Tidal deformability of rotating neutron stars

GWsNS School: Roscoff

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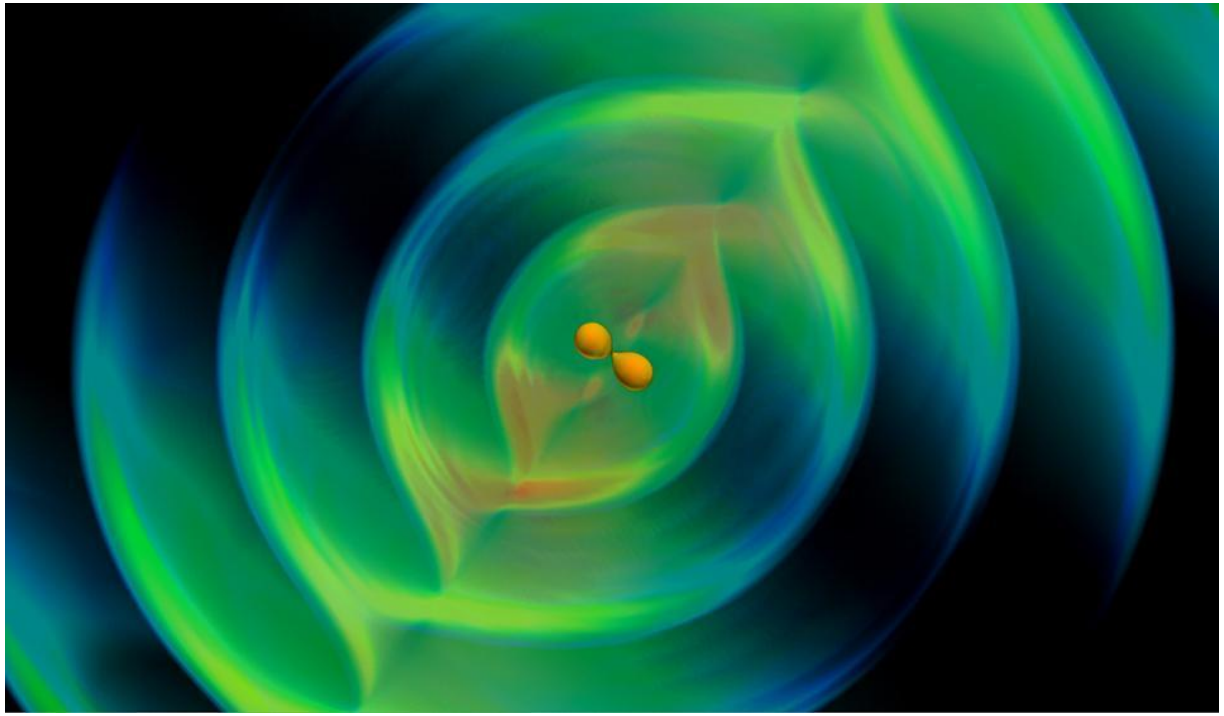
Motivation

- For BNS, the spin and tidal post-Newtonian orders are inherent.
    - However treated as separate, uncoupled phenomena.
- Spin biases in Bayesian analysis impact the measurement of Λ_2.
    - Large χ extends upper limit for Λ_2 inference.
- This suggests that spin impacts the tidal deformability – hinting at a tidal-spin coupling in waveform modelling.

- The question is: **HOW** does spin impact the tides?



T. Dietrich, AEI Potsdam-Golm and Uni Potsdam

- The NS's induced quadrupole is linearized using Lagrange multipliers, with each gradient defining a tidal deformability response.

$$\mathcal{Q}_{ij} = \left(\frac{\partial \mathcal{Q}_{ij}}{\partial \mathcal{G}_{ij}}\right)\mathcal{G}_{ij} + \left(\frac{\partial \mathcal{Q}_{ij}}{\partial \mathcal{B}_{ij}}\right)\mathcal{B}_{ij}$$

- The *l=2* magnetic tidal field is revised in terms of an *octupolar* tide, which is physically relevant when there is angular momentum.

$$\mathcal{Q}_{ij} = \lambda_{(2)}\mathcal{G}_{ij} + \lambda_{(2,3)}\mathcal{J}^k\mathcal{H}_{ijk}$$

- Extracting tidal information from this linearized quadrupole is just as strategic as it is difficult.

$$\left(\frac{\partial \mathcal{Q}_{ij}}{\partial \mathcal{E}_{ij}}\right)_{\rm total} = \left(\frac{\partial \mathcal{Q}_{ij}}{\partial \mathcal{E}_{ij}}\right)_{\rm static} + \left(\frac{\partial \mathcal{Q}_{ij}}{\partial \mathcal{E}_{ij}}\right)_{\rm rotational}$$

$$\implies \Lambda_2^{\rm total} = \Lambda_2^{(0)} + \chi\Lambda_2^{(1)}.$$

$$\left(\frac{\partial \mathcal{Q}_{ij}}{\partial \mathcal{E}_{ij}}\right)_{\rm total} = \lambda_{(2)} + \lambda_{(2,3)}\left(\frac{\partial \mathcal{B}_{ij}}{\partial \mathcal{E}_{ij}}\right)$$

$$\implies \Lambda_2^{\rm total} = \Lambda_2^{(0)} + \chi^2\Lambda_2^{(2)}.$$

- **ANSATZ**: The spin-enhanced tidal deformability is a polynomial in χ:

$$\lambda_{l=2}^{\text{total}} = \lambda_0 + \chi\lambda_1 + \chi^2\lambda_2 + \mathcal{O}(\chi^3).$$

Static tidal number $2k_2R^5/3$ (5PN)

LO rotational tidal number (more informative for slow-spins, 6.5PN)

NLO rotational tidal number (less informative for slow-spins, 8PN)

- arXiv:1509.02171 readily provides LO rotational tidal numbers, but under a convention not mutual to original static formula (Symmetric relations nonetheless offered in arXiv:2103.16595).

- The presented work provides LO and NLO rotational tidal numbers (see bonus slides for NLO, i.e. 2nd order spin).

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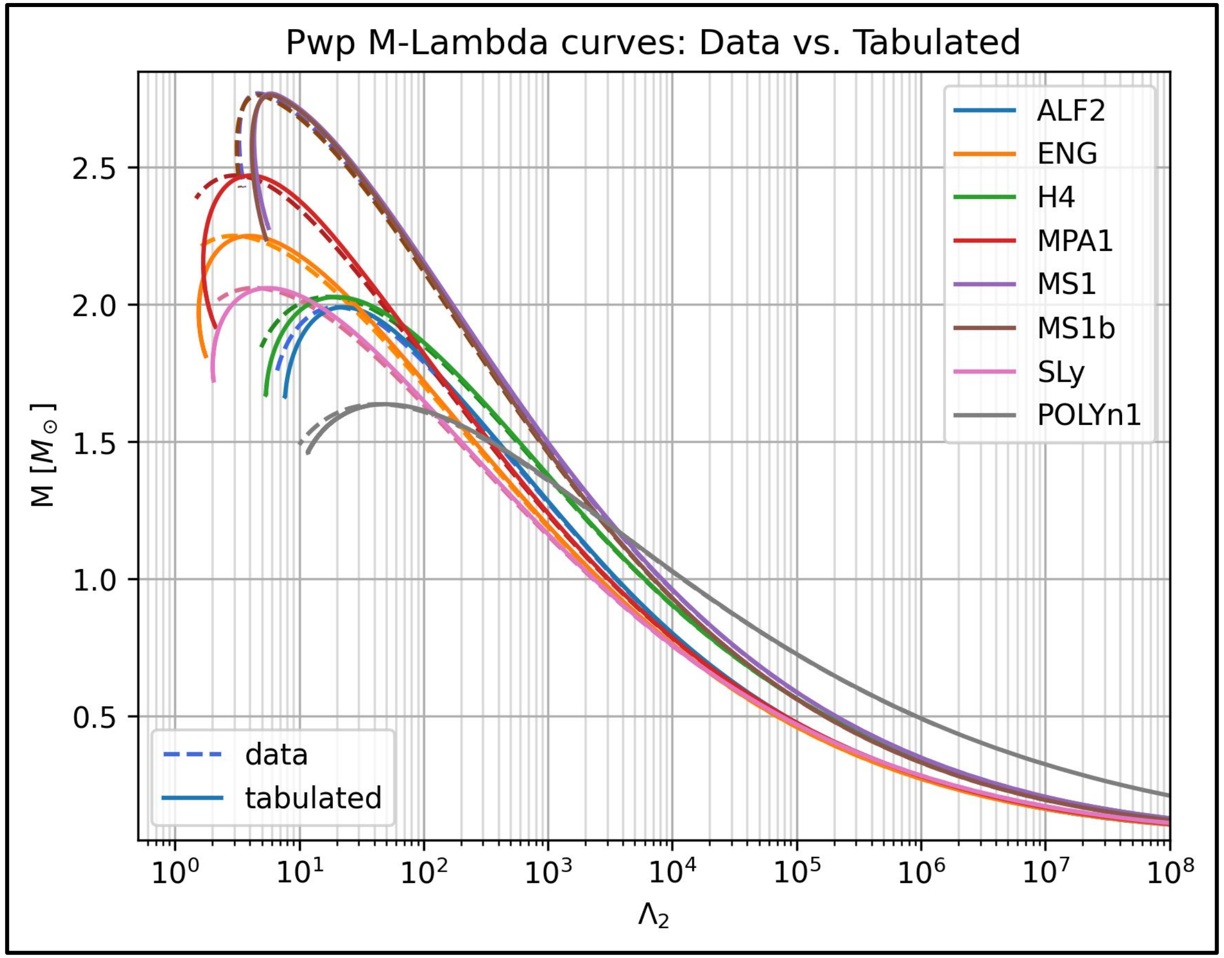
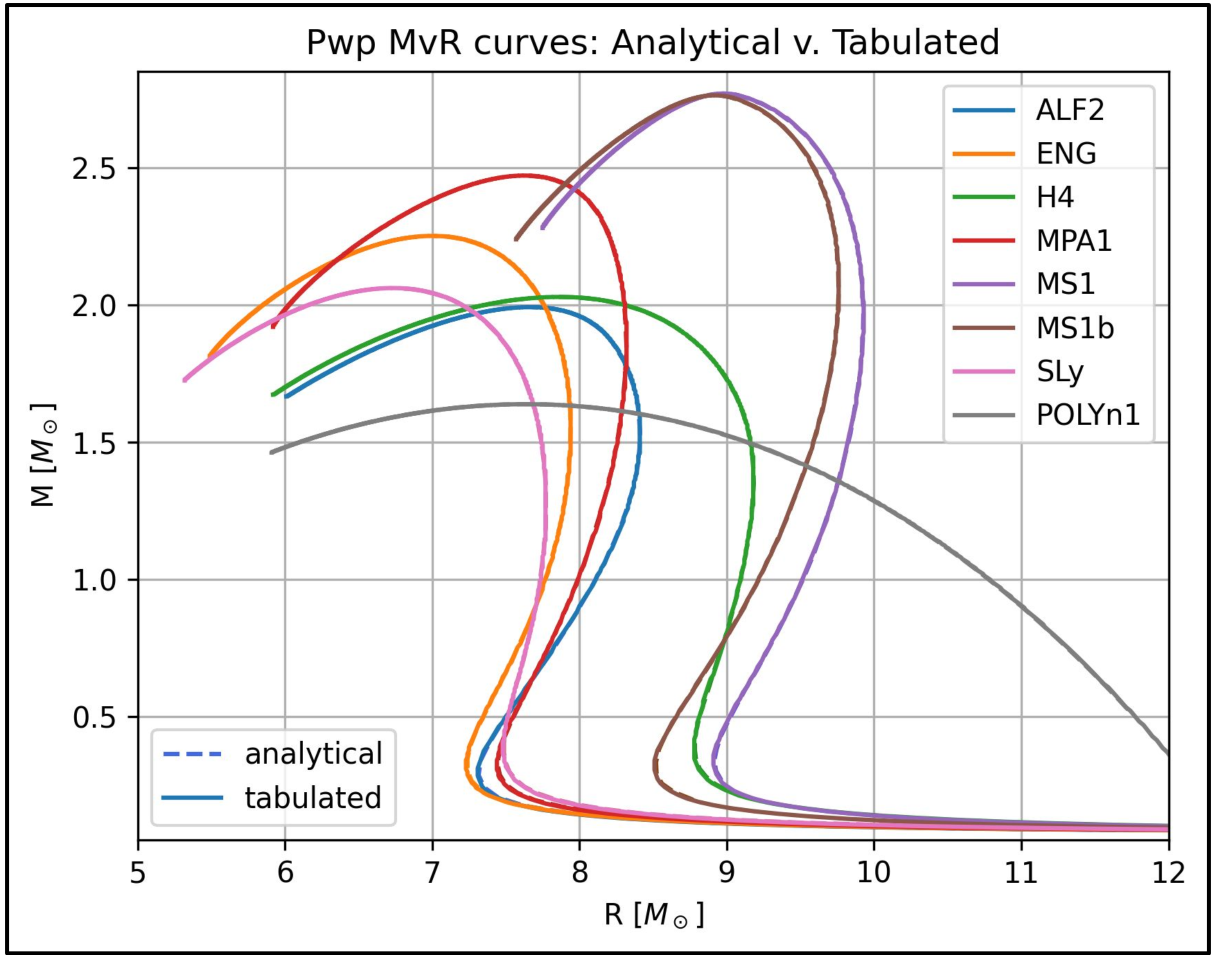
Results





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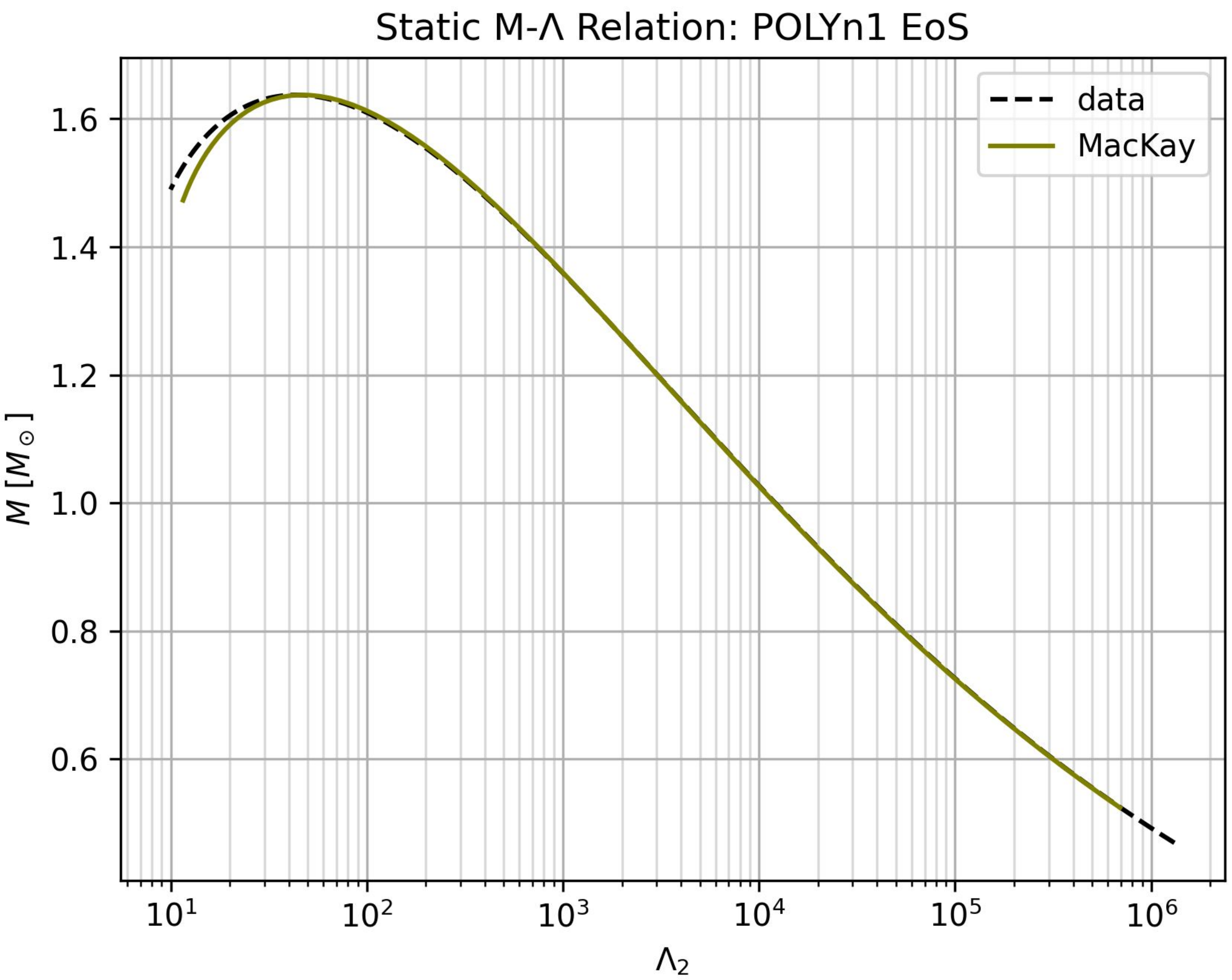
Representative EoS: Simple Polytrope "POLYn1" (K=100, Γ=2 i.e. n=1)

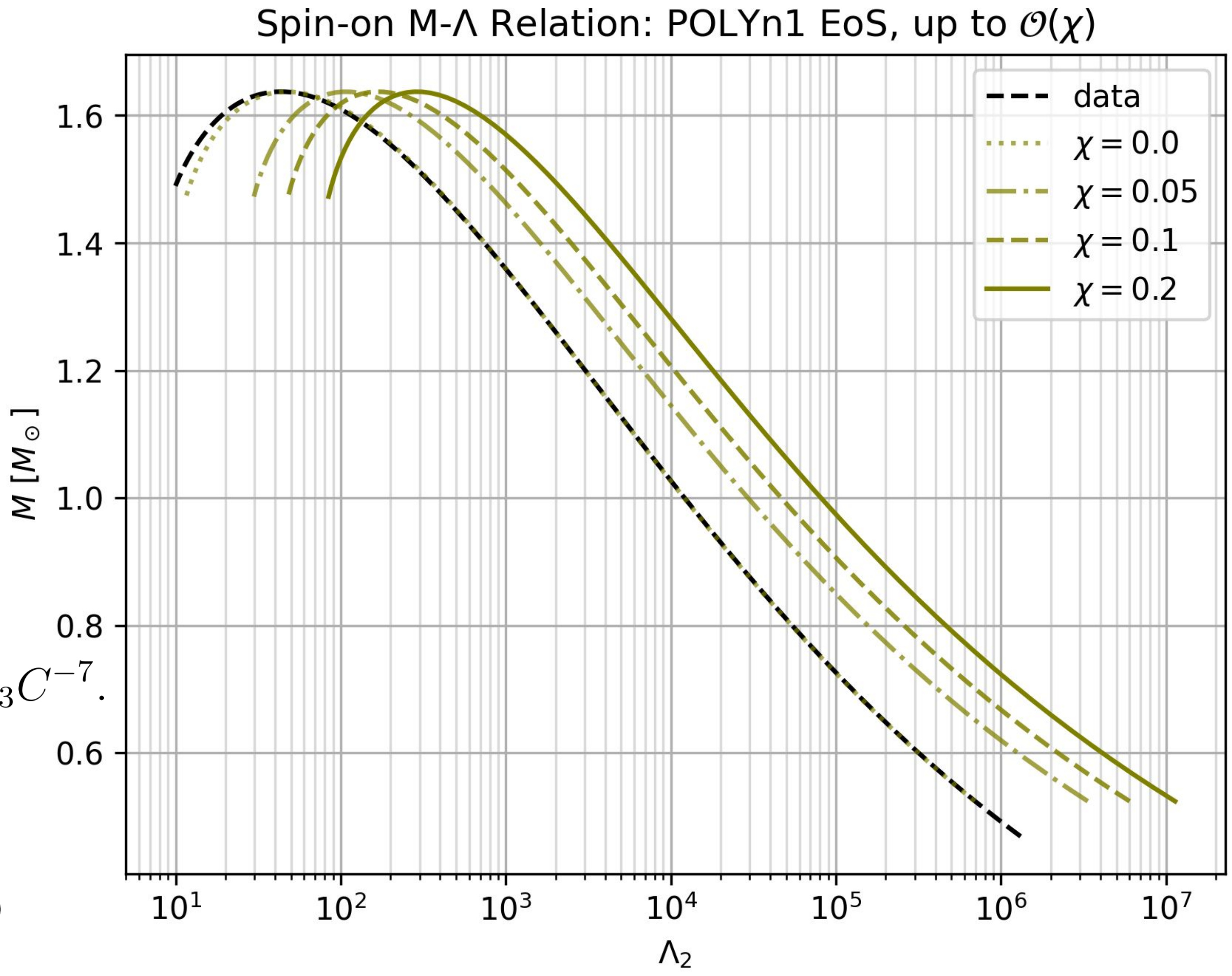


Representative EoS: Simple Polytrope “POLYn1” (*K*=100, *Γ*=2 i.e. *n=1*)

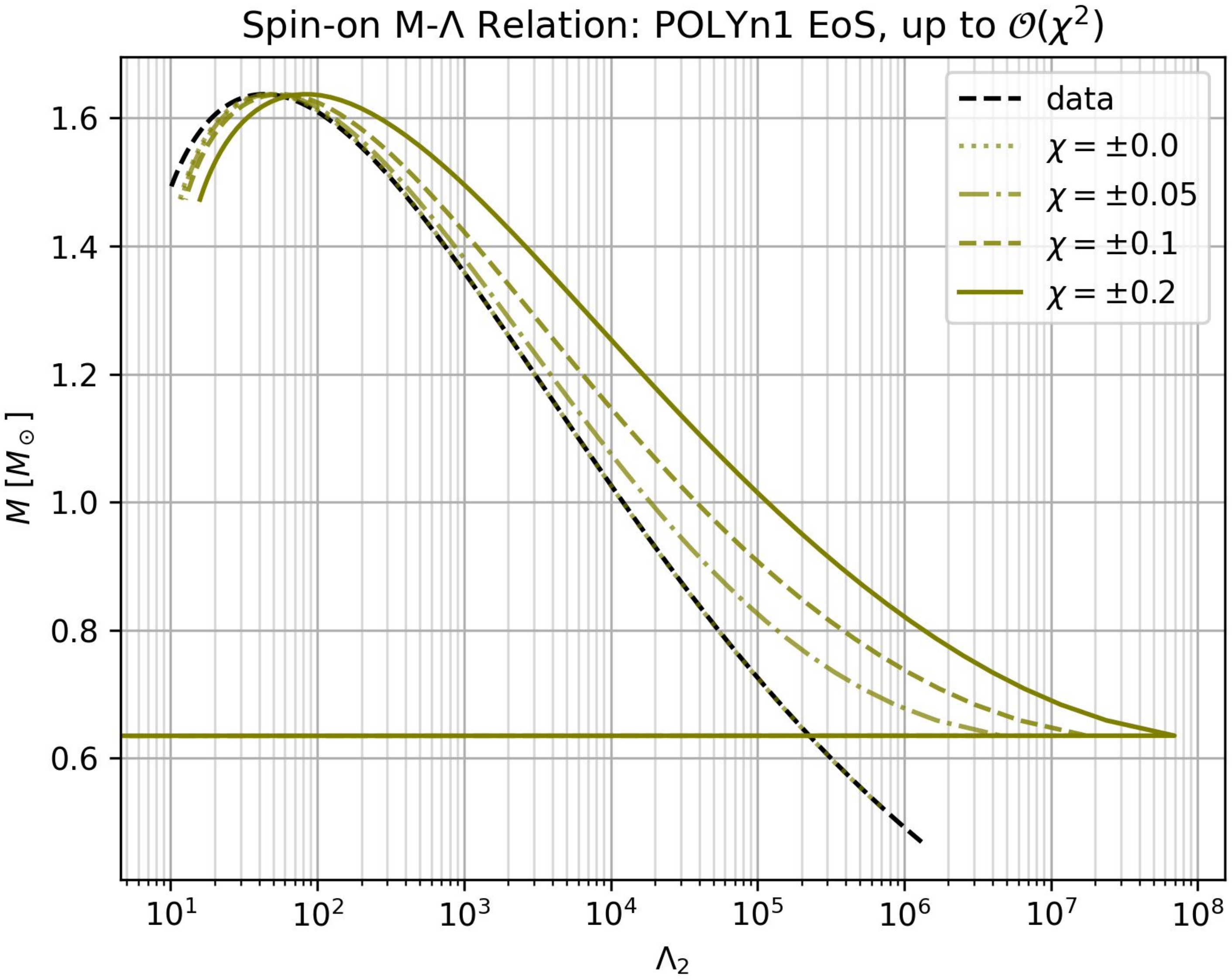
Exercise for viewer:
Find the issue!

$$\Lambda_2^{\text{total}} = \left(\frac{2}{3} - \chi\frac{1}{6}p_{32}\right) k_2 C^{-5} - \chi\left(\frac{700}{27}\sqrt{\frac{7}{20}}\frac{k_3^*}{k_3} - \frac{37}{160}\right) k_3 C^{-7}.$$

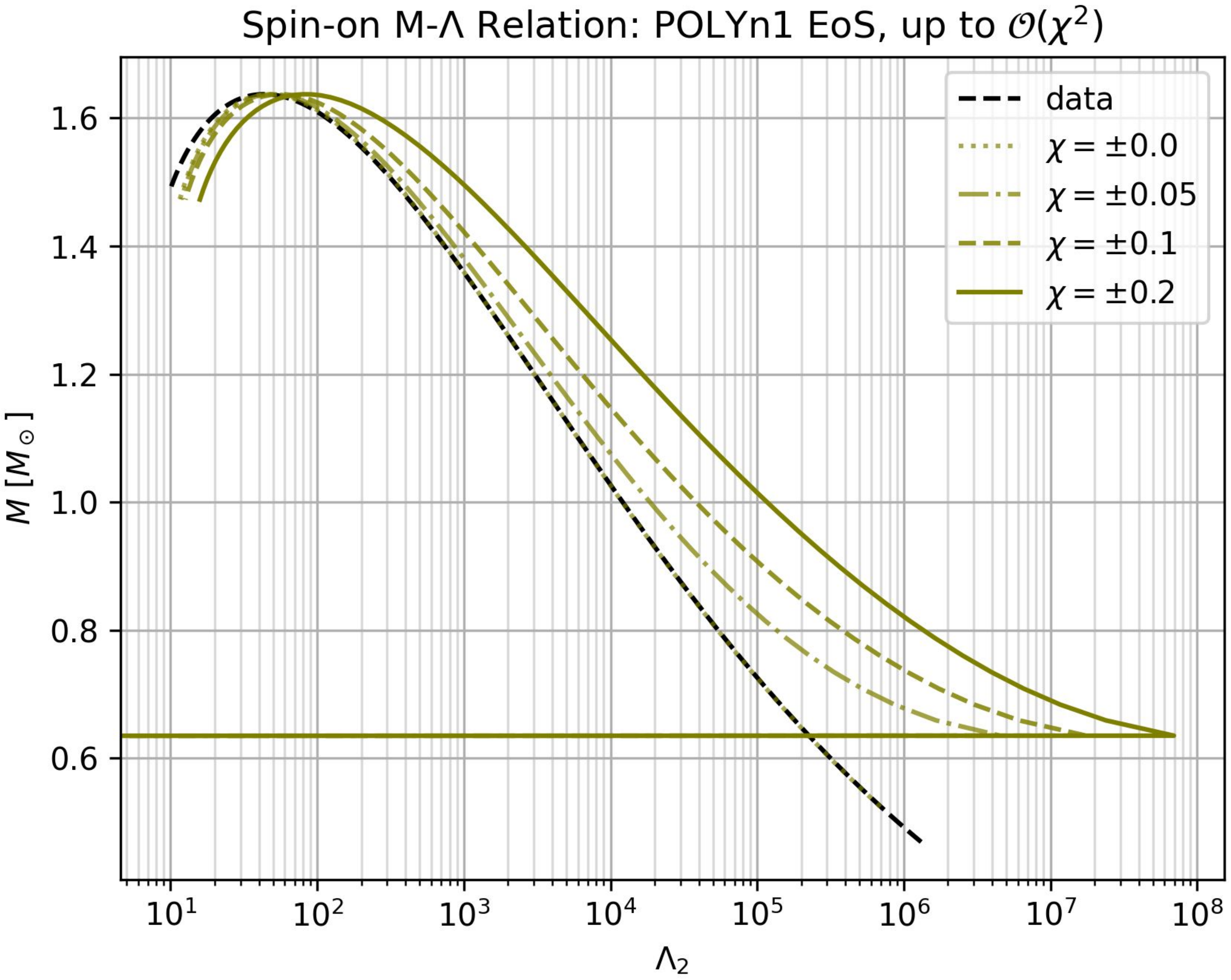
(see the last bonus slide for p_{32} term)



Representative EoS: Simple Polytrope "POLYn1" (K=100, Γ=2 i.e. $n=1$)



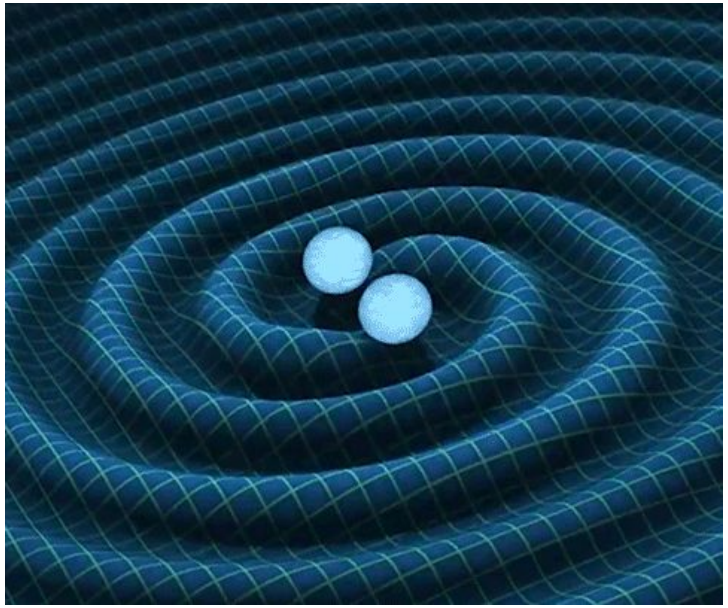
Representative EoS: Simple Polytrope “POLYn1” (K=100, Γ=2 i.e. n=1)



Discussion of (both) results:

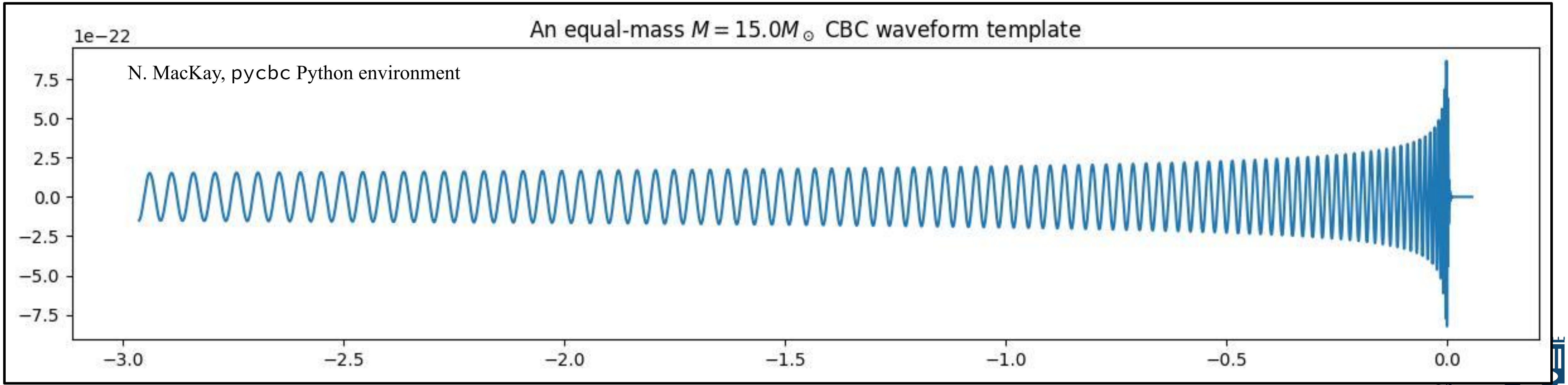
- Large χ shifts the M-Λ curve translationally and/or extremely.
- No RNS code implied (no equatorial mass increase with rotations)
- Compiled using a TOV solver with tidal and slow-spin layers (somewhat relevant for slow-spins only)

What is Anticipated?



Take this total tidal coefficient to current waveform models:

- How does the spin-enhanced tide influence the waveform structure?
- Can we implement tidal-spin coupling (TSC) with what we currently have…
  - …or must we obtain a new waveform that better suits TSC?

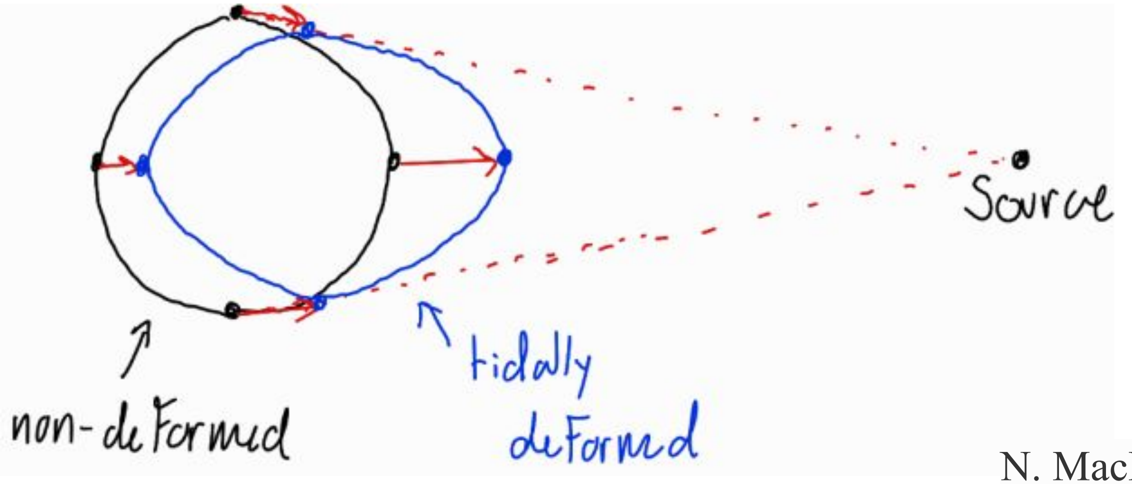


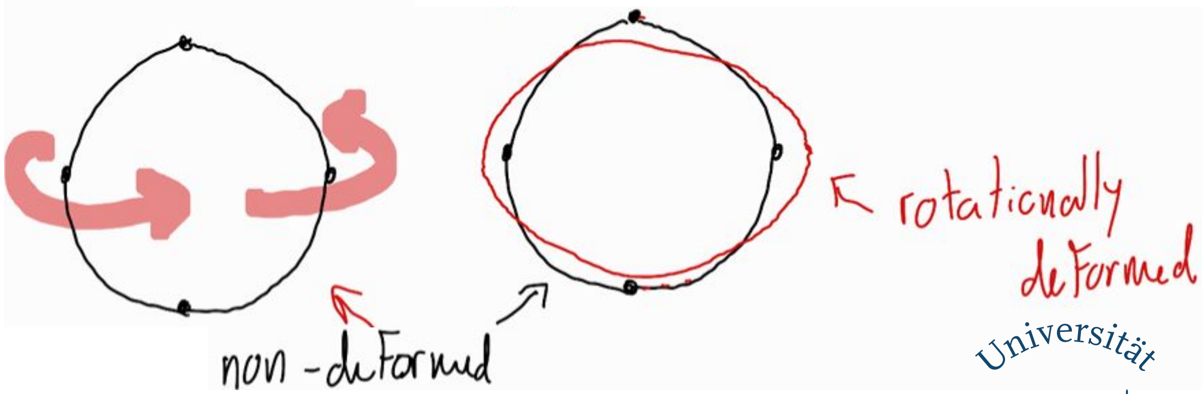
Bonus Slides

Intuition: Oblation under tidal and axial effects

- The Riemann tensor embeds the Ricci and Weyl curvature tensors.
  - Ricci → matter, Weyl → energy.

- The Weyl tensor, in GR language, has an "electric" component and a "magnetic" component:
  - "Electric" induces external tidal fields → spherically deforms its binary partner (if made of matter).
  - "Magnetic" is purely rotational: present if source is rotating → oblates the source regardless of its composition.





N. MacKay, hand drawn

- The NS's angular momentum $\mathfrak{J}_k$ generates a second, purely rotational tidal field with its own tidal response coefficient.

$$\mathcal{Q}^{ij} = \lambda_2 \mathcal{G}^{ij} + \lambda_{23} \mathfrak{J}_k \mathcal{H}^{ijk}.$$

G. Castro, L. Gualtieri and P. Pani, Phys. Rev. D **104**, 044052 (2021) [arXiv:2103.16595]

- The spherical-harmonic perturbations, here, come with baggage: an extra harmonic number.

(skipping a lot of steps. Important: each tide has a unique *l* number with *m=0* as an axisymmetric gauge)

$$\Longrightarrow \; Q^{20} = \lambda_2 G^{20} + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} H^{30}$$

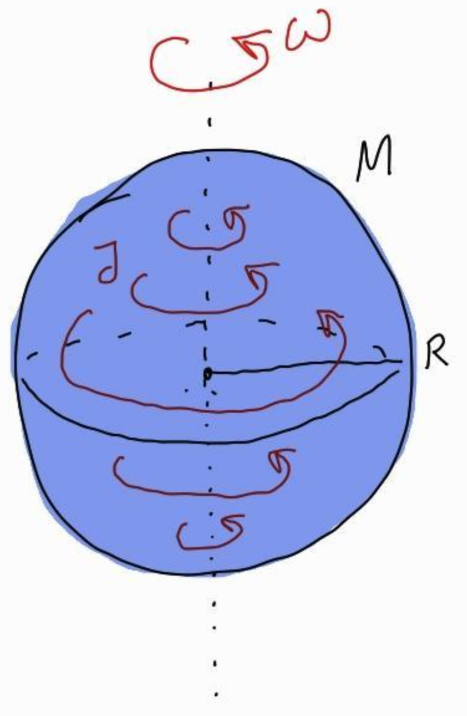
- Furthermore, we differentiate the quadrupole expression w.r.t the external tidal field and determine a net-worth of the tidal response coefficient.

$$\frac{\partial Q^{20}}{\partial G^{20}} = \lambda_2 + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} \frac{\partial H^{30}}{\partial G^{20}}$$

*my derivation, to define response coefficients in terms of instantaneous changes (gradients)

Methods

N. MacKay, hand drawn



- With added rotations, we recognize that the internal mass *M* of the NS is "churned" due to rotations and the direction of spin.

- This generates a mass-current *J*, which induces the rotational tidal field.

- Drawing an analogy to electrodynamic charge and charge-currents:
  - the external tidal field resembles an "electric field" of the mass, and
  - the rotational tidal field a "magnetic field" of the mass-current.

P. Pani, L. Gualtieri and V. Ferrari, Phys. Rev. D **92**, 124003 (2015) [arXiv:1509.0217]

- These "electromagnetic" tidal fields linearize the NS mass "moment" in the same fashion as the NS's mass quadrupole.

$$M_2 = \lambda_E^{(2)} E_0^{(2)} + \lambda_E^{(23)} B_0^{(3)}$$

- The mass quadrupole and the mass moment frameworks happen to have a hidden symmetry!

$$\Longrightarrow \quad Q^{20} = \lambda_2 G^{20} + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} H^{30}$$

$$M_2 = \lambda_E^{(2)} E_0^{(2)} + \lambda_E^{(23)} B_0^{(3)}$$

G. Castro, L. Gualtieri and P. Pani, Phys. Rev. D **104**, 044052 (2021) [arXiv:2103.16595]

- Thus, whatever Pani has calculated before in the mass-moment formalism is **directly applicable** in the quadrupole moment formalism, *given relevant symmetric formulae*.

$$\frac{\partial Q^{20}}{\partial G^{20}} = \lambda_2 + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} \frac{\partial H^{30}}{\partial G^{20}}$$

Results: Analytical

- The quadrupole gradient expression extracts useful insight…

$$\frac{\partial Q^{20}}{\partial G^{20}} = \lambda_2 + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} \frac{\partial H^{30}}{\partial G^{20}}$$

P. Pani, L. Gualtieri and V. Ferrari, Phys. Rev. D **92**, 124003 (2015) [arXiv:1509.0217]

$$\frac{\partial Q^{20}}{\partial G^{20}} = -\frac{4}{3}\sqrt{\frac{\pi}{5}} \frac{\partial M_2}{\partial E_0^{(2)}} = \frac{2}{3} k_2 R^5$$

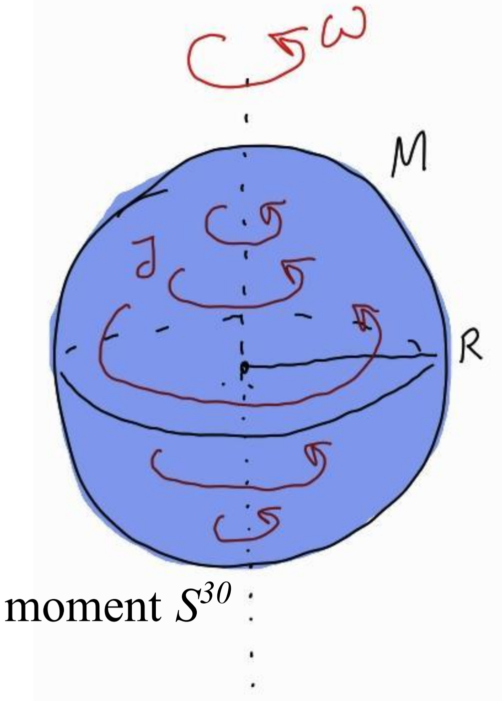
T. Hinderer, 2008 ApJ **677** 1216 (erratum: 2009 *ApJ* **697** 964: [arXiv:0711.2420])

- …the left-hand side remarkably recovers Hinderer's expression.

Results: Analytical

- The quadrupole gradient expression extracts useful insight…

$$\frac{\partial Q^{20}}{\partial G^{20}} = \lambda_2 + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} \frac{\partial H^{30}}{\partial G^{20}}$$



- The gradient of the rotational tide w.r.t the external tide intertwines with how the mass-current moment S^{30} affects each tidal field concurrently.

$$\lambda_{23}\left(\frac{\partial H^{30}}{\partial S^{30}} \cdot \frac{\partial S^{30}}{\partial G^{20}}\right) \Rightarrow -\chi M^{-2} k_2 R^5 \frac{5}{7}\left(\frac{\left(75\sqrt{35} - 224\frac{\gamma_{23}^*}{\gamma_3^*}\right)\left(88\sqrt{35} - 42\frac{\gamma_{32}}{\gamma_2}\right)}{1008\sqrt{35}}\right)$$

*my derivation, but one inserts terms found in
P. Pani, L. Gualtieri and V. Ferrari, Phys. Rev. D **92**, 124003 (2015) [arXiv:1509.0217]

- The tidal gradient is a negative, spin-dependent contribution that is consistently quadrupolar!

Results: Analytical

- The quadrupole gradient expression extracts useful insight…

$$\frac{\partial Q^{20}}{\partial G^{20}} = \lambda_2 + \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} \frac{\partial H^{30}}{\partial G^{20}}$$

- As it so turns out, the total tidal deformation is overall additive, and independent of spin direction.

$$\begin{aligned} \implies \quad \lambda_2 &= \frac{\partial Q^{20}}{\partial G^{20}} - \chi M^2 \frac{\sqrt{35}}{10} \lambda_{23} \frac{\partial H^{30}}{\partial G^{20}} \\ &= k_2 R^5 \left[\frac{2}{3} + \frac{\chi^2}{7056} \left(75\sqrt{35} - 224 \frac{\gamma_{23}^*}{\gamma_3^*} \right) \left(44\sqrt{35} - 21 \frac{\gamma_{32}}{\gamma_2} \right) \right] . \end{aligned}$$

TidalLoveSlowRot_NS.nb - Wolfram Mathematica 13.1

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

Tidal Love Numbers of a Slowly-Rotating Neutron Star

References:

[1] P. Pani , L. Gualtieri, V. Ferrari; “Tidal deformations of a slowly-spinning neutron start” (2015)

[2] P. Pani, L. Gualtieri, A. Maselli, V. Ferrari; “Tidal deformations of a spinning compact object”, (2015)

Review: [3] Paolo Pani, “Advanced methods in Black-Hole Perturbation Theory” (2013)

```
Quit[];

In[ ]:= Put[10!, LocalObject["TidalLoveSlowRot_NS.nb"]]

Out[ ]= LocalObject[file:///home/user/.Wolfram/Objects/TidalLoveSlowRot_NS.nb]

(*Insert /home/user/Downloads/arXiv-1509.02171v3/anc/ for all Get →
 Import commands*)

Import: Import called with 0 arguments; 1 or 2 arguments are expected.

Out[ ]= Import[]
```

Collect equations

```
$Assumptions = {L > 1};

In[ ]:= background =
  Import["/home/user/Downloads/arXiv-1509.02171v3/anc/background.m"];
ruleP =
  Union[
   Import[
    "/home/user/Downloads/arXiv-1509.02171v3/anc/final_eqs_polar-led.m"],
   {m → 0}];
ruleA =
  Import[
   "/home/user/Downloads/arXiv-1509.02171v3/anc/final_eqs_axial-led.m"];
```

Love solver.nb - Wolfram Mathematica 13.1

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Love number solver

- Adopt Hinderer’s recipe:
  - Redefine all perturbation function in terms of variable change y->x+1
  - Define the ratio H'/H for each respective tidal field, and bring it to the surface r=R
  - Denote the ratio (1+x)H’/H as a ratio factor y, and set the equation
  - Solve for k_l by the equation y == (whole expression for ratio)
  - Via Pani II paper (https://arxiv.org/abs/1509.02171),

```
In[ ]:= H20[y_] :=
  a2*y (y - 2) - g2 (-3 + 1 / (2 - y) - 1 / y + 3 y + (3 / 2) * (y - 2) y*Log[1 - 2 / y])
H30[y_] :=
 (2 y^2 (y - 2)^2 (y - 1) a3 +
    (2 (-2 - 10 y - 65 y^2 - 60 y^3 + 15 y^4) +
       15 (y - 2)^2 (y - 1) y^2*Log[1 - 2 / y]) g3) / (2 y (y - 2))
h30[y_] :=
 -M
   (-64 y^3 (8 - 10 y + 3 y^2) as3 +
      3 (8 + 20 y + 60 y^2 - 210 y^3 + 90 y^4 +
         15 y^3 (8 - 10 y + 3 y^2) Log[1 - 2 / y]) gs3) / (192 y)
dh30[y_] := - ((M*chi) / (6720 y^2))
   (-128 Sqrt[35] y (5 y - 4) a2 +
     3
      (8 Sqrt[35] (-16 - 44 y + 90 y^2 + 270 y^3 - 945 y^4 + 405 y^5 +
           y (64 - 80 y + 1080 y^3 - 1350 y^4 + 405 y^5) ArcTanh[1 / (1 - y)]) g2) +
     35 y (8 + 20 y + 60 y^2 - 210 y^3 + 90 y^4 +
        15 y^3 (8 - 10 y + 3 y^2) Log[1 - 2 / y]) g32)
```

$$\Lambda_2^{\text{total}} = \Bigg\{ \frac{2}{3} + \chi^2 \Bigg[\left(\frac{1375}{84} - \frac{3663}{12544}\sqrt{\frac{5}{7}\frac{k_3}{k_3^*}} \right) + p_{32}\left(\frac{5}{27} + \frac{165}{784}\sqrt{\frac{5}{7}\frac{k_2}{k_3^*}}C^2 - \frac{37}{2240\sqrt{35}}\frac{k_3}{k_3^*} \right) + p_{32}^2\left(\frac{1}{84\sqrt{35}}\frac{k_2}{k_3^*}C^2 \right) \Bigg] \Bigg\} k_2 C^{-5}.$$

- The spin-corrected terms are expressed as an expansion in the tidal-coupled polynomial p_{32}.
- 0-th order in p_{32} is calculable, but it'd be an incomplete / partial calculation.
- p_{32} in itself is a calculation dependent on non-linear, coupled ODEs to solve simultaneously with the TOV equations – we need a TOV solver, first.