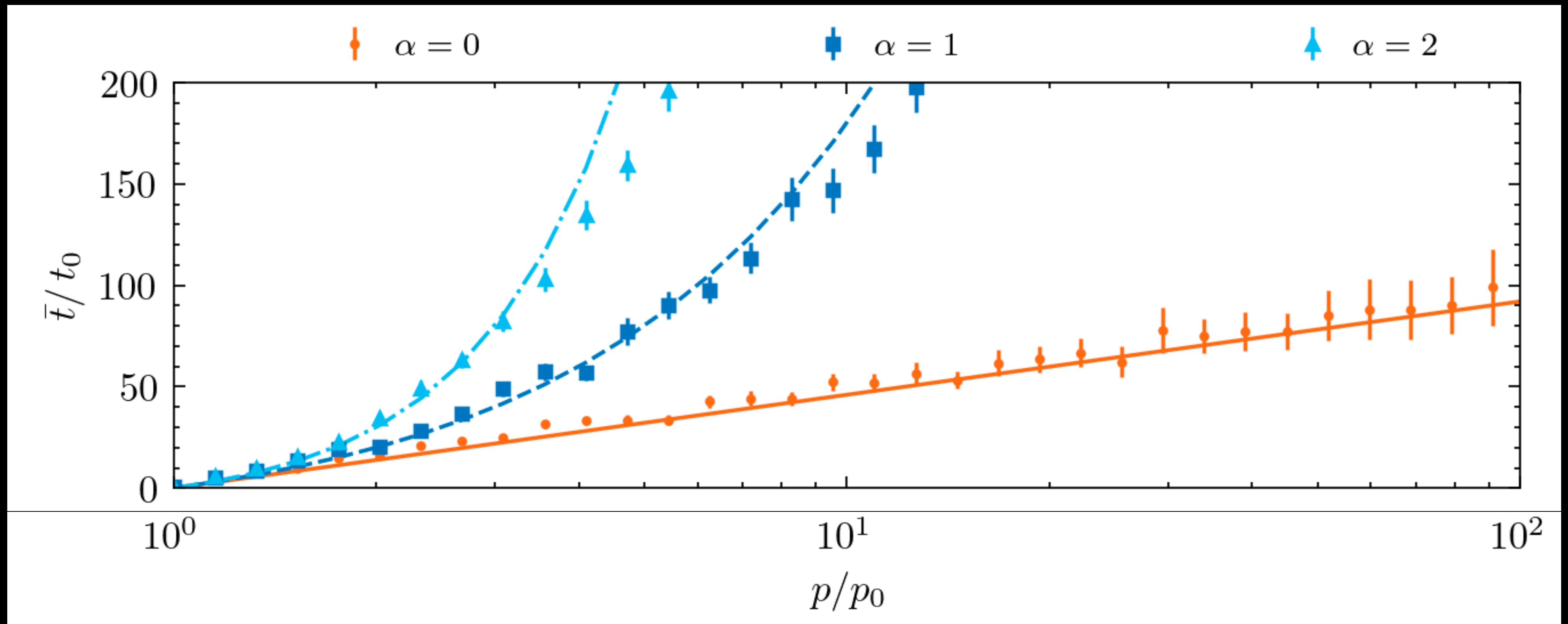
SA, et al., A&A (2025)

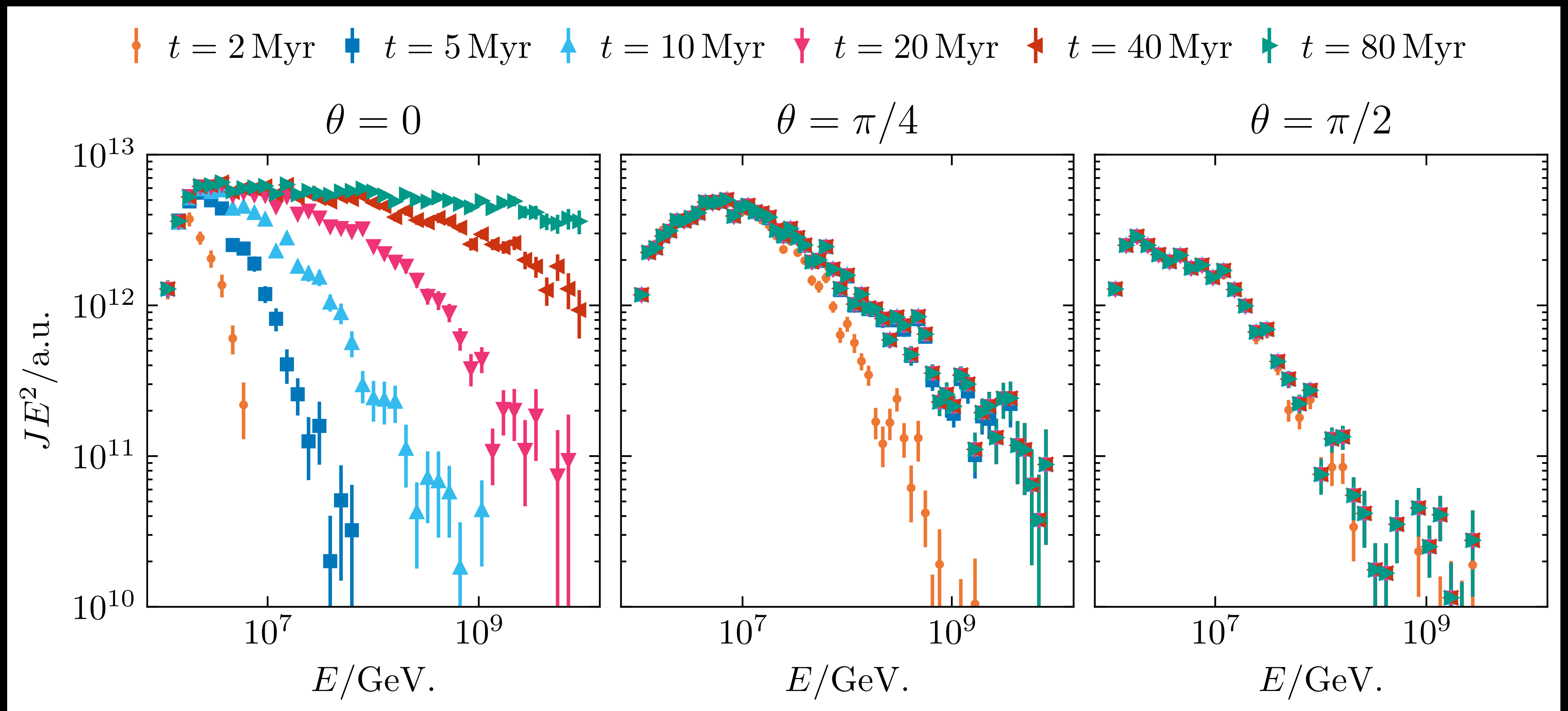
TIME-DEPENDENT DSA

FURTHER EXAMPLES







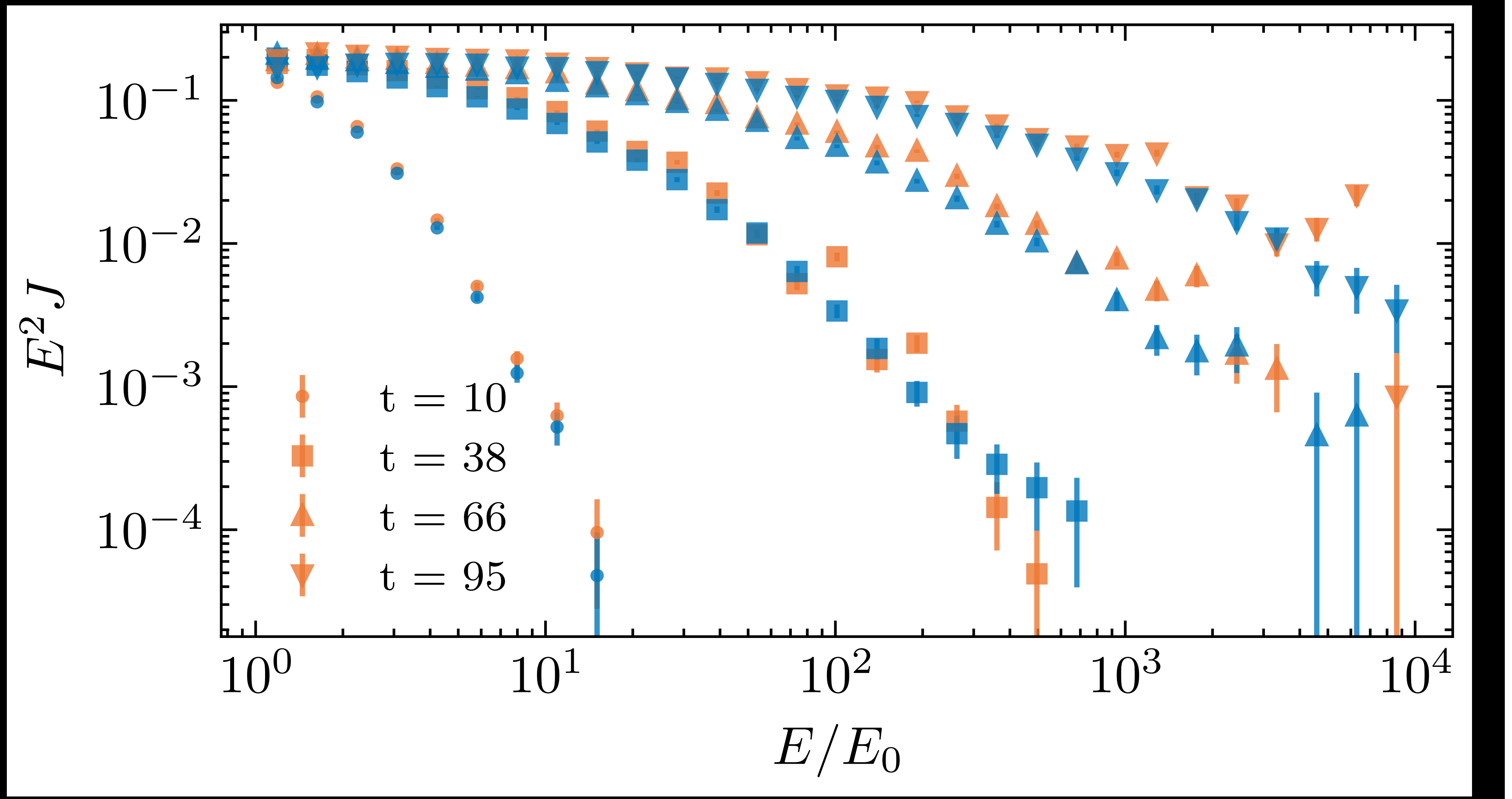


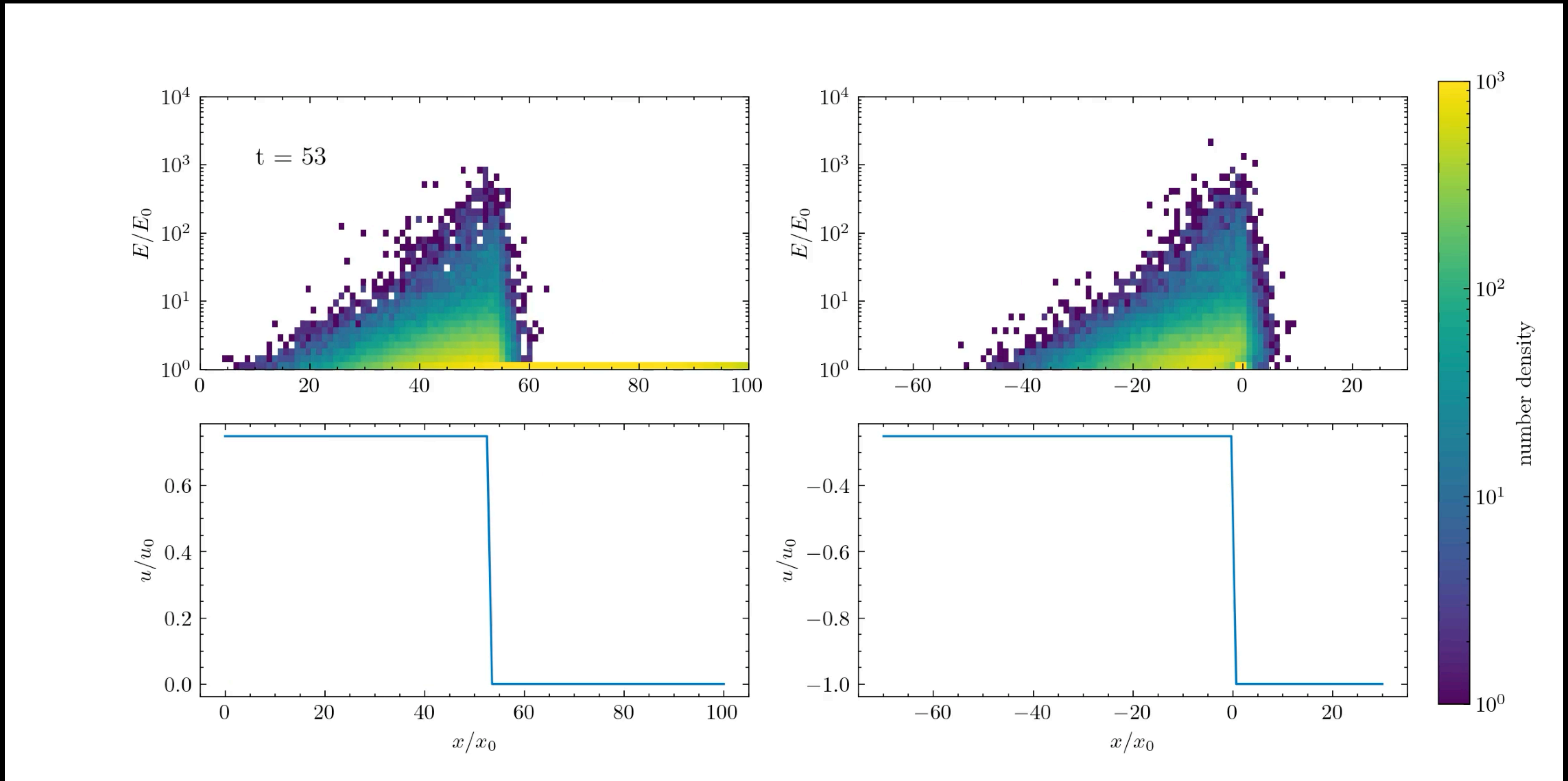
SA, et al., JCAP (2024)

MOVING SHOCKS

FROM STATIONARY TO LAB FRAME

Time evolution of energy spectra at the (moving) shock





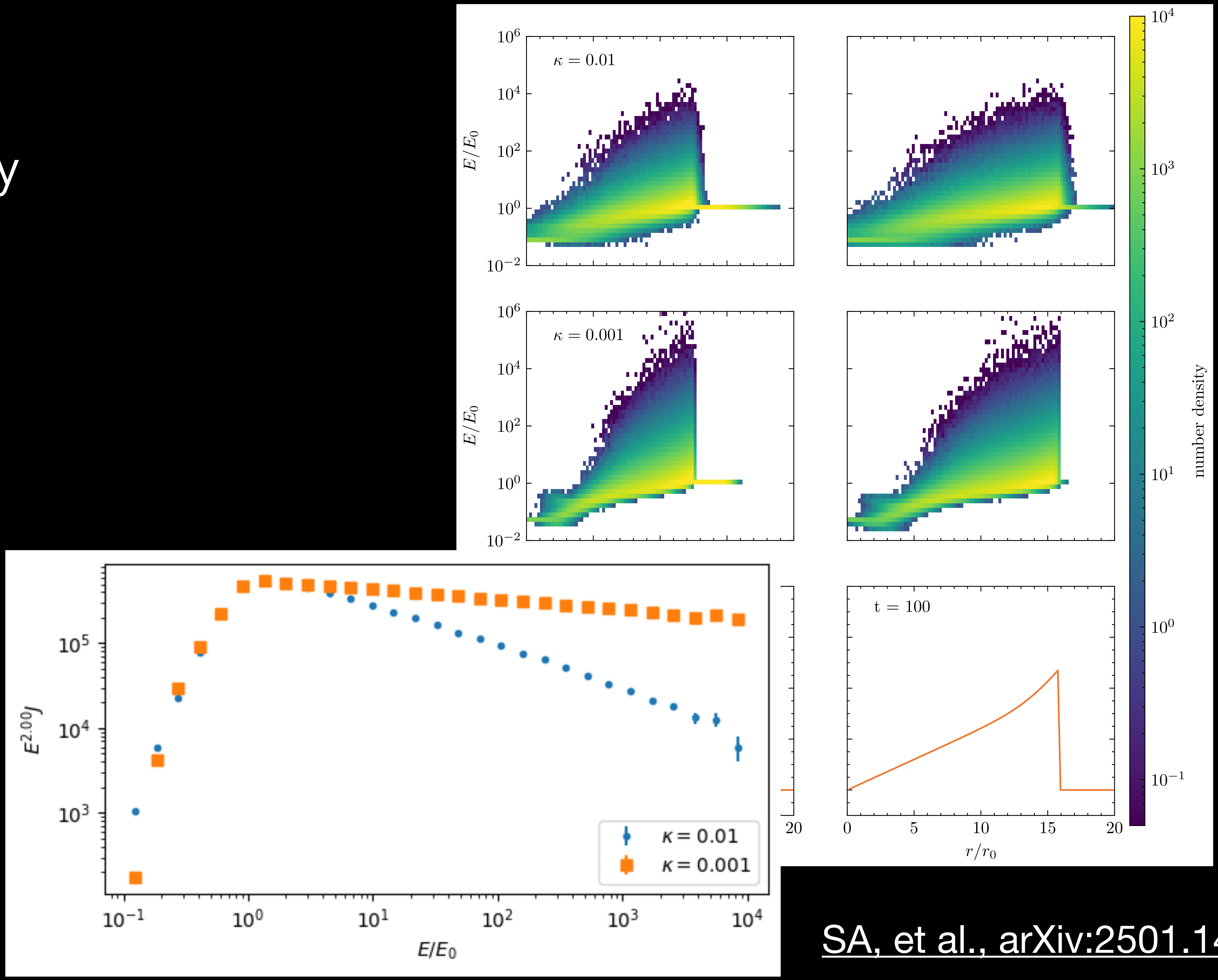
SA, et al., arXiv:2501.14331

SPHERICAL BLAST WAVE

TIME-DEPENDENT EFFECTS

- Time-dependent advection $u(t, r)$ given by Sedov-Taylor similarity solution
- Corrections to the spectral slope $f \propto p^{-a}$ (Drury, 1983)

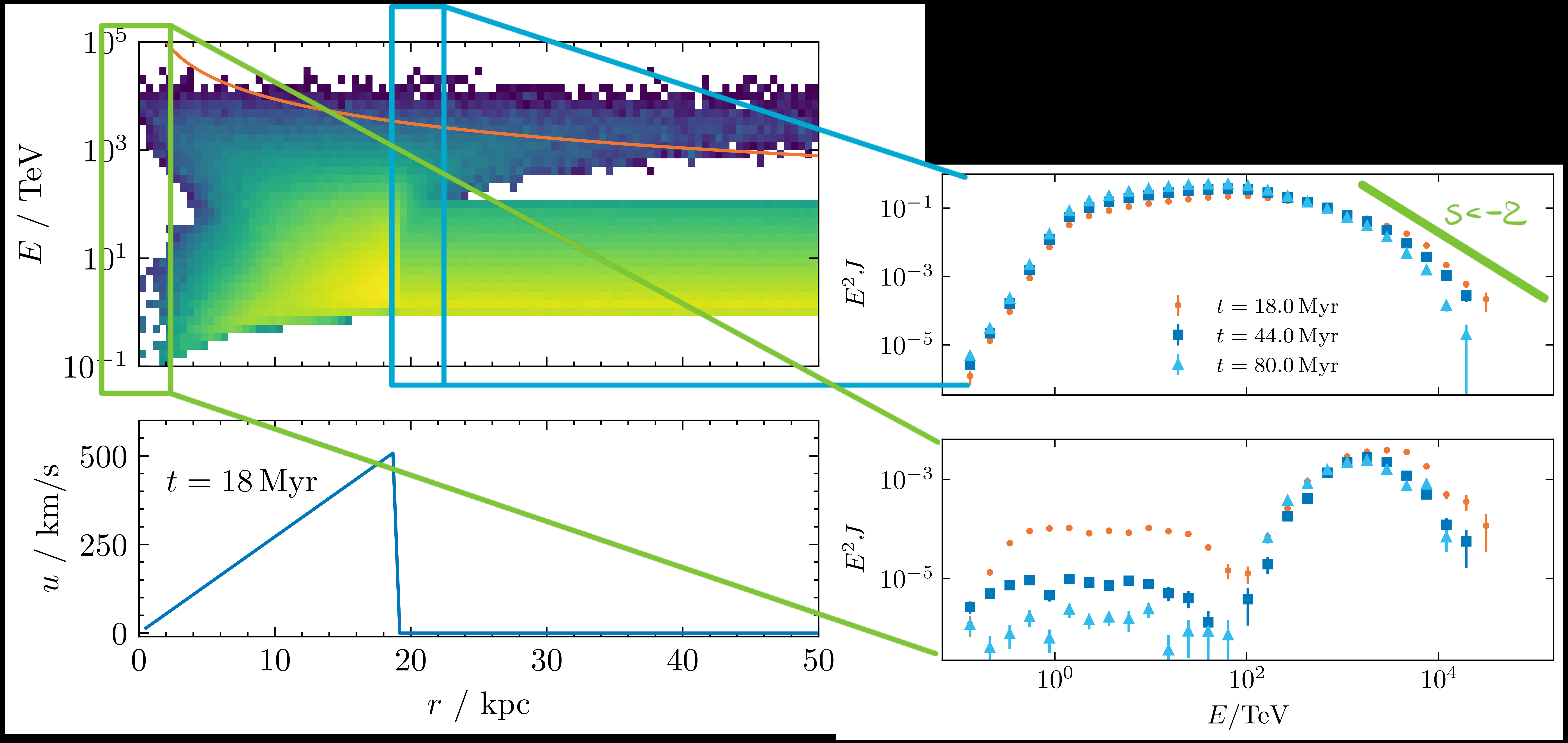
$$a = 4\left[1 + (3+b)\left(\frac{\kappa_1}{Ru_1} + \frac{\kappa_2}{Ru_2}\right) + \frac{\kappa_2}{Ru_2}\right] + O(\epsilon^2)$$



SA, et al., arXiv:2501.14331

MOVING SHOCKS

SPHERICAL BLAST WAVES



SA, et al., 2025