



ECT*
EUROPEAN CENTRE
FOR THEORETICAL STUDIES
IN NUCLEAR PHYSICS AND RELATED AREAS

QUANTUM FRACTALS

Bira van Kolck



2025 PROGRAM OF ACTIVITIES

FEBRUARY 17-21	Key Reactions in Nuclear Astrophysics A. TUMINO (Università degli Studi di Enna "Kore" & INFN-LNS), C. BERTULANI (Texas A&M University-Commerce), R. DIEHL (Max Planck Institut für Experimentelle Physik), J. JORIS (Technical University of Catalonia), L. TRACHE (IFIN-HH)	21-25	Penetrating Probes of Hot High-μB Matter: Theory Meets Experiment E. SCOMPARIN (INFN Torino), T. GALATYUK (TU Darmstadt), M. LOMBARDO (INFN Firenze), R. RAPP (Texas A&M University), G. USAI (Università di Cagliari)
MARCH 3-7	Scale Setting: Precision Lattice QCD for Particle and Nuclear Physics A. BAZAVOV (Michigan State University), S. COLLINS (University of Regensburg), R. SOMMER (DESY & Humboldt-Universität zu Berlin)	JUL/AUG 28-01	New Developments in the Charge Radii Determination for Light Nuclei M. KRAUS (GSI Helmholtz & PSI Villigen), L. GASTALDO (Kierchhoff-Institut für Physik, Universität Bonn), A. ATHENODOROU (The Cyprus Institute), J. JORIS (Technical University of Catalonia), M. KAMINSKI (University of Alabama), K. LANDSTEINER (IFT Madrid), K. KHARZEEV (Stony Brook University/BNL), U. GURDUR (University of California, Los Angeles)
MAR/APR 31-04	Holographic Perspectives on Chiral Transport and Spin Dynamics M. KAMINSKI (University of Alabama), K. LANDSTEINER (IFT Madrid), K. KHARZEEV (Stony Brook University/BNL), U. GURDUR (University of California, Los Angeles)		
	Mechanical Properties of Hadrons: Structure and Visualization P. SCHWEITZER (University of Connecticut), K. JDO (University of Connecticut), B. PASQUINI (University of Connecticut)		
APRIL 14-18	Lepton Flavour Change in Neutrino Oscillations K. BENNACEUR (IP2I, IN2P3), S. J. JORIS (Technical University of Catalonia)		
MAY 05-09	Quantum Science Generation A. BALDIZZI (University of Trento), C. CAPECCI (University of Trento), DE BERNARDIS (INFN-CNR Pisa), F. MANTEGAZZINI (INFN-CNR Pisa)	22-26	Analytic Structure of QCD and Yang-Lee Edge Singularity V. SKOKOV (North Carolina State University), G. BASAR (University of North Carolina Chapel Hill), C. SCHMIDT (Universität Bielefeld)
12-16	Nonequilibrium Phenomena in Strongly Interacting Systems: Atomic Nuclei, Liquid Helium, Ultracold Gases, and Neutron Stars P. MAGIERSKI (Warsaw University of Technology), B. HASKELL (Nicolaus Copernicus Astronomical Center), G. ROATI (University of Florence, INFN), G. WLADZIMIRSKI (Warsaw University of Technology)		
19-23	The Complex Structure of Strong Interactions in Euclidean and Minkowski Space J. SKULLERUD (National University of Ireland), M. GOMEZ-ROCHA (Universidad de Granada), T. FREDERICO (Instituto Tecnológico de Aeronáutica), O. OLIVEIRA (Universidade de Coimbra), M. PELEAZ (Universidad de la República Uruguay), P. SILVA (Universidade de Coimbra), F. SIRINGO (Università di Catania)	13-24	Attractors and Thermalization in Nuclear Collisions and Cold Quantum Gases M. HELLER (Ghent University), J. BERGES (Heidelberg University), J. BREWER (University of Oxford), T. LAPP (University of Jyväskylä), M. SPALINSKI (National Centre for Nuclear Research)
JUNE 09-13	Universality in Strongly-Interacting Systems: from QCD to Atoms J. KRISCHER (SRM University AP), D. BLUME (The University of Oklahoma), R. BRICENO (UC Berkeley), L. CONTESSI (Université Paris-Saclay, JCLab, CNRS), M.P. VALDERAMA (Beihang University)	OCTOBER 06-09	Superconducting Devices for Quantum Optics and Quantum Simulations F. MANTEGAZZINI (INFN-CNR Pisa), N. RICH (CNRS-Institut Néel), M. ESPOSITO (CNR-SPIN), N. CRESCINI (FBK), F. AHNES (FBK)
JUNE/JULY 16-04	Doctoral Training Program/TALENT School: Quantum Computing for Nuclear Physics A. BAZAVOV (Michigan State University), Z. DAVIDOVI (University of Maryland), M. HARTHA-JENSEN (University of Oslo and Michigan State University), R. LAROSE (Michigan State University), D. LEE (Michigan State University), A. ROGGERO (University of Trento)	NOVEMBER 03-07	Pan-American Few-Body Physics Boot Camp: Fostering Collaboration G. HUPIN (JCLab), T. FREDERICO (Instituto Tecnológico de Aeronáutica), S. KANG (North Carolina State University), A. LEFINE-SZILY (University of São Paulo)
07-09	Theory Service for the Low Energy Nuclear Physics Community: a Hands-on Workshop G. COLO (University of Milano & INFN), J. DUDEK (JCLab, CNRS, Strasbourg), M. RODRIGUEZ-GALLARDO (University of Sevilla)	DECEMBER 01-05	Neutron-Capture Reactions for Astrophysical Processes A. SPYROU (Michigan State University), S. GORIELY (IAA-ULB), A. LARSEN (University of Oslo), S. LIDDICK (Michigan State University), D. MÜCHER (University of Cologne), M. WIEDEKING (Lawrence Berkeley National Lab)
14-18	Next Generation Ab Initio Nuclear Theory C. BARBIERI (Università di Milano), E. EPELBAUM (Ruhr Universität Bochum), R. FURNSTADL (Ohio State University), S. PASTORE (Washington University in St. Louis)	17-21	ISNET-11: Information and Statistics in Nuclear Experiment and Theory A. EKSTROM (Chalmers University of Technology), D. PHILLIPS (Ohio State University), A. LOVELL (Los Alamos National Laboratory)

The ECT* is part of the Fondazione Bruno Kessler. The Centre is funded by the Autonomous Province of Trento, funding agencies of EU Member and Associated states, and by INFN-TIFPA and has the support of the Department of Physics of the University of Trento. The Director of ECT* is Prof. Ulfarsson van Kolck. For information: staff@ectstar.eu | www.ectstar.eu



Wikipedia



All aspects
of strong
interactions
and related
areas!



Outline

- Emergence of structure
- Discrete scale invariance
- Bosonic clusters
- Nucleons
- Where is QCD?
- Conclusion

Emergence of quantum structures

“reduction”:
what are the
building blocks?



1 keV

1 GeV

p

atoms,
molecules

hadrons,
nuclei

“elementary”
particles

$$pr \sim 1$$

uncertainty principle

Heisenberg '27

2 Å

0.2 fm

r



“emergence”:
how do the building
blocks fit together?

Here: $\hbar = 1, c = 1$

$$[m] = [E] = [p] = [r]^{-1} = [t]^{-1}$$

Structures and scales

“... were the world to be made between now and tomorrow 100 or 1,000 times larger or smaller than it is at present, all its parts being enlarged or diminished proportionally, everything would appear tomorrow exactly as now, just as though nothing had been changed.”

Nicolas Oresme
a commentary on Aristotle's *De caelo et mundo*, 1377

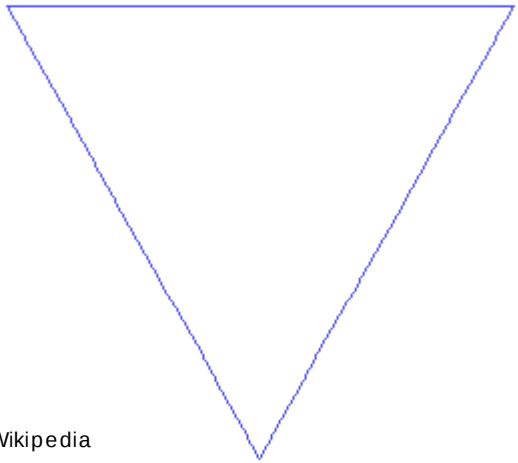


scale transformation

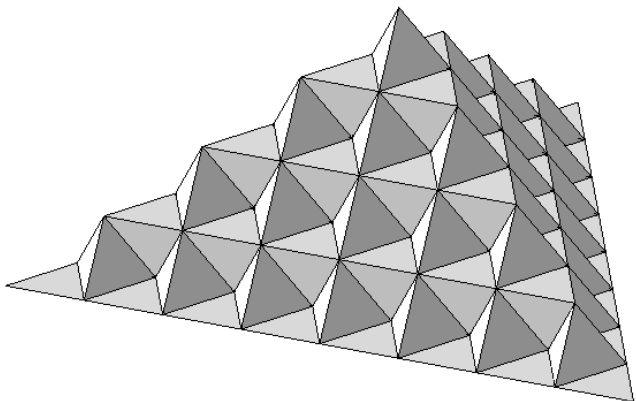
$$\begin{array}{ccc} r \rightarrow \alpha r & \longleftrightarrow & p \rightarrow \alpha^{-1} p \\ \alpha \geq 0 & \text{(quantum)} & \updownarrow \text{(nonrelativistic)} \\ t/m \rightarrow \alpha^2 t/m & \longleftrightarrow & mE \rightarrow \alpha^{-2} mE \end{array}$$

Simplest “complex” structures: one scale

von Koch 1904



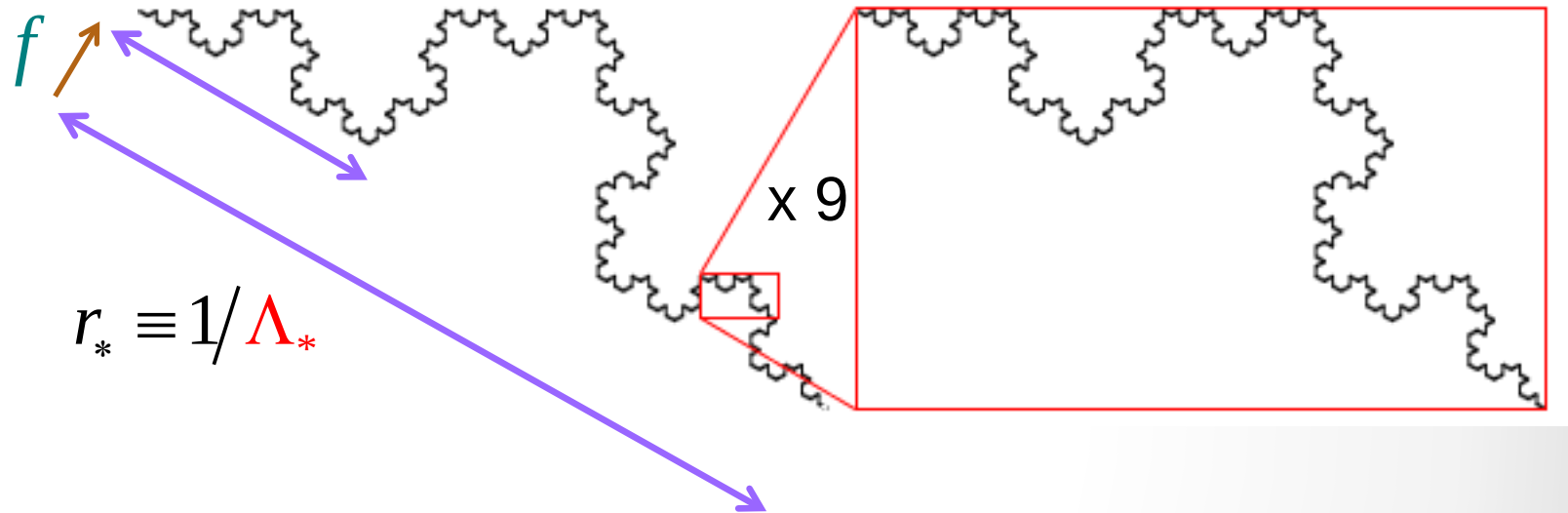
Wikipedia



Wikipedia

(Scaling) Fractal

yozh.org/2010/10/21/mset001/



$$r_* \equiv 1/\Lambda_*$$

$$r \rightarrow \alpha_n r = f^n r$$

$$\left\{ \begin{array}{l} f \text{ real} \\ n \text{ integer} \end{array} \right.$$

Discrete scale invariance

romanesco or “fractal broccoli”



New Scientist

Nuclear physics: nucleons (proton or neutron) with spin $S=1/2$, nearly the same mass
 four-component fermions $m_N \approx 940 \text{ MeV}$

lightest
exchanged
particles:
pions
 $m_\pi \approx 140 \text{ MeV}$

$$V(r) = - \underbrace{\frac{g_{\pi N}^2}{m_N}}_{\text{range}} \frac{e^{-m_\pi r}}{r^3} S_{12} + \dots$$

range $R \sim m_\pi^{-1} \approx 1.4 \text{ fm}$
 Yukawa '35

proton + neutron, $S=1$: deuteron

$$B_2 \approx 2.2 \text{ MeV} \Rightarrow \frac{R}{a_2} \approx \frac{\sqrt{m_N B_2}}{m_\pi} \approx \frac{1}{3}$$

binding energy scattering length
~ system size

Atomic physics: neutral atoms, mass $m_{at} \approx Am_N$
 fermions or bosons

lightest
exchanged
particles:
two photons
 $m_\gamma = 0$

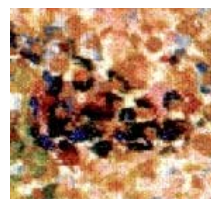
$$V(r) = - \underbrace{\frac{l_{\text{vdW}}^4}{4\pi m_{at}}}_{\text{"range"}} r^{-6} + \dots$$

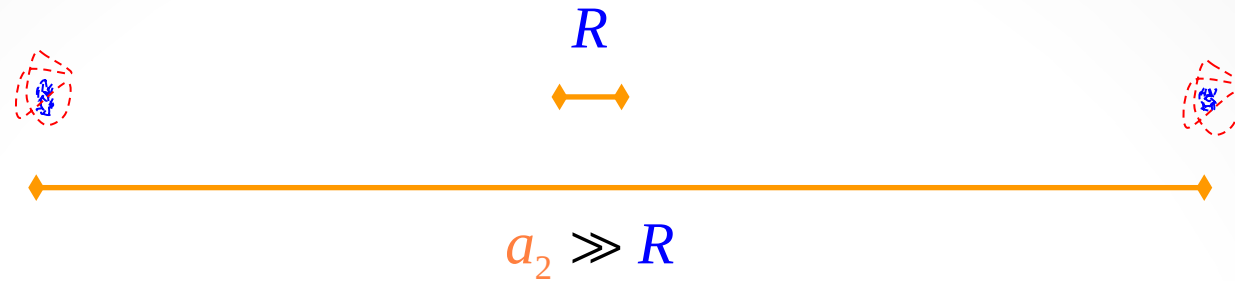
"range" $R \sim l_{\text{vdW}}$
 v.d. Waals 1873



two bosonic ^4He atoms, $S=0$: ^4He dimer

$$B_2 \approx 1.3 \text{ mK}, \quad l_{\text{vdW}} \approx 5.4 \text{ \AA} \Rightarrow \frac{R}{a_2} \approx \sqrt{m_N B_2} l_{\text{vdW}} \approx \frac{1}{20}$$





multipole expansion
of interactions

“Short-Range
Effective Field Theory”

vK '97'98

Bedaque + vK '97

Kaplan, Savage, Wise '98

...

cf. Bethe, Peierls '35

$$V_2(\vec{r}; \Lambda) = \frac{4\pi}{m} \left\{ \overset{\text{leading order}}{C_0(\Lambda)} \delta_{\Lambda}^{(3)}(\vec{r}) + \overset{\text{next-to-leading order}}{C_2(\Lambda)} \nabla^2 \delta_{\Lambda}^{(3)}(\vec{r}) + \dots \right\}$$

cutoff-dependent parameters smeared delta function

↓ renormalization

S matrix in expansion in powers of R/r ,
insensitive to *arbitrary* choice of regularization
as long as $\Lambda^{-1} \lesssim R$

LO point limit

$$R \rightarrow 0$$

scale
invariance

unitarity limit

$$a_2 \rightarrow \infty$$



$$mE_2 = 0 \Leftrightarrow \alpha^{-2} mE_2 = 0$$

no scale!

no two-body bound states...

cf. Oresme

Two-component
fermions

no N -body bound states either
unless
scale invariance broken by external interaction/trap

$$\text{e.g. } \left. \frac{E_N^{(0)}}{N} \right|_{N \rightarrow \infty} = \xi \frac{3(3\pi^2 \rho)^{2/3}}{10m}$$

Other cases?

universal number
Bertsch '99

free-gas energy,
uniform density

Bosons,
more-component
fermions

Three-body system

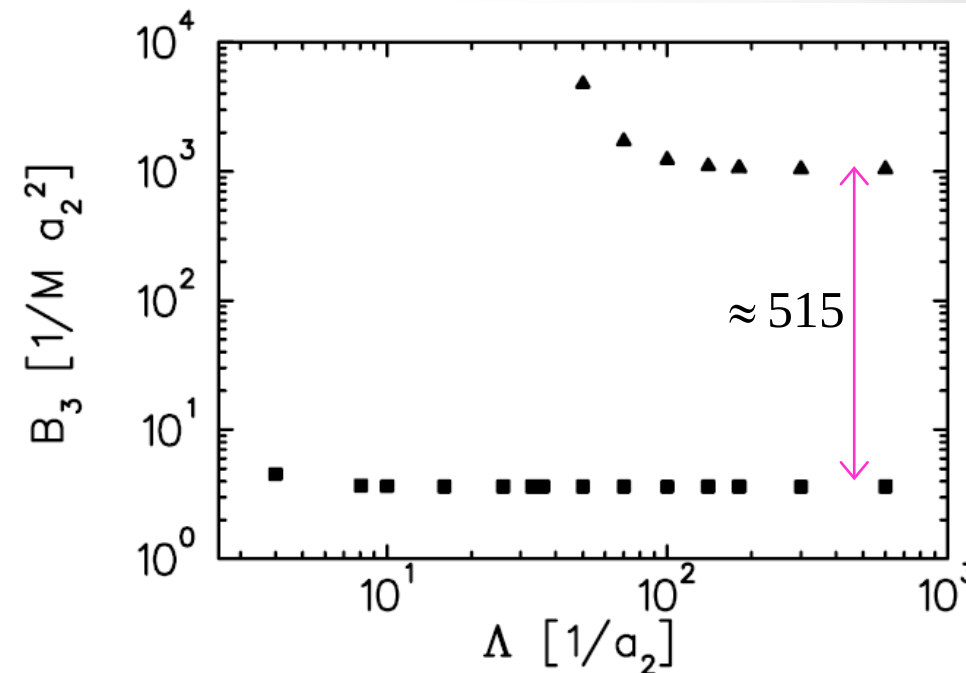
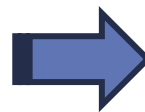
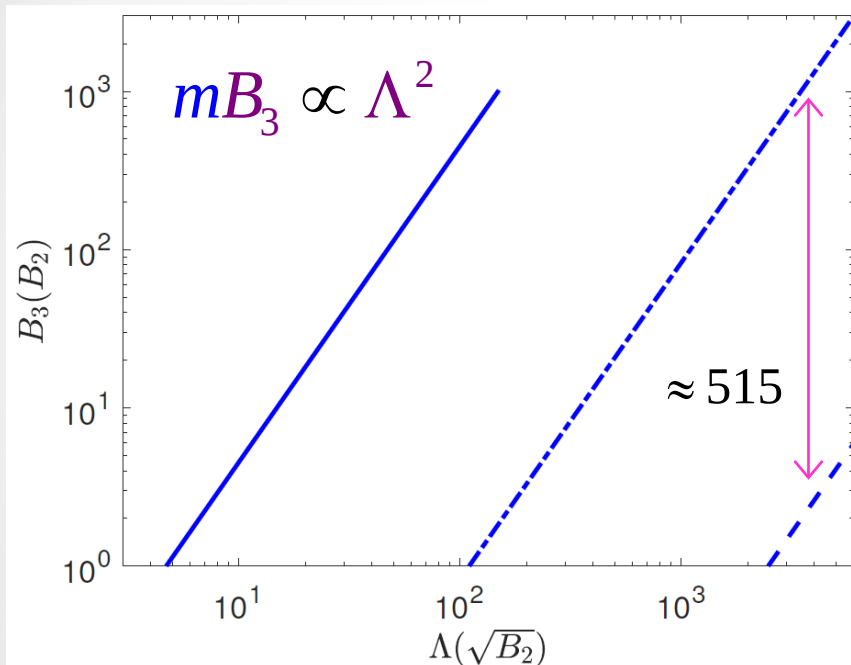
LO

Bedaque, Hammer + vK '99 '00



...

Feng Wu '22



Bedaque, Hammer, vK, Nucl. Phys. A 676 (2000) 357

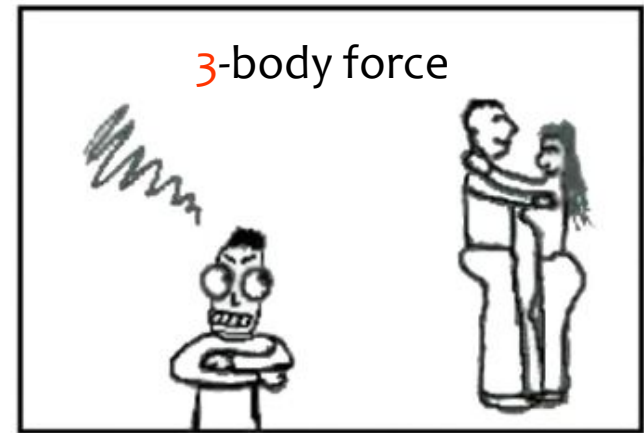
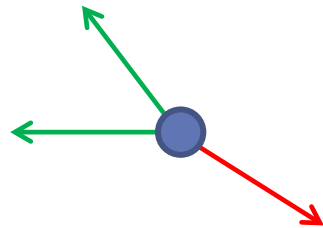
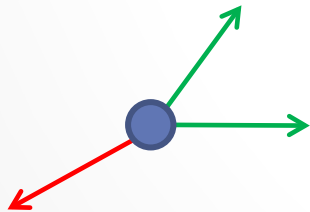
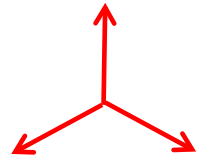
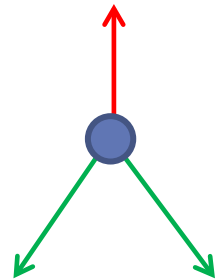
“Thomas collapse”

Thomas '35

$$V_3(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3) = \frac{(4\pi)^2}{m} D_0(\Lambda) \delta_{\Lambda}^{(3)}(\vec{r}_1 - \vec{r}_2) \delta_{\Lambda}^{(3)}(\vec{r}_2 - \vec{r}_3)$$

one parameter

determined by one three-body datum



Wikipedia

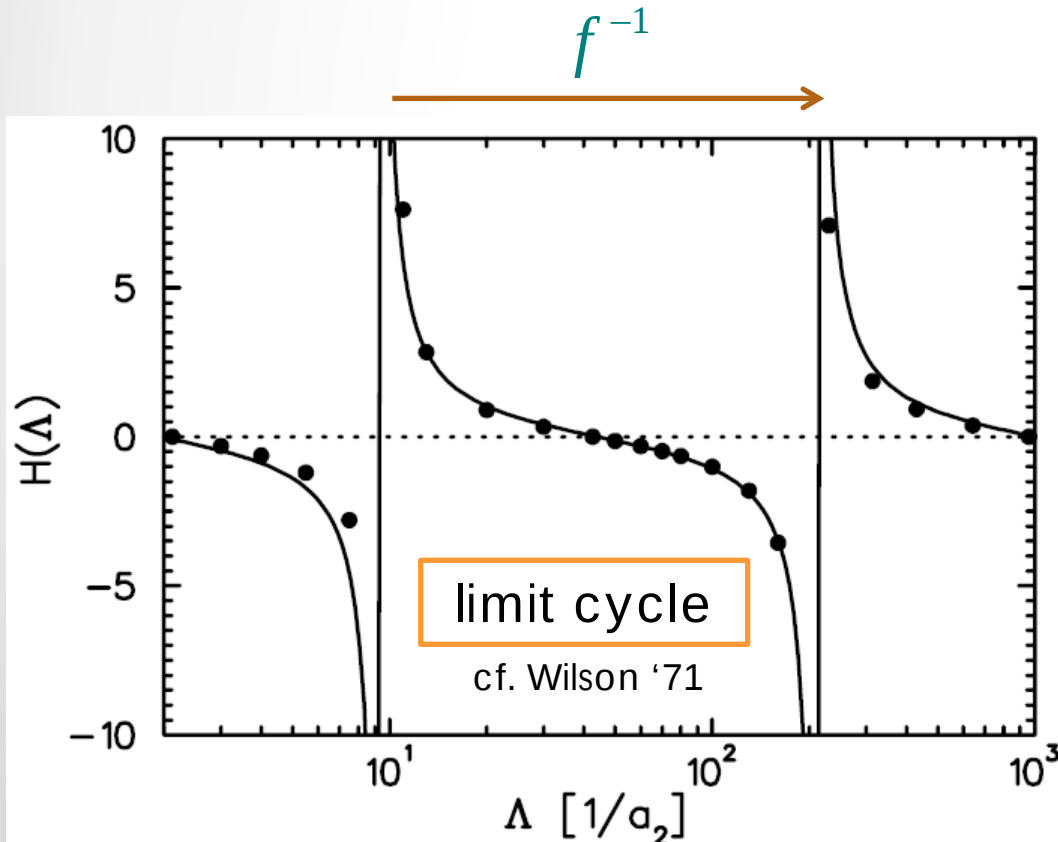
$$H(\Lambda) \equiv \frac{\Lambda^2 D_0(\Lambda)}{m C_0^2(\Lambda)} \simeq \frac{\sin(s_0 \ln(\Lambda_*/\Lambda) - \arctan(s_0^{-1}))}{\sin(s_0 \ln(\Lambda_*/\Lambda) + \arctan(s_0^{-1}))}$$

dimensionful parameter

$$s_0 \simeq 1.00624$$

anomalous breaking of
(continuous) scale invariance

quantum phenomenon!



$$\Lambda \rightarrow \alpha_n^{-1} \Lambda = f^{-n} \Lambda$$

$$f^{-1} = \exp(\pi/s_0) \simeq 22.7$$

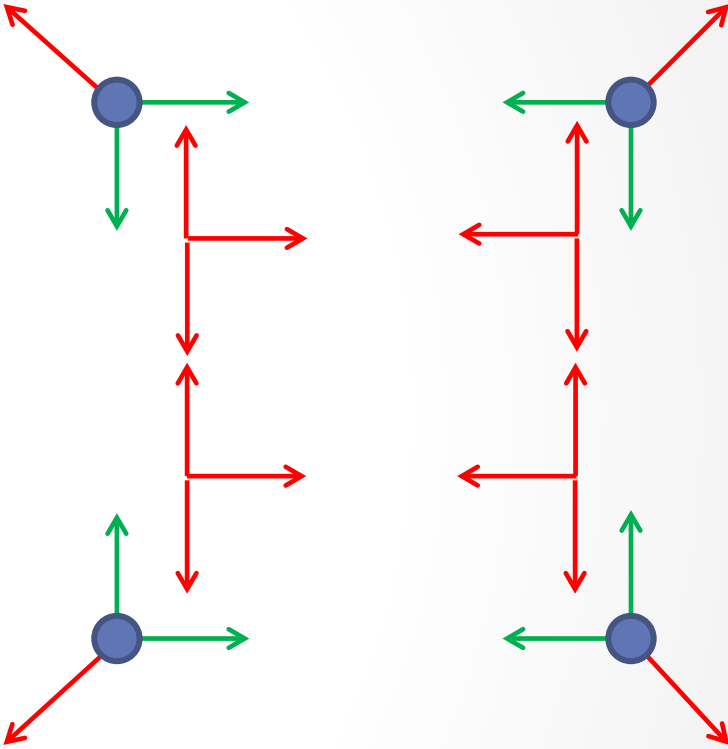
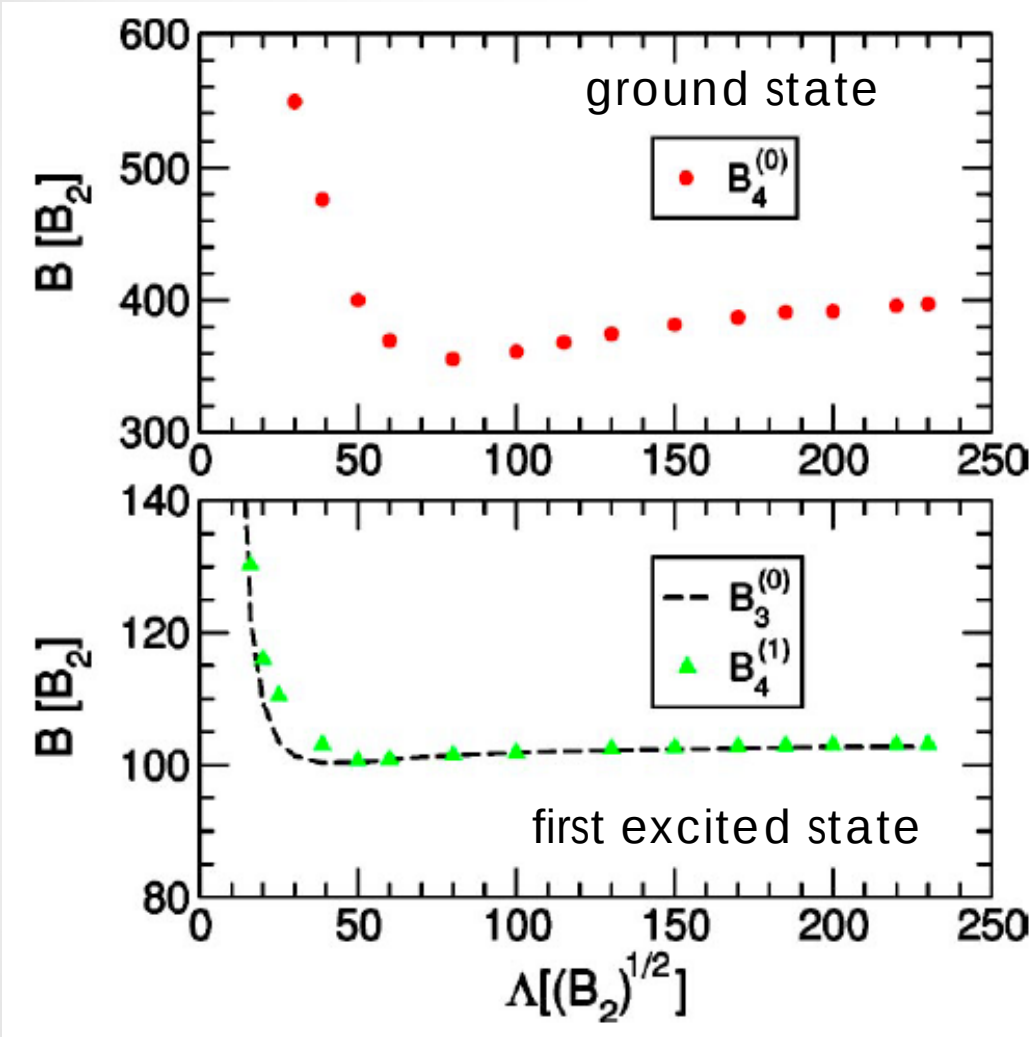
STRUCTURE
FROM A
SINGLE SCALE

Discrete
scale invariance



LO

Four-body system



No 4-body force...

... and no other scale at leading order!

Two consequences

1) Towers of excited states

$$mB_{A,n}^{(0)} \rightarrow \alpha_l^{-2} mB_{A,n}^{(0)} = mB_{A,n+l}^{(0)}$$

$$\Rightarrow mB_{A,n}^{(0)}(\Lambda_*) = mB_{A,0}^{(0)}(\Lambda_*) \exp(-2n\pi/s_0)$$

ground
state

fixes
tower position

A = 3

Efimov '70

A = 4

Hammer, Platter '07

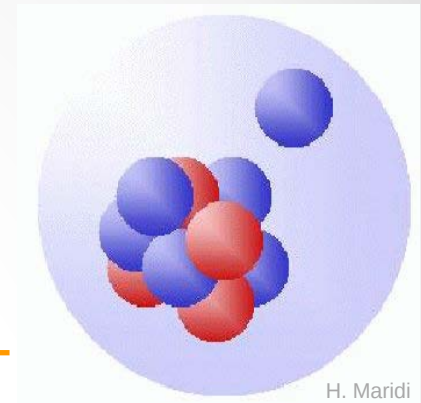
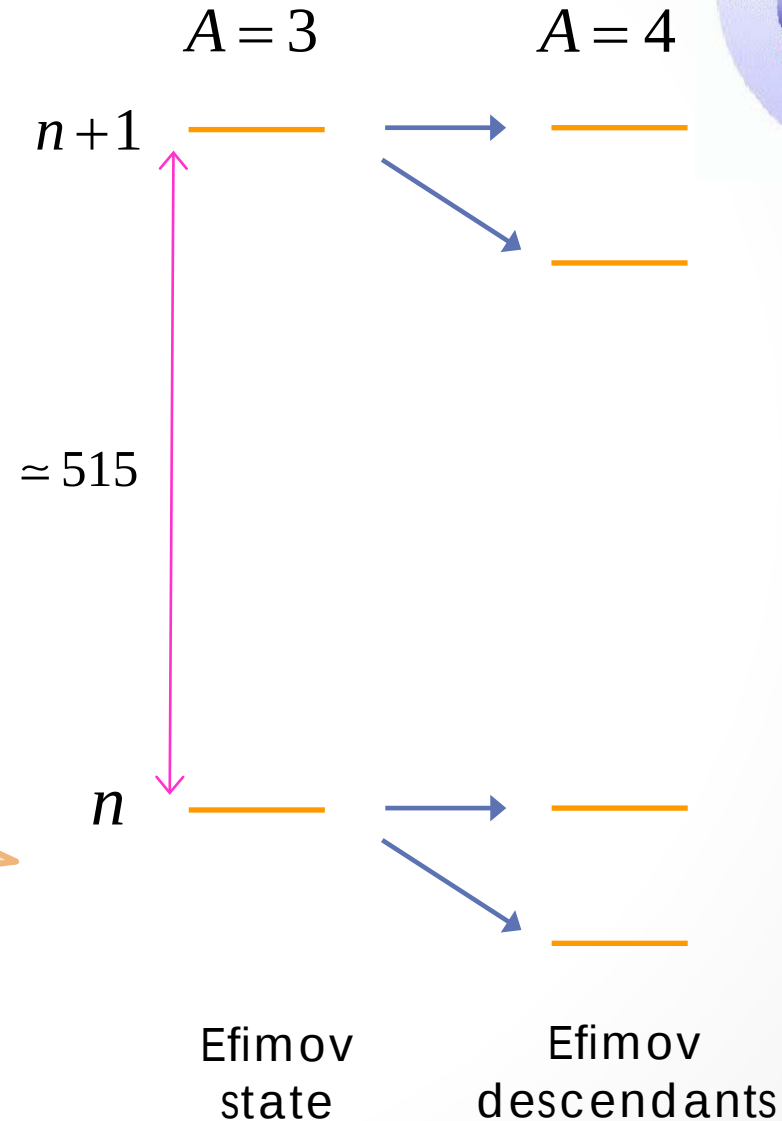
A = 5, 6

bosons

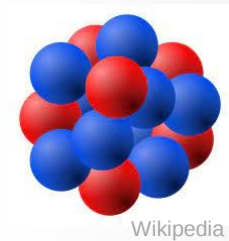
von Stecher '10'11
Gattobigio, Kievsky,
Viviani '11'12

Towers
upon
towers!

- origin?
- rules?



Wu,
Frederico,
Higa
+ vK '24



≈ 22.7

2) Ground-state correlations

single scale $\rightarrow$ $\frac{B_{A,0,0}^{(0)}(\Lambda_*)}{A} = \kappa_A \frac{B_{3,0}^{(0)}(\Lambda_*)}{3}$

universal numbers

$$\left\{ \begin{array}{l} \kappa_2 \equiv 0 \\ \kappa_3 \equiv 1 \\ \kappa_4 \simeq 3.5 \\ \kappa_{A \geq 5} \simeq ? \end{array} \right.$$

Hammer, Platter '07

von Stecher '10

...
Carlson, Gandolfi, Vitiello + vK '17

varying Λ_*

$A = 4$

Tjon line

Platter, Hammer, Meißner '05

Tjon '75

Nakaichi, Akaishi, Tanaka, Lim '78

$A = 5, 6$

bosons

Generalized Tjon lines

Bazak, Eliyahu + vK '16

Nakaichi, Akaishi, Tanaka, Lim '79'80

fixing Λ_*

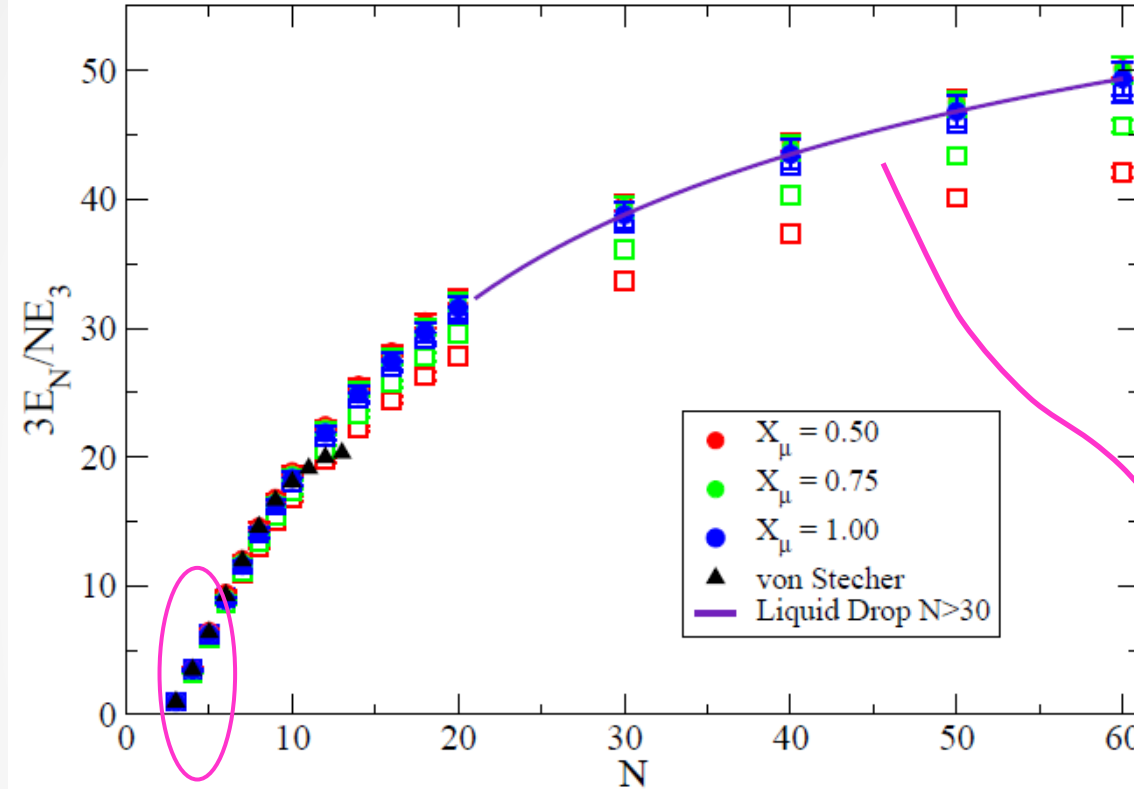
$$\frac{B_{A,0,0}^{(0)}(B_{3,0})}{A} = \kappa_A \frac{B_{3,0}}{3}$$

N unitary bosons

Gandolfi, Carlson, Vitiello + vK '17



Carlson, Gandolfi, Vitiello, vK, *Phys. Rev. Lett.* **119** (2017) 223002



↑ increasing Λ

LO

Quantum
Monte
Carlo

saturation!

cf. Piatecki + Krauth '14
Comparin + Krauth '16

$$\kappa_N \approx \frac{3}{N} (N-2)^2$$

Bazak, Eliyahu + vK '16

$$\kappa_N = \kappa_\infty \left[1 - \eta_s N^{-1/3} + \mathcal{O}(N^{-2/3}) \right]$$

$$\kappa_\infty = 90 \pm 10 \quad \eta_s = 1.7 \pm 0.3$$

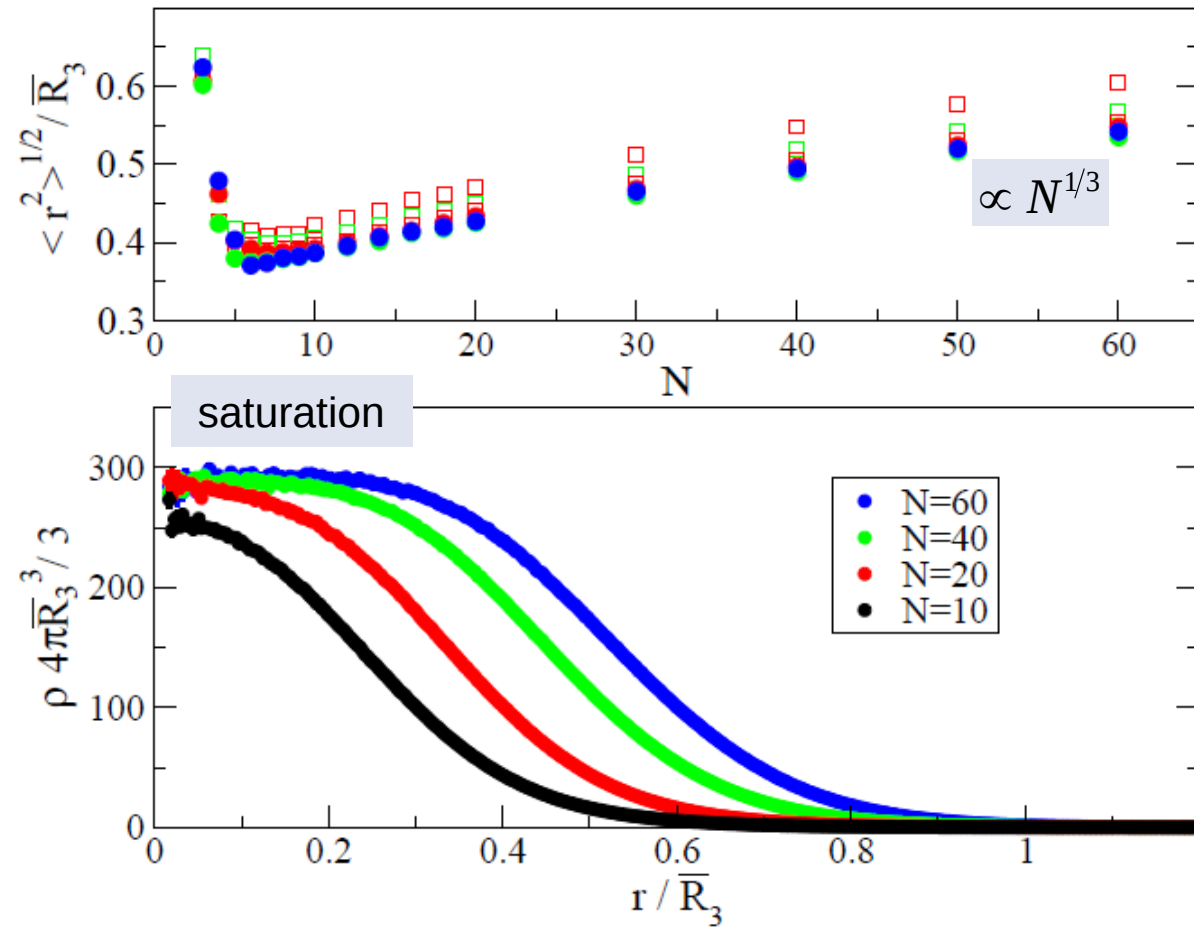
cf. ${}^4\text{He}$

$$\kappa_\infty \simeq 182$$

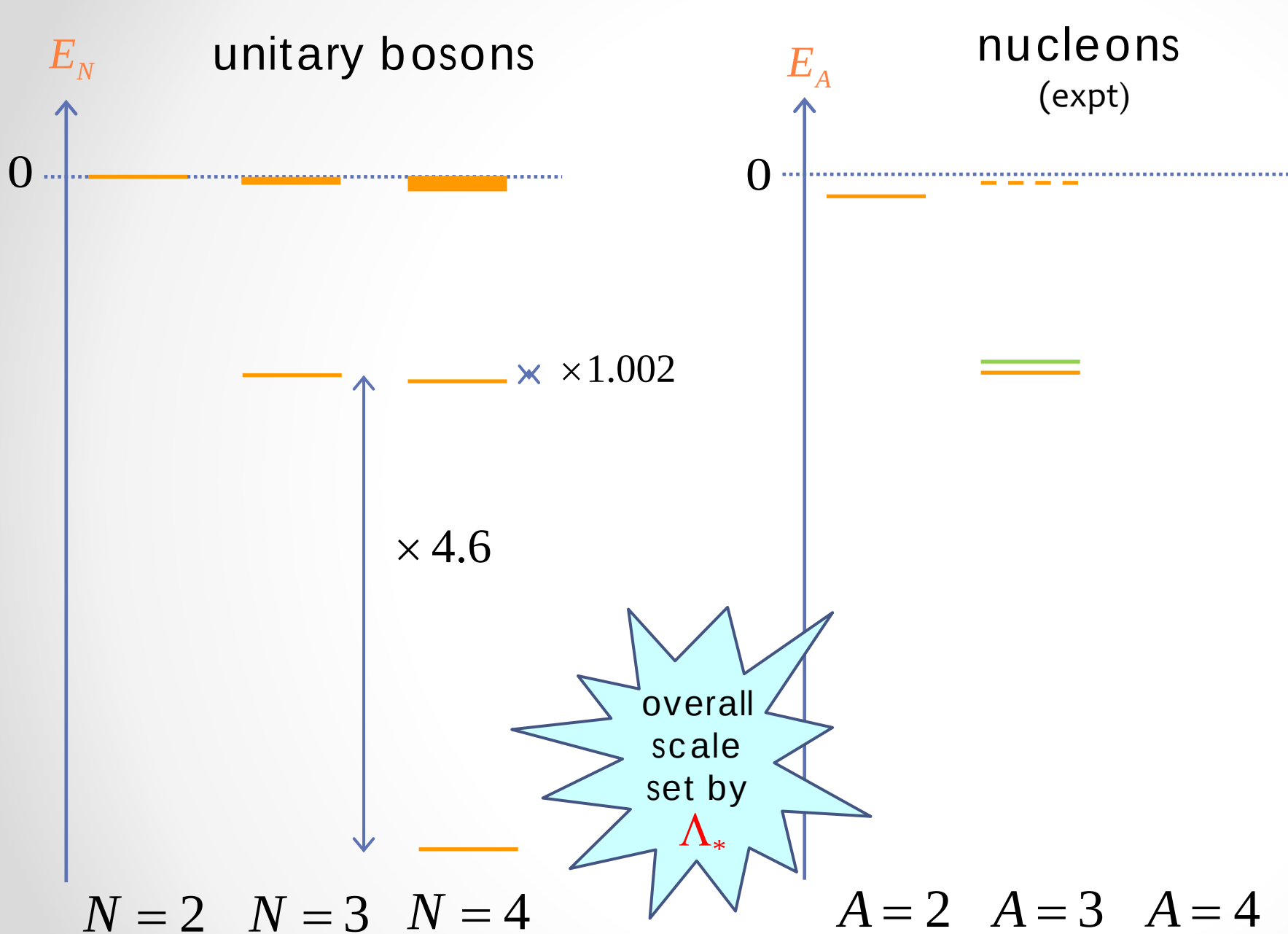
$$\eta_s \simeq 2.7$$

Pandharipande *et al.* '83

A liquid
indeed...



$$\bar{R}_3 \equiv (2mB_3)^{-1/2}$$



$$\frac{3|B_2|}{2|B_3|} \simeq 0.4 < 1$$

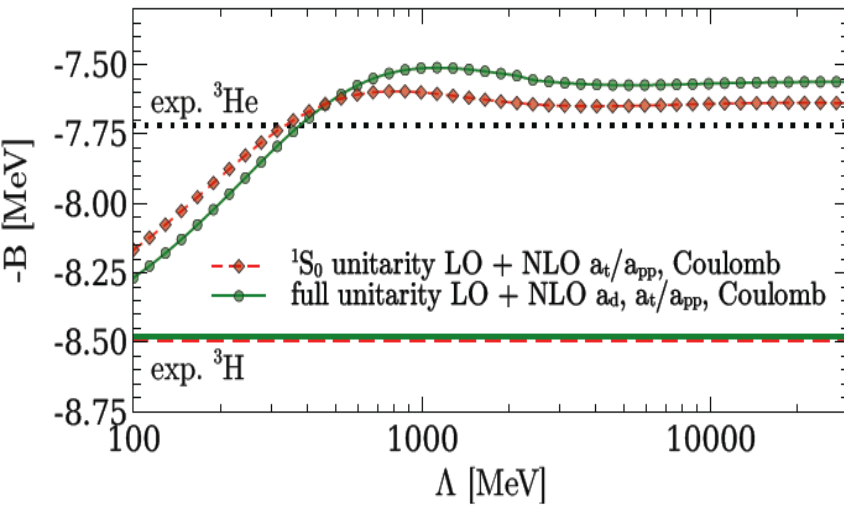
subleading

virtual state in
neutron-deuteron
scattering
Rupak, Vaghani,
Higa + vK '18

${}^3\text{He}$ - ${}^3\text{H}$
isospin splitting
obtained at NLO
König, Griebhammer,
Hammer + vK '16

Nucleons around unitarity

Helion-triton splitting



$$A = 3$$

NLO

$$B_h^{(1)} - B_t \simeq -(0.92 \pm 0.18) \text{ MeV}$$

vs.

$$-0.764 \text{ MeV (exp)}$$

LO

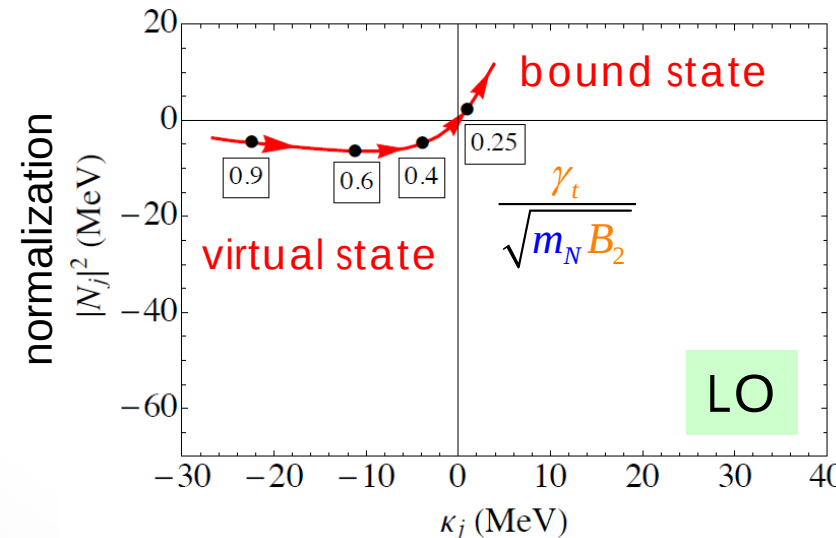
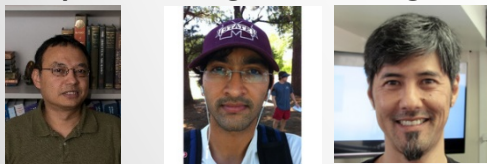
König, Grießhammer, Hammer + vK '16



König, Grießhammer, Hammer, vK, *Phys. Rev. Lett.* **118** (2017) 202501

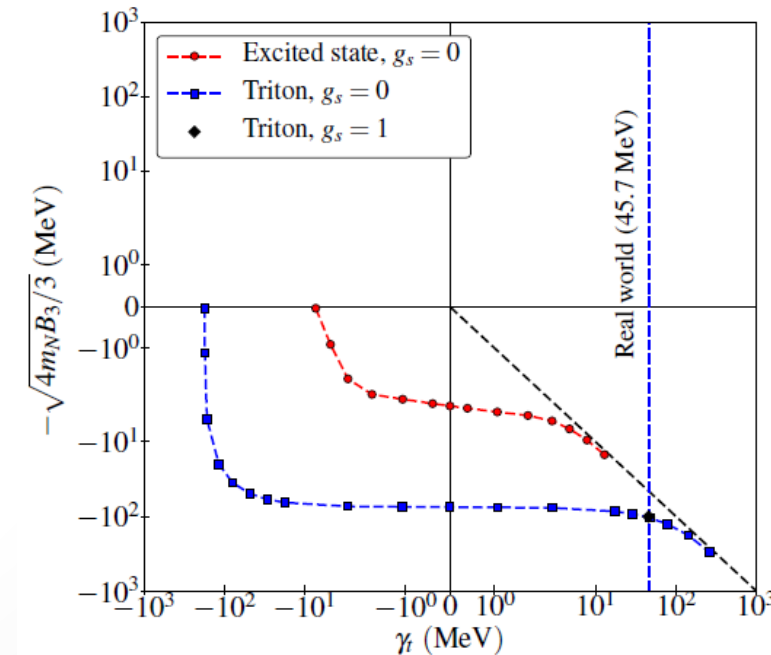
First-excited state of triton

Rupak, Vaghani, Higa + vK '18

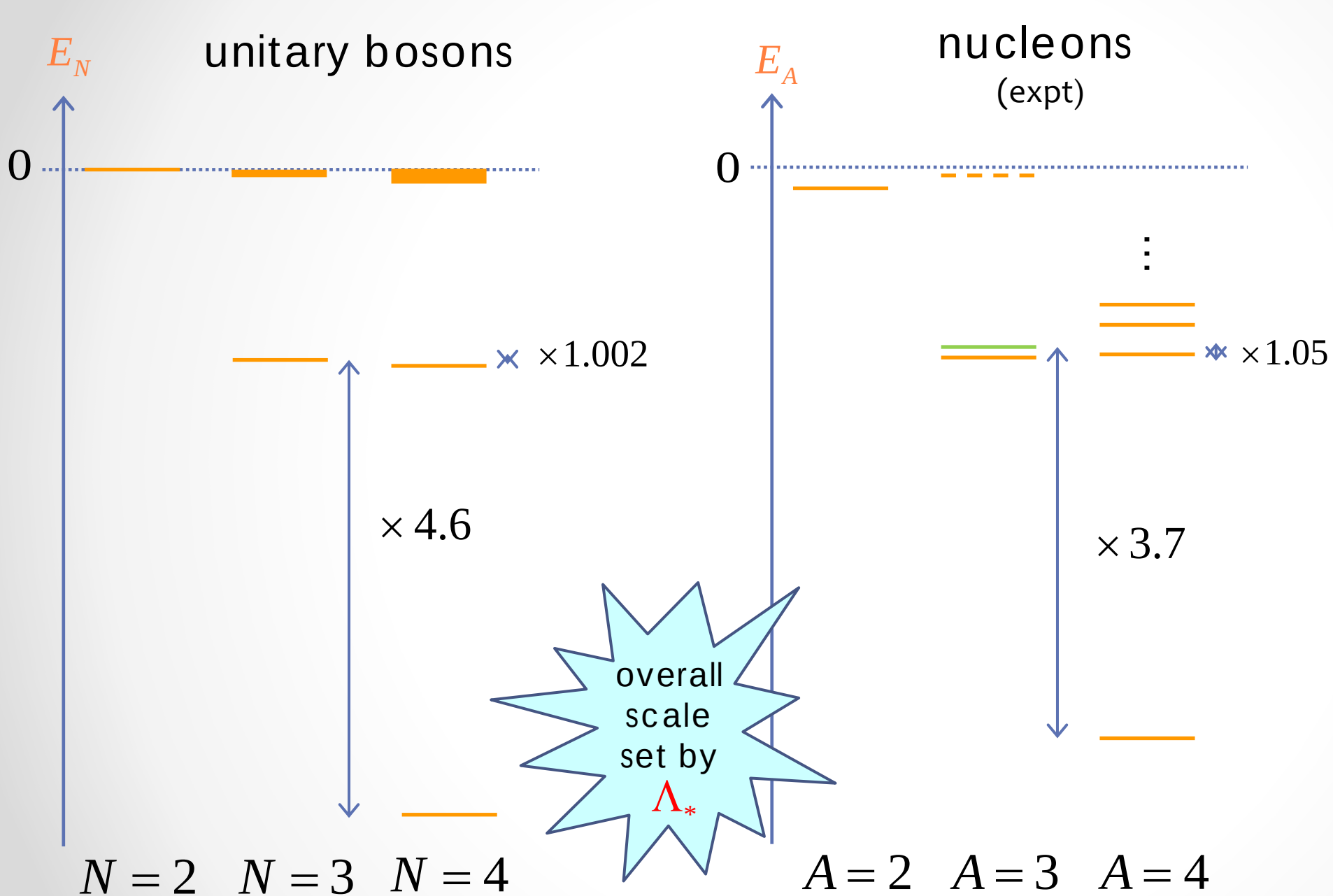


binding momentum

Rupak, Vaghani, Higa, vK, *Phys. Lett. B* **791** (2019) 414



deuteron binding momentum



$$\frac{3|B_2|}{2|B_3|} \simeq 0.4 < 1$$

subleading

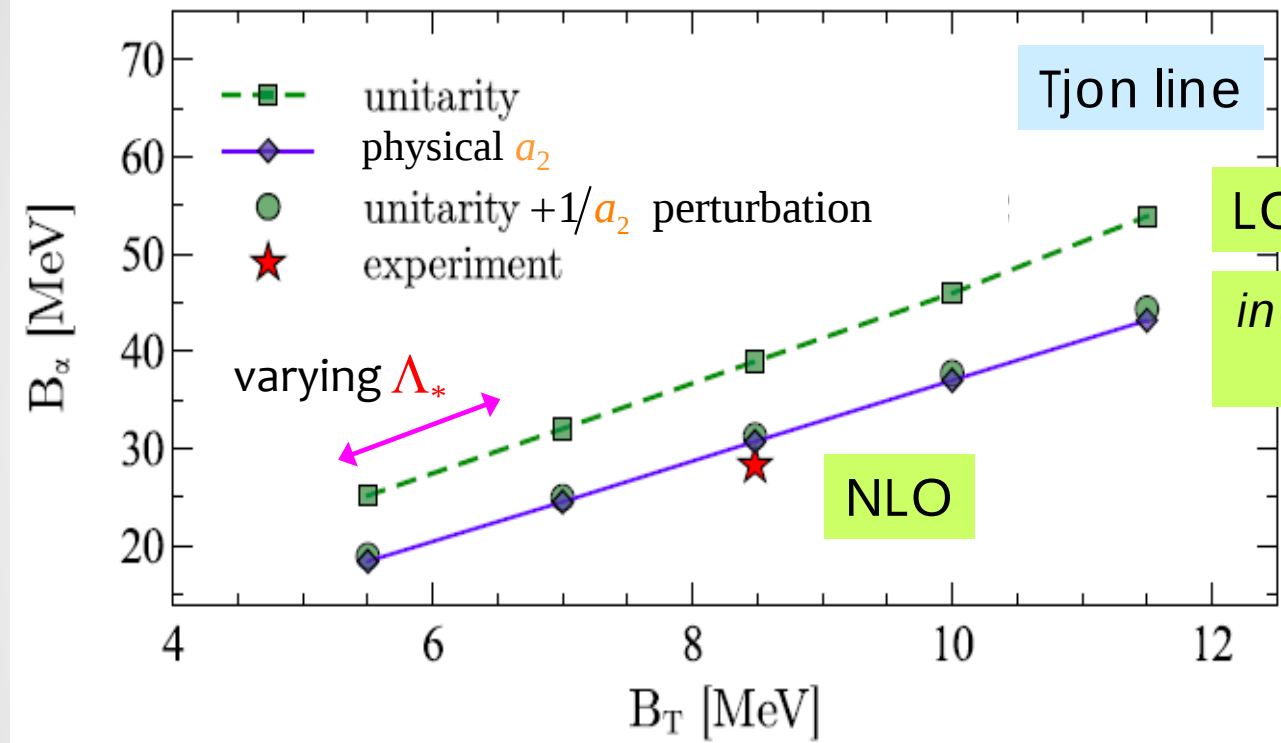
virtual state in
neutron-deuteron
scattering
Rupak, Vaghani,
Higa + vK '18

${}^3\text{He}$ - ${}^3\text{H}$
isospin splitting
obtained at NLO
König, Grießhammer,
Hammer + vK '16

correction
obtained at NLO
König, Grießhammer,
Hammer + vK '17

Alpha particle

$A = 4$



First-excited state

$$\frac{B_{\alpha^*}^{(0)}}{B_t} = \frac{B_{\alpha^*}^{(0)}}{B_h^{(0)}} \approx 1.0023$$

Hammer, Platter '07

...
Deltuva '10

vs.

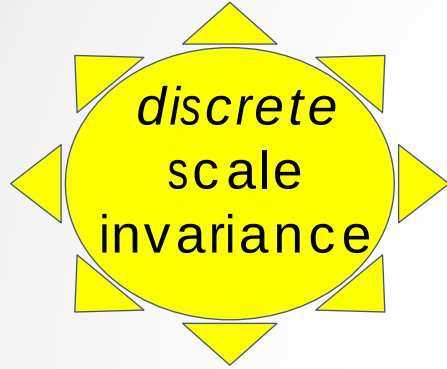
$$\left\{ \begin{array}{l} \frac{B_{\alpha^*}}{B_h} = 1.05 \text{ MeV (exp)} \\ \frac{B_{\alpha^*}}{B_t} = 0.95 \text{ MeV (exp)} \end{array} \right.$$



but needs full NLO

Similar for ^4He atoms Wu, Lin, König + vK, in progress

More nucleons around unitarity



Multiple towers of excited states?

Ground states

single scale



$$\frac{B_A^{(\text{LO})}(\Lambda_*)}{A} = \kappa_A \frac{B_3(\Lambda_*)}{3}$$

$$\left\{ \begin{array}{l} \kappa_2 \equiv 0 \\ \kappa_3 \equiv 1 \\ \kappa_4 \simeq 3.5 \\ \kappa_{A \geq 5} \simeq ? \end{array} \right.$$

= bosons

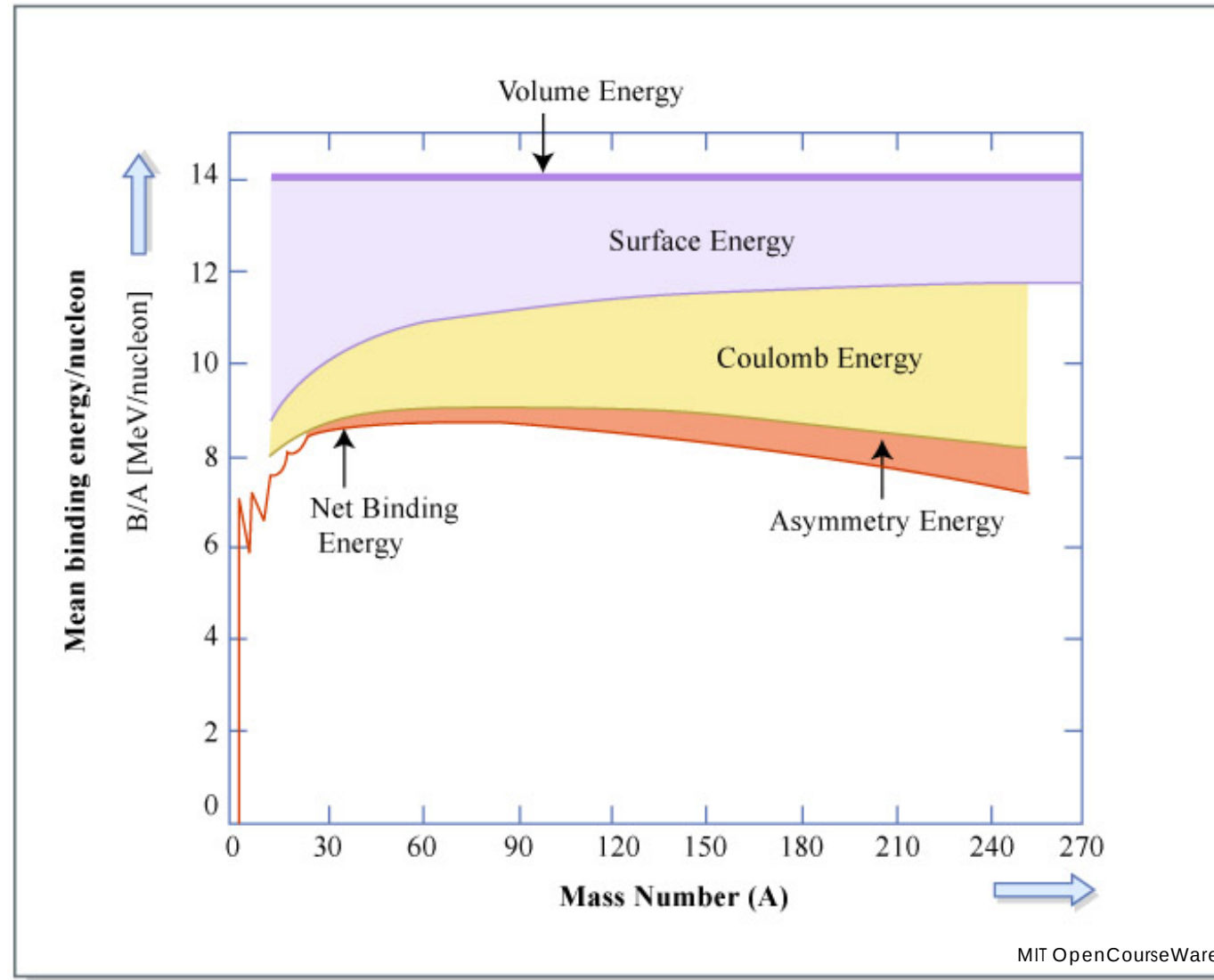
grows slower
than bosons?

nuclear structure
from a
single parameter?

Semi-empirical mass formula

Von Weizsäcker '35

$$\kappa_A = \kappa_\infty \left[1 - \eta_s A^{-1/3} - \eta_c A^{-4/3} Z(Z-1) - \eta_A \left(1 - 2Z/A \right)^2 + \dots \right]$$



$$\kappa_\infty \approx 5.6$$

$$\eta_s \approx 1.2$$

$$\eta_c \approx 0.05$$

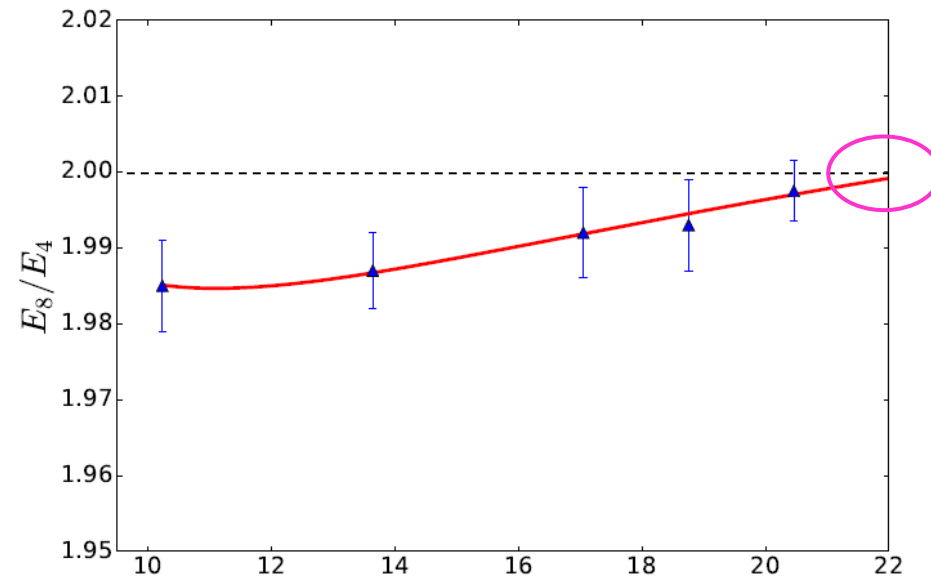
$$\eta_A \approx 1.5$$

$$\frac{B_3(\Lambda_*)}{3} \approx 2.8 \text{ MeV}$$



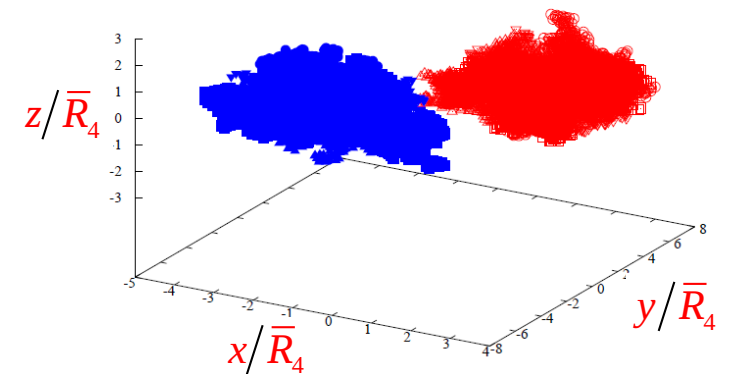
$$A=8$$

LO



$$\bar{R}_4 \equiv (\bar{R}_4 \Lambda)^{-1/2}$$

$$\kappa_8 \simeq \kappa_4 ?$$



consistent with ${}^8\text{Be}$

Clustering a universal property of multi-component unitary fermions?

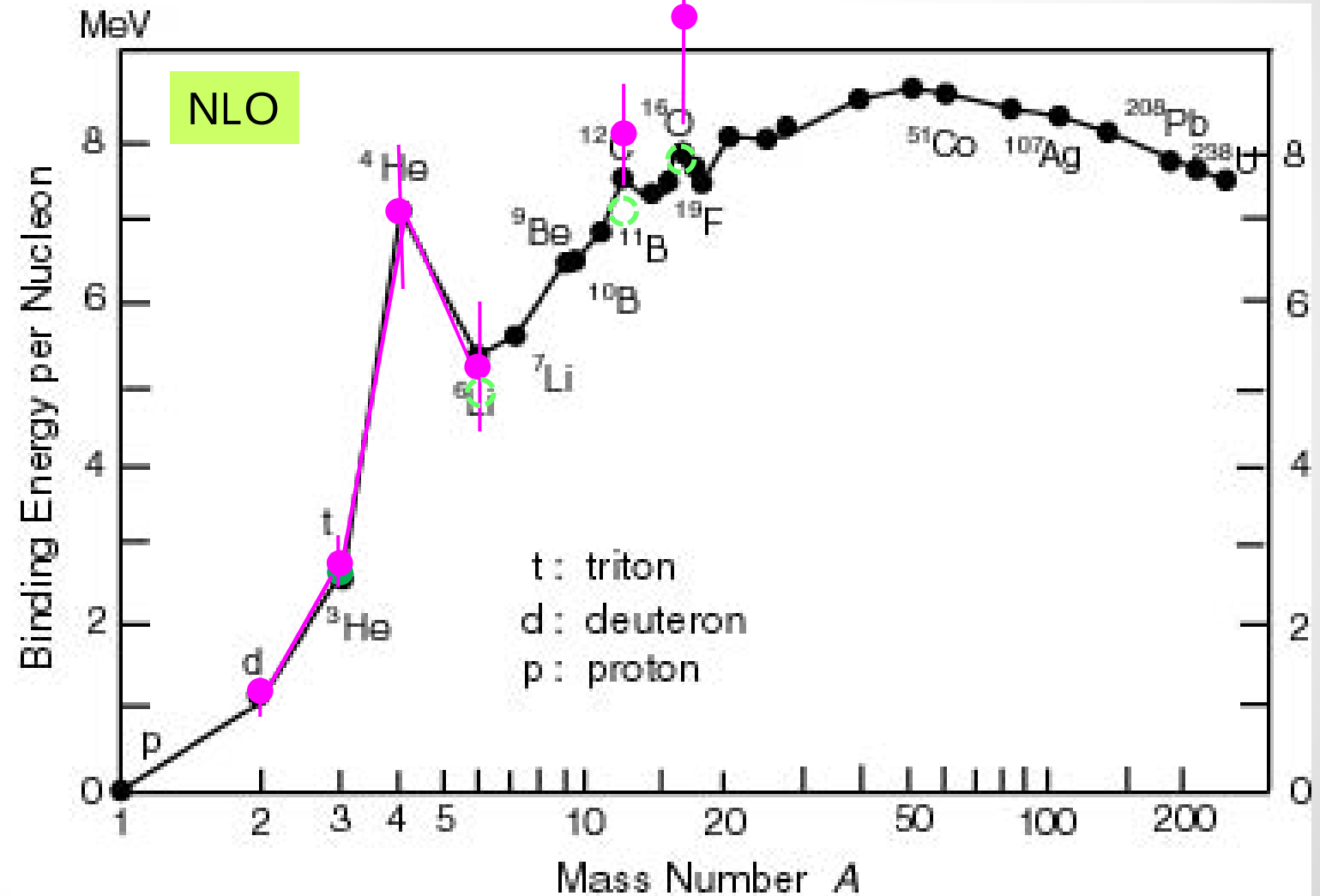
See also Schäfer, Contessi, Kirscher, Mareš '20



$$\frac{3B_A^{(1)}(\Lambda_*, \dots)}{AB_3(\Lambda_*)} \equiv \kappa_A^{(1)}(\dots)$$

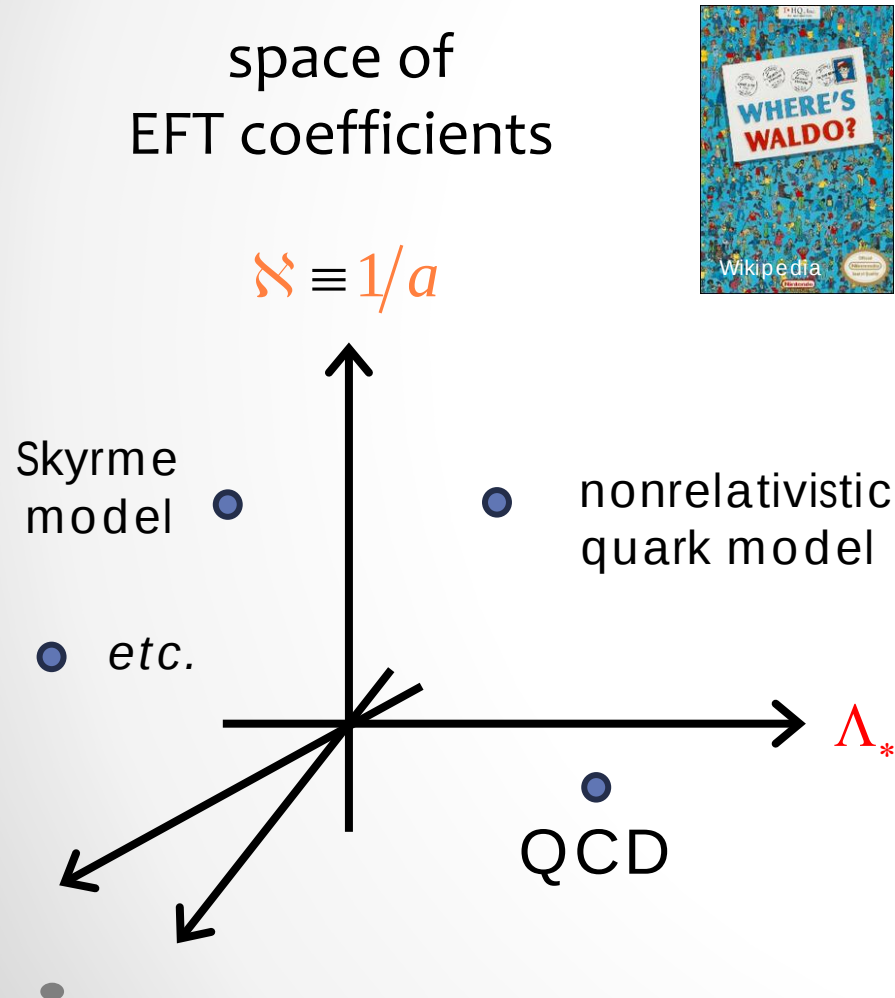
A	$\kappa_A^{(1)}$	$\kappa_A^{(\text{exp})}$
4	2.50	2.50
6	1.86	1.89
12	2.9	2.72
16	3.5	2.81

nuclear structure from
one essential,
three-body parameter!?



Universality but no chiral symmetry...

Where is QCD?



Unitarity: how?

Beane, Bedaque, Savage + vK '02

Beane + Savage '03

Epelbaum + Meißner '03

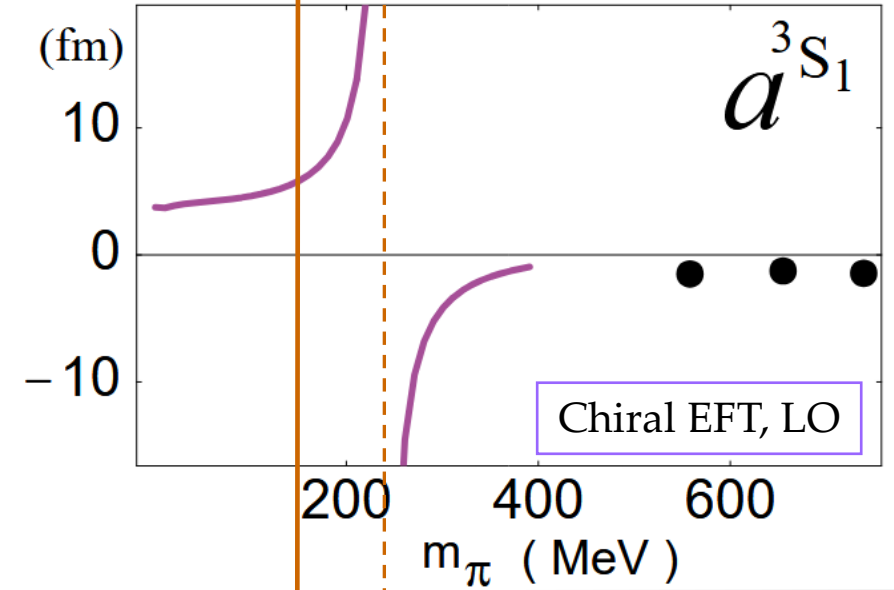
...

real world $m_\pi \approx 140$ MeV

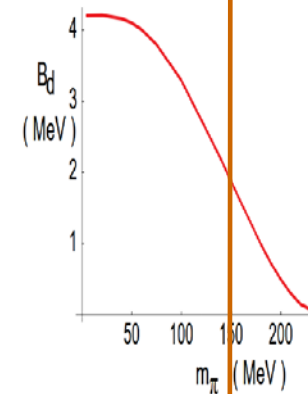
unitarity limit $|a| \rightarrow \infty$

$m_\pi^* (M_{QCD})$

A conjecture



deuteron binding energy



analogous to Feshbach resonance

$$B \rightarrow m_\pi$$

Conclusion

Quantum systems near unitarity can be described by essentially *one* (three-body!) parameter Λ_*

Fractal-like structure emerges from discrete scale invariance

Bosons saturate and form a quantum liquid

Multi-component fermions tend to cluster

Can we explain nuclear matter with a *perturbative* expansion around unitarity?