QUANTUM FRACTALS

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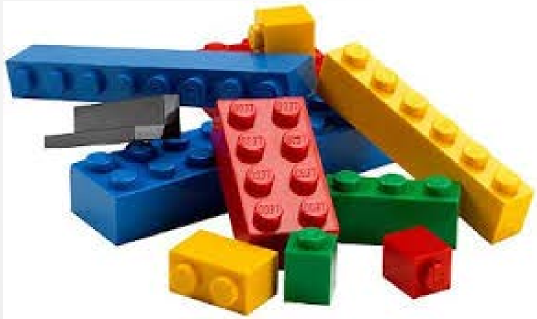
All aspects of strong interactions and related areas!

Outline

- Emergence of structure
- Discrete scale invariance
- Bosonic clusters
- Nucleons
- Where is QCD?
- Conclusion

Emergence of quantum structures

"reduction":
what are the
building blocks?



	atoms, molecules		
1 keV		$2\,\mathring{A}$	
	hadrons, nuclei		
1 GeV		$0.2\,\mathrm{fm}$	
	"elementary" particles		

$p \downarrow$ $\qquad r \uparrow$

$$pr \sim 1$$

uncertainty principle

Heisenberg '27

"emergence":
how do the building
blocks fit together?

Here: $\hbar = 1, c = 1$

$$[m] = [E] = [p] = [r]^{-1} = [t]^{-1}$$

Structures and scales

"… were the world to be made between now and tomorrow 100 or 1,000 times larger or smaller than it is at present, all its parts being enlarged or diminished proportionally, everything would appear tomorrow exactly as now, just as though nothing had been changed."

Nicolas Oresme

a commentary on Aristotle's *De caelo et mundo*, 1377

scale transformation

$$r \to \alpha r \qquad \longleftrightarrow \qquad p \to \alpha^{-1} p$$

$$\alpha \geq 0$$

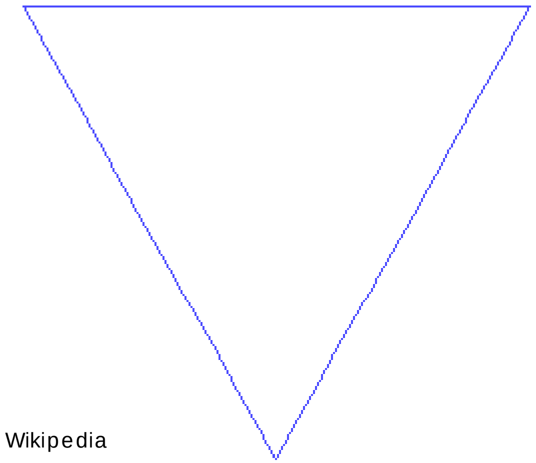
(quantum)

(nonrelativistic)

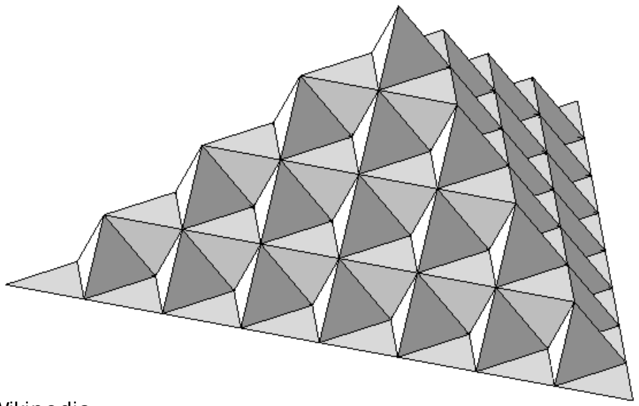
$$t/m \to \alpha^2 t/m \qquad \longleftrightarrow \qquad mE \to \alpha^{-2} mE$$

Simplest "complex" structures: one scale

von Koch 1904



Wikipedia

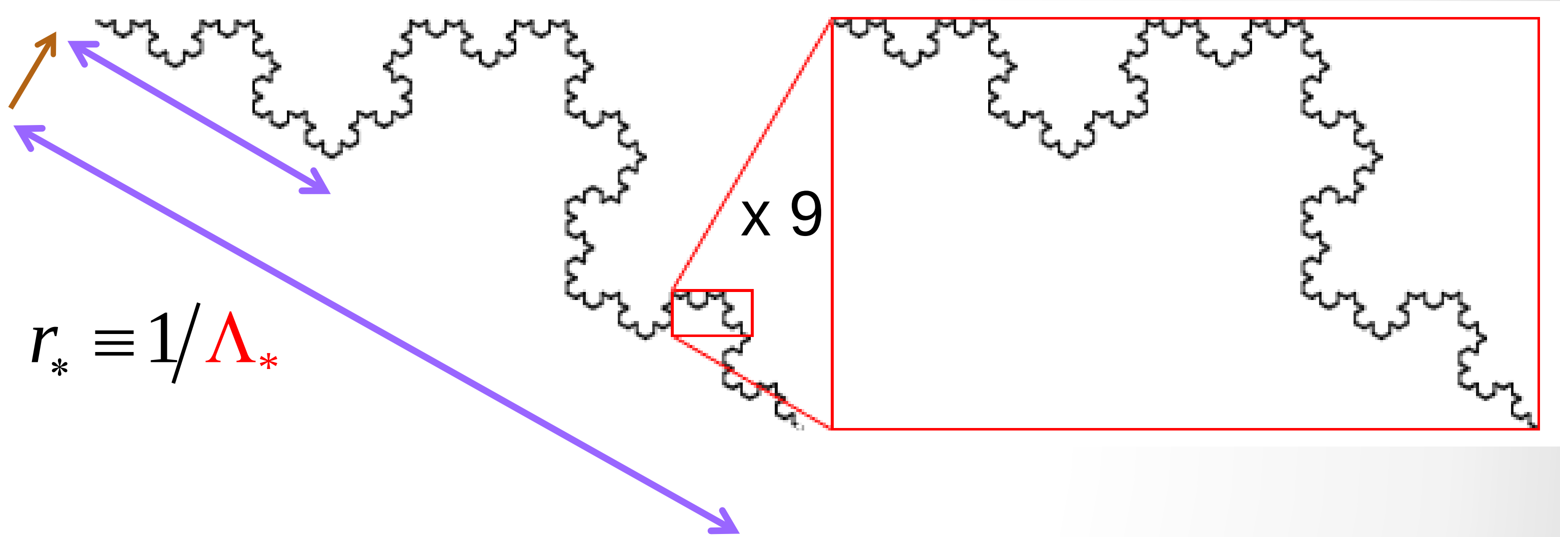


Wikipedia

(Scaling) Fractal

yozh.org/2010/10/21/mset001/

f



$$r_* \equiv 1/\Lambda_*$$

$$r \to \alpha_n r = f^n r$$

$$\begin{cases} f \text{ real} \\ n \text{ integer} \end{cases}$$

romanesco or "fractal broccoli"

Discrete scale invariance

Nuclear physics: nucleons (proton or neutron) with spin S=½, nearly the same mass

four-component fermions $m_N \simeq 940$ MeV

lightest exchanged particles: pions

$m_\pi \simeq 140$ MeV

$$V(r) = -\underbrace{\frac{g_{\pi N}^2}{m_N}\frac{e^{-m_\pi r}}{r^3}}_{}S_{12} + \ldots$$

range $R \sim m_\pi^{-1} \simeq 1.4$ fm

Yukawa '35

proton + neutron, S=1: deuteron

$B_2 \simeq 2.2$ MeV $\Rightarrow$ $\dfrac{R}{a_2} \simeq \dfrac{\sqrt{m_N B_2}}{m_\pi} \approx \dfrac{1}{3}$

binding energy

scattering length ~ system size

Atomic physics: neutral atoms, mass $m_{at} \approx A m_N$

fermions or bosons

lightest exchanged particles: two photons

$m_\gamma = 0$

$$V(r) = -\underbrace{\frac{l_{\rm vdW}^4}{4\pi m_{\rm at} r^6}}_{} + \ldots$$

"range" $R \sim l_{\rm vdW}$

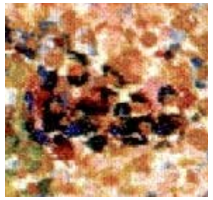
v.d. Waals 1873

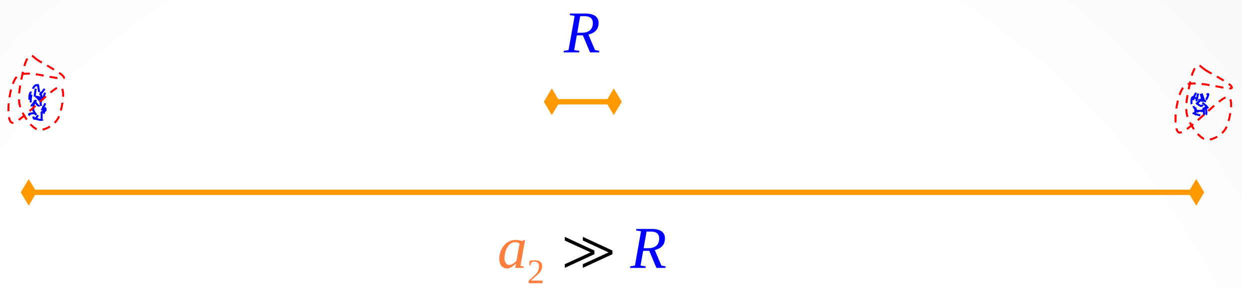
two bosonic ^{4}He atoms, S=0: ^{4}He dimer

$B_2 \simeq 1.3$ mK

$l_{\rm vdW} \simeq 5.4$ Å

$\Rightarrow$ $\dfrac{R}{a_2} \simeq \sqrt{m_N B_2}\, l_{\rm vdW} \approx \dfrac{1}{20}$





leading order

next-to-leading order

multipole expansion of interactions

$$V_2(\vec{r};\Lambda) = \frac{4\pi}{m}\left\{ C_0(\Lambda)\ \delta_\Lambda^{(3)}(\vec{r}) + C_2(\Lambda)\ \nabla^2 \delta_\Lambda^{(3)}(\vec{r}) + \ldots \right\}$$

cutoff-dependent parameters

smeared delta function

renormalization

S matrix in expansion in powers of R/r,
insensitive to *arbitrary* choice of regularization
as long as $\Lambda^{-1} \lesssim R$

"Short-Range Effective Field Theory"

vK '97'98
Bedaque + vK '97
Kaplan, Savage, Wise '98
…
cf. Bethe, Peierls '35

LO point limit $R \to 0$

scale invariance

unitarity limit $a_2 \to \infty$ ⇒ $mE_2 = 0 \Leftrightarrow \alpha^{-2} mE_2 = 0$

no scale!

no two-body bound states...

cf. Oresme

Two-component fermions

no *N*-body bound states either
unless
scale invariance broken by external interaction/trap

$$\text{e.g.} \quad \left.\frac{E_N^{(0)}}{N}\right|_{N\to\infty} = \xi \, \frac{3(3\pi^2\rho)^{2/3}}{10m}$$

universal number
Bertsch '99

free-gas energy, uniform density

Other cases?

Bosons,
more-component
fermions

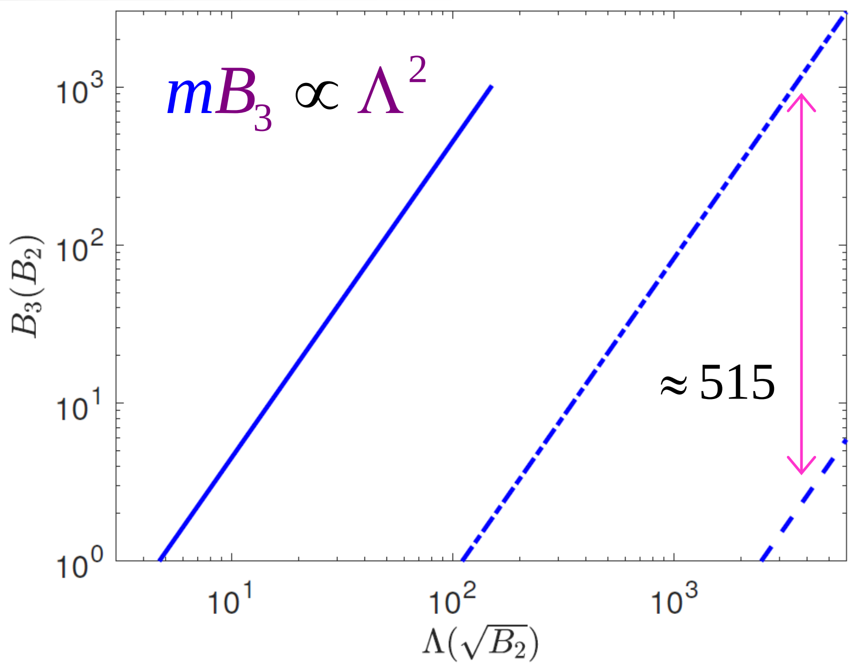
Three-body system

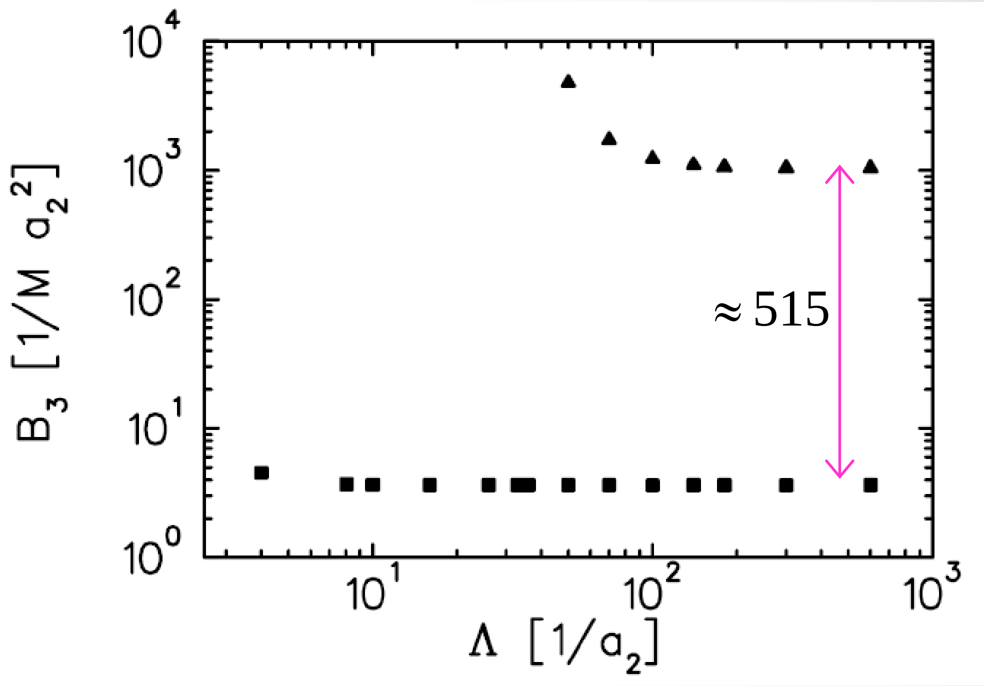
Bedaque, Hammer + vK '99 '00

...

LO

Feng Wu '22





Bedaque, Hammer, vK, *Nucl. Phys. A* **676** (2000) 357

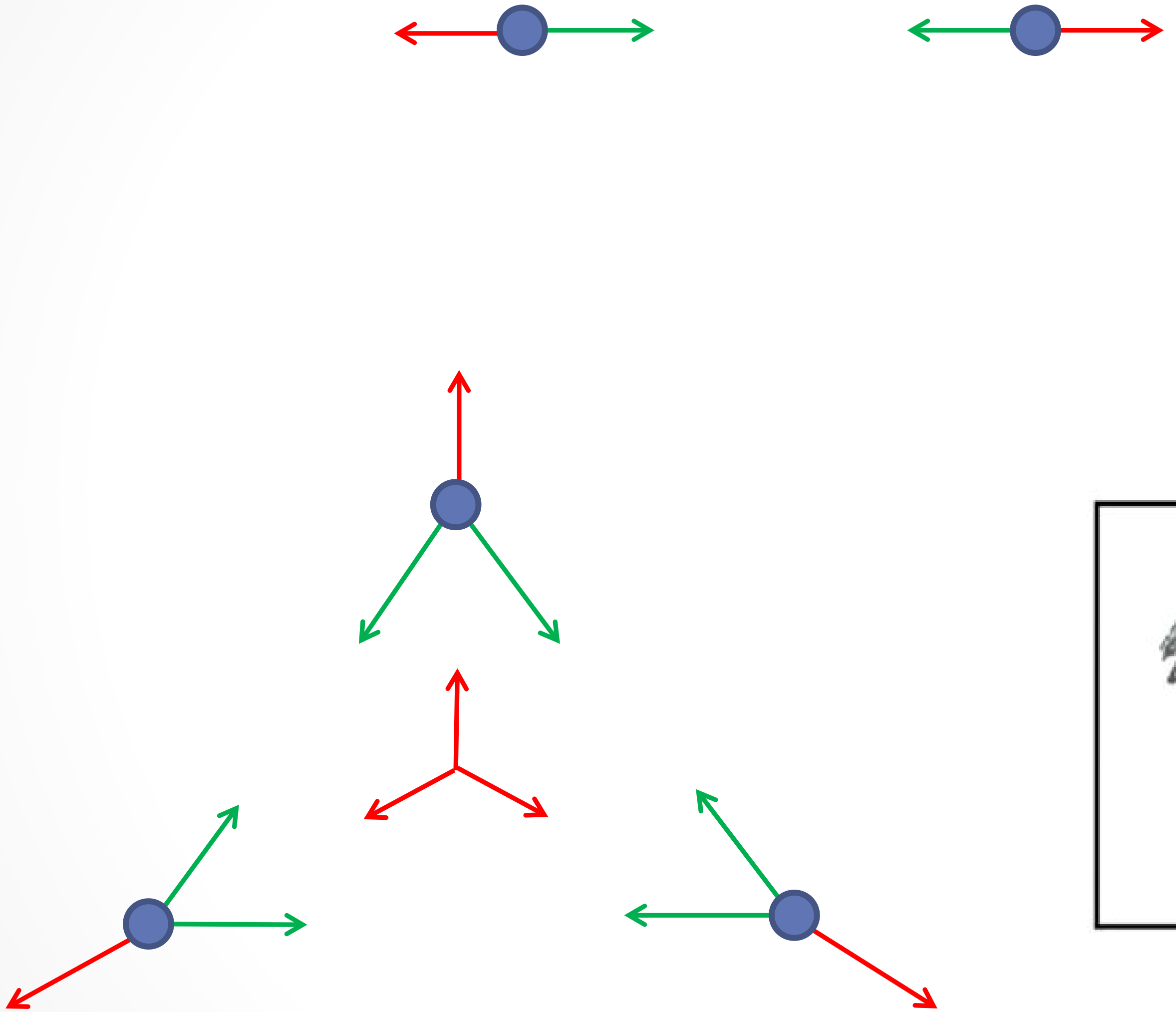
"Thomas collapse"

Thomas '35

$$V_3(\vec{r}_1-\vec{r}_2,\vec{r}_2-\vec{r}_3)=\frac{(4\pi)^2}{m}D_0(\Lambda)\,\delta^{(3)}_\Lambda(\vec{r}_1-\vec{r}_2)\,\delta^{(3)}_\Lambda(\vec{r}_2-\vec{r}_3)$$

one parameter

determined by one three-body datum




3-body force
Wikipedia

Bedaque, Hammer + vK '99 '00
...

$$H(\Lambda) \equiv \frac{\Lambda^2 D_0(\Lambda)}{m C_0^2(\Lambda)} \simeq \frac{\sin\left(s_0 \ln(\Lambda_*/\Lambda) - \arctan(s_0^{-1})\right)}{\sin\left(s_0 \ln(\Lambda_*/\Lambda) + \arctan(s_0^{-1})\right)}$$

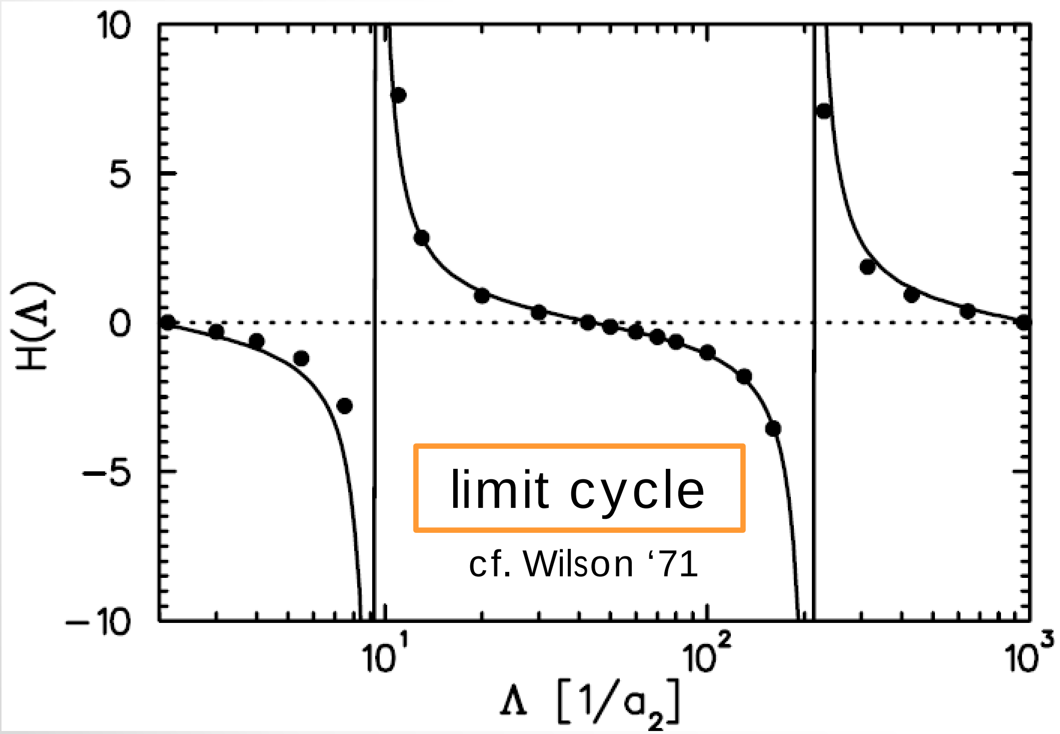
anomalous breaking of (continuous) scale invariance

dimensionful parameter

$s_0 \simeq 1.00624$

quantum phenomenon!

f^{-1}



limit cycle

cf. Wilson '71

Bedaque, Hammer, vK, *Phys. Rev. Lett.* **82** (1999) 463

$$\Lambda \to \alpha_n^{-1}\Lambda = f^{-n}\Lambda$$

$$f^{-1} = \exp\left(\pi/s_0\right) \simeq 22.7$$

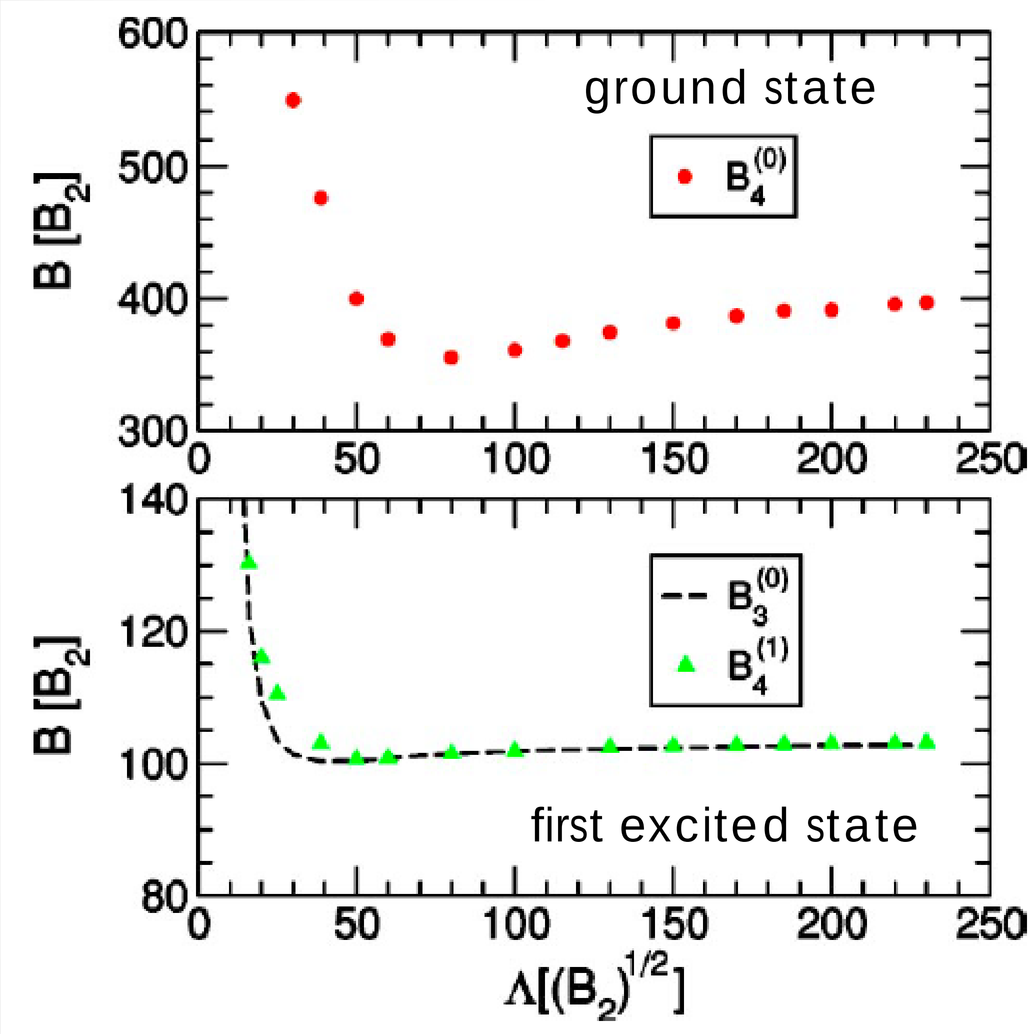
STRUCTURE FROM A SINGLE SCALE

Discrete scale invariance

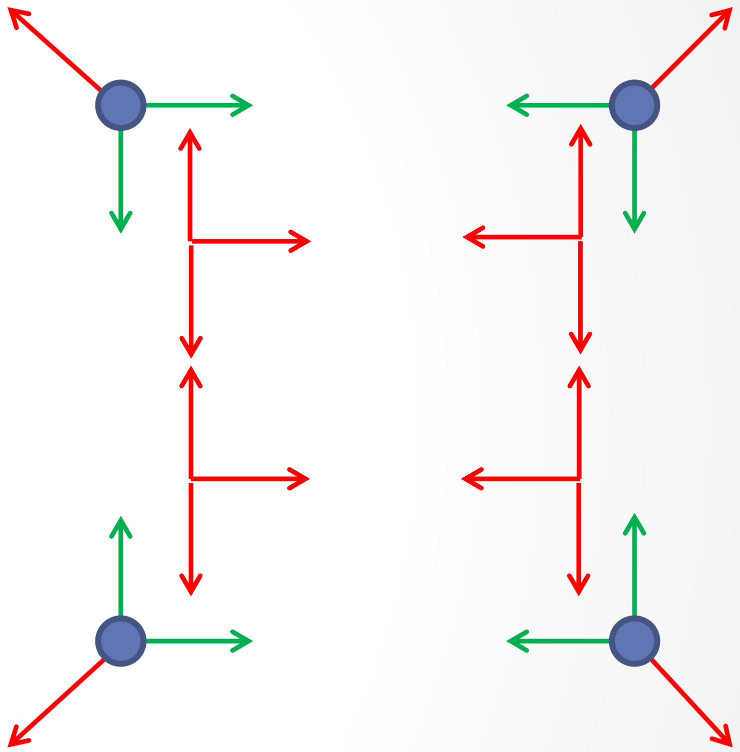
Platter, Hammer + Meißner '04 ...

LO

Four-body system



Platter, Hammer, Meißner, *Phys. Rev. A* **70** (2004) 052101



No 4-body force...

... and no other scale at leading order!

Two consequences

halo!

Wu, Frederico, Higa + vK '24

1) Towers of excited states

$$mB^{(0)}_{A,n} \to \alpha_l^{-2}\, mB^{(0)}_{A,n} = mB^{(0)}_{A,n+l}$$

$$\Rightarrow \quad mB^{(0)}_{A,n}(\Lambda_*) = mB^{(0)}_{A,0}(\Lambda_*)\exp\left(-2n\pi/s_0\right)$$

ground state

fixes tower position

$A=3$ Efimov '70

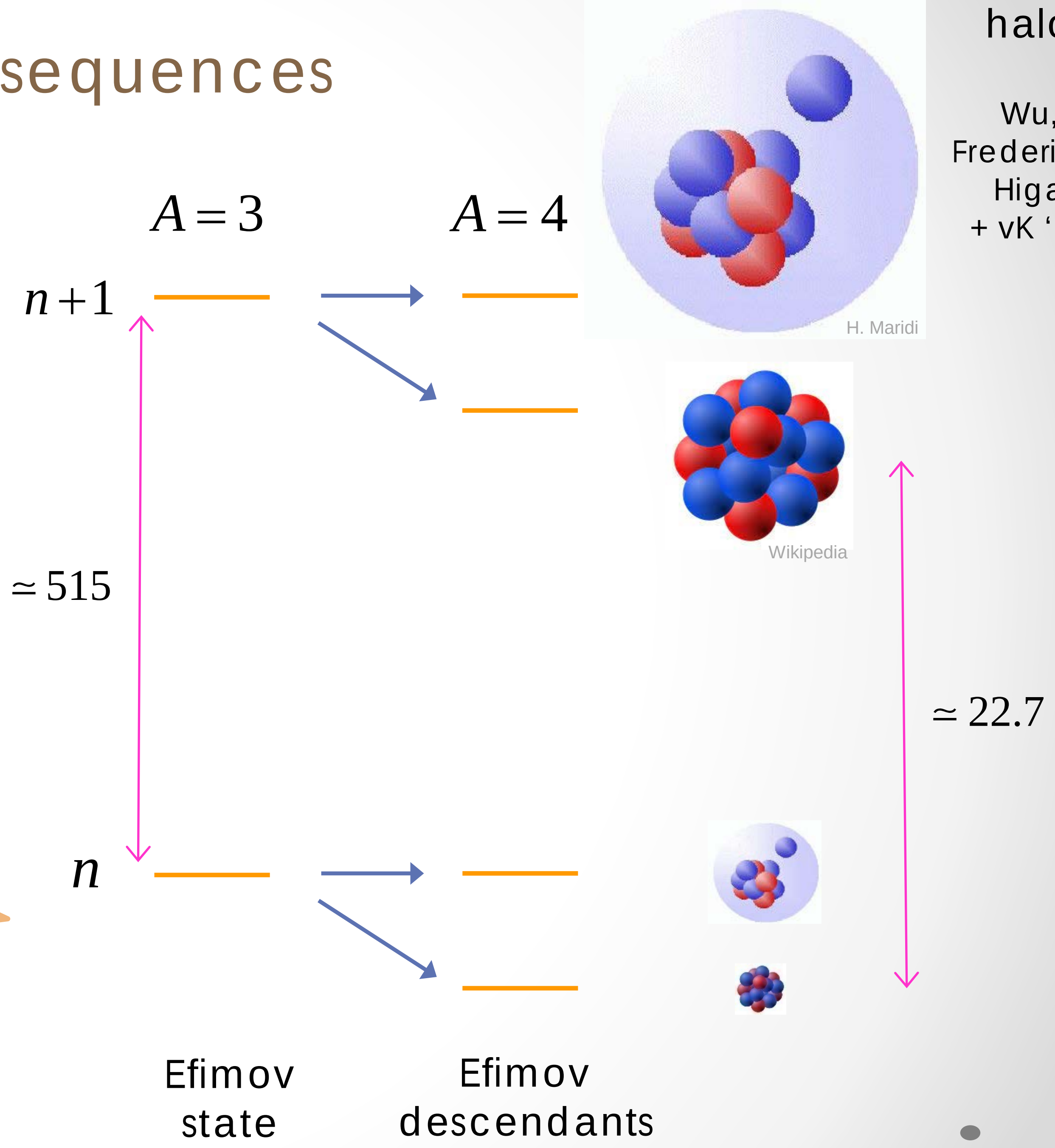
$A=4$ Hammer, Platter '07

$A=5,6$ von Stecher '10'11, Gattobigio, Kievsky, Viviani '11'12

bosons

Towers upon towers!

- origin?
- rules?



2) Ground-state correlations

single scale → $$\frac{B^{(0)}_{A,0,0}(\Lambda_*)}{A} = \kappa_A \frac{B^{(0)}_{3,0}(\Lambda_*)}{3}$$

universal numbers

$$\left\{ \begin{array}{l} \kappa_2 \equiv 0 \\ \kappa_3 \equiv 1 \\ \kappa_4 \simeq 3.5 \\ \kappa_{A\geq 5} \simeq ? \end{array} \right.$$

Hammer, Platter ’07

von Stecher ’10

...

Carlson, Gandolfi, Vitiello + vK ‘17

varying Λ_*

$A=4$

Tjon line

Platter, Hammer, Meißner ’05

Tjon ’75

Nakaichi, Akaishi, Tanaka, Lim ’78

$A=5,6$

bosons

Generalized Tjon lines

Bazak, Eliyahu + vK ’16

Nakaichi, Akaishi, Tanaka, Lim ’79’80

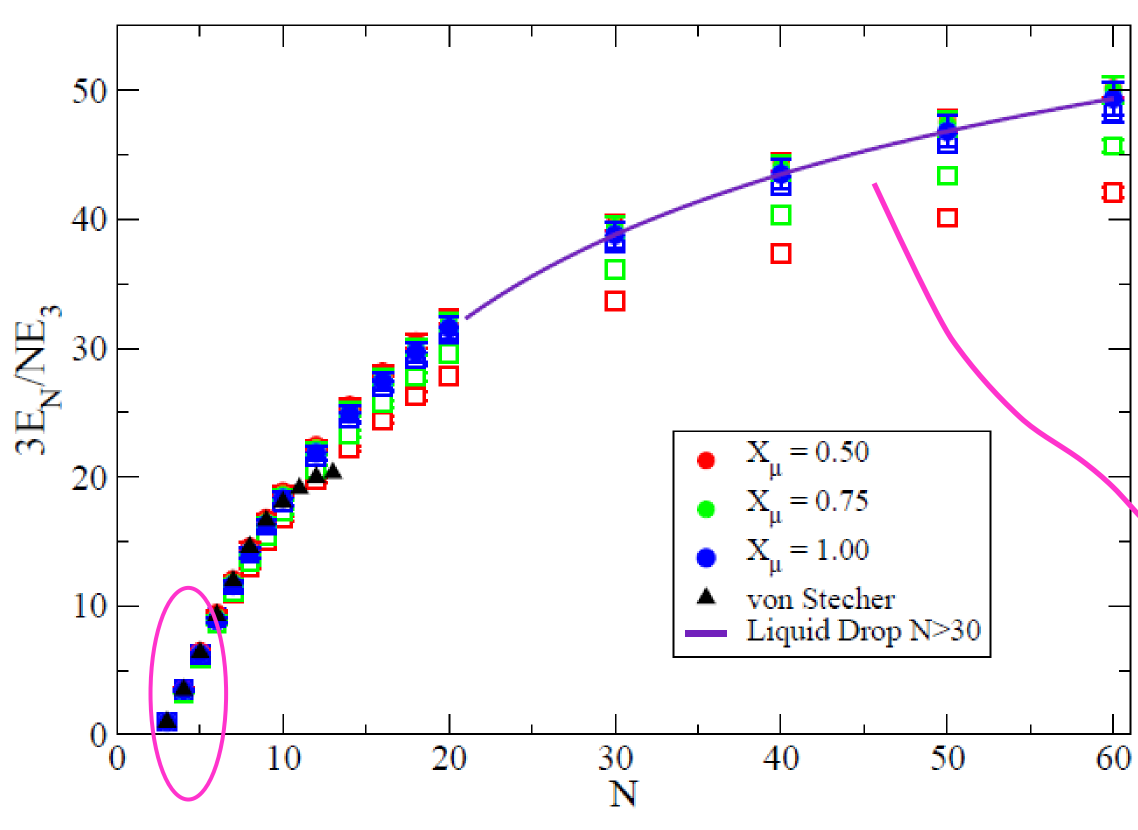
fixing Λ_*

$$\frac{B^{(0)}_{A,0,0}(B_{3,0})}{A} = \kappa_A \frac{B_{3,0}}{3}$$

N unitary bosons

Gandolfi, Carlson, Vitiello + vK '17

Carlson, Gandolfi, Vitiello, vK, *Phys. Rev. Lett.* **119** (2017) 223002



increasing Λ

LO

Quantum Monte Carlo

saturation!

cf. Piatecki + Krauth '14
Comparin + Krauth '16

$$\kappa_N = \kappa_\infty \left[1 - \eta_S N^{-1/3} + \mathcal{O}\left(N^{-2/3}\right) \right]$$

$$\kappa_N \approx \frac{3}{N}(N-2)^2$$

Bazak, Eliyahu + vK '16

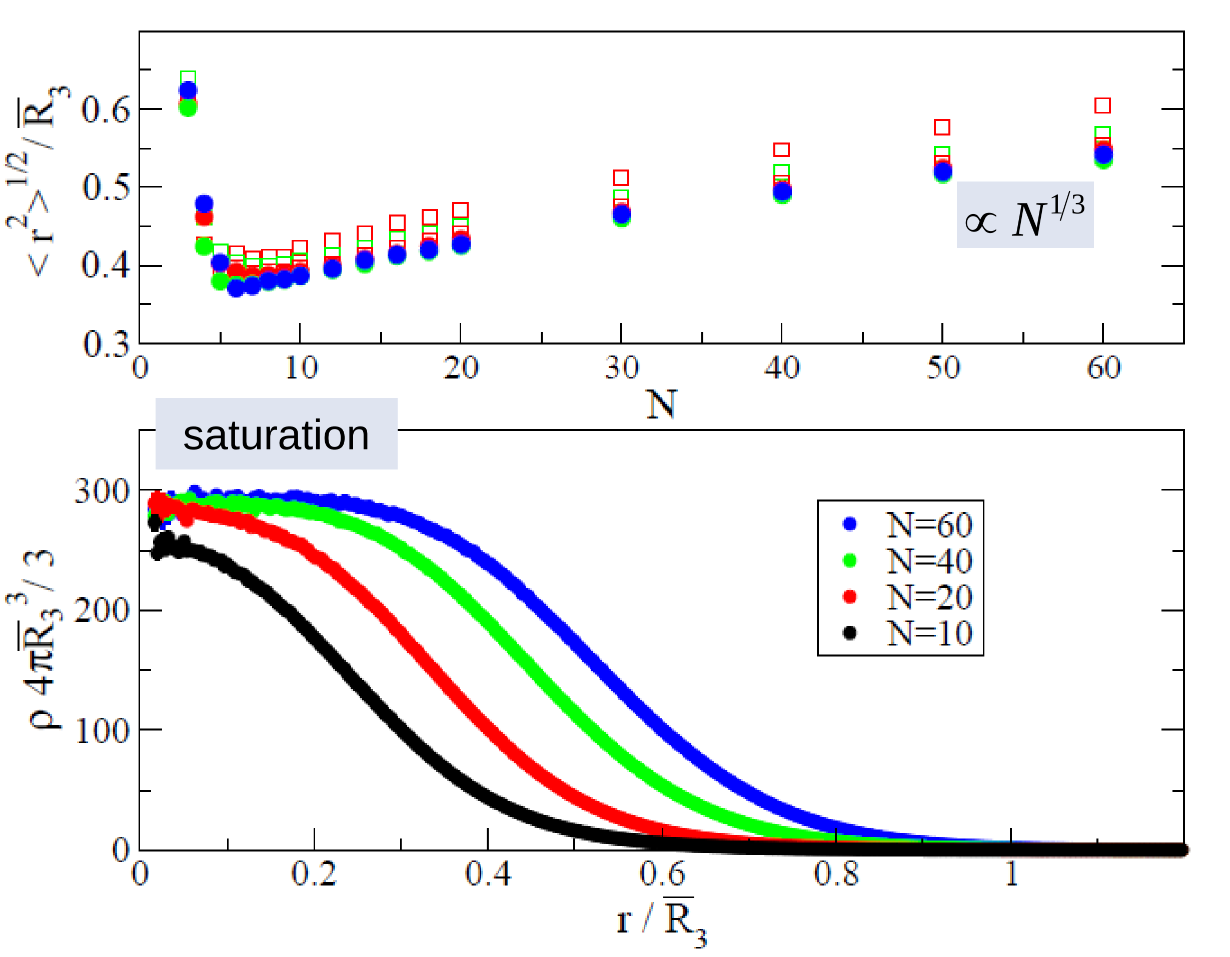
$$\kappa_\infty = 90 \pm 10 \qquad \eta_S = 1.7 \pm 0.3$$

cf. ^{4}He $\quad \kappa_\infty \simeq 182 \qquad \eta_S \simeq 2.7$

Pandharipande *et al.* '83

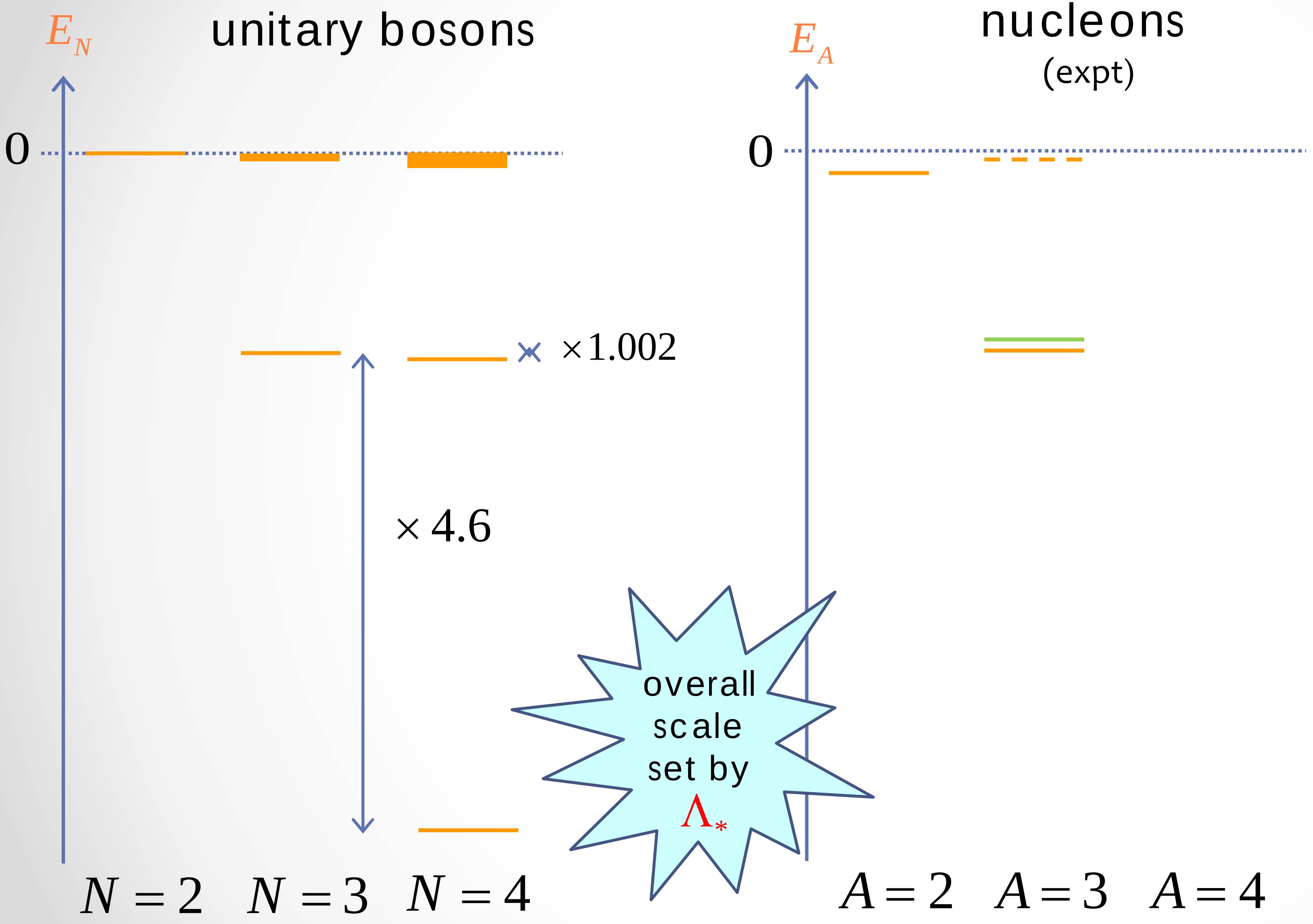
Gandolfi, Carlson, Vitiello + vK ’17

A liquid indeed…



$$\overline{R}_3 \equiv \left(2mB_3\right)^{-1/2}$$

Carlson, Gandolfi, Vitiello, vK, *Phys. Rev. Lett.* **119** (2017) 223002



$$\frac{3|B_2|}{2|B_3|} \simeq 0.4 < 1$$

subleading

virtual state in neutron-deuteron scattering

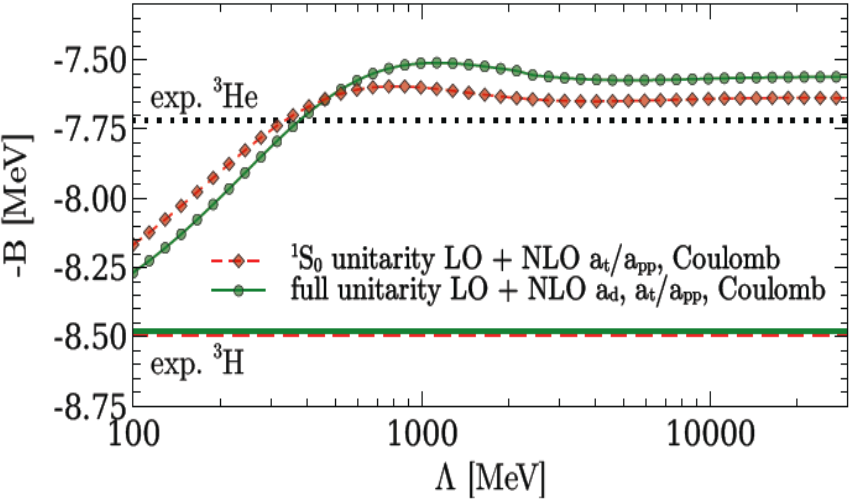
Rupak, Vaghani, Higa + vK '18

^{3}He-^{3}H isospin splitting obtained at NLO

König, Grießhammer, Hammer + vK '16

Nucleons around unitarity

Helion-triton splitting



König, Grießhammer, Hammer, vK, *Phys. Rev. Lett.* **118** (2017) 202501

$A = 3$

NLO

$$B_h^{(1)} - B_t \simeq -(0.92 \pm 0.18)\ \text{MeV}$$

vs.

$$-0.764\ \text{MeV (exp)}$$

LO

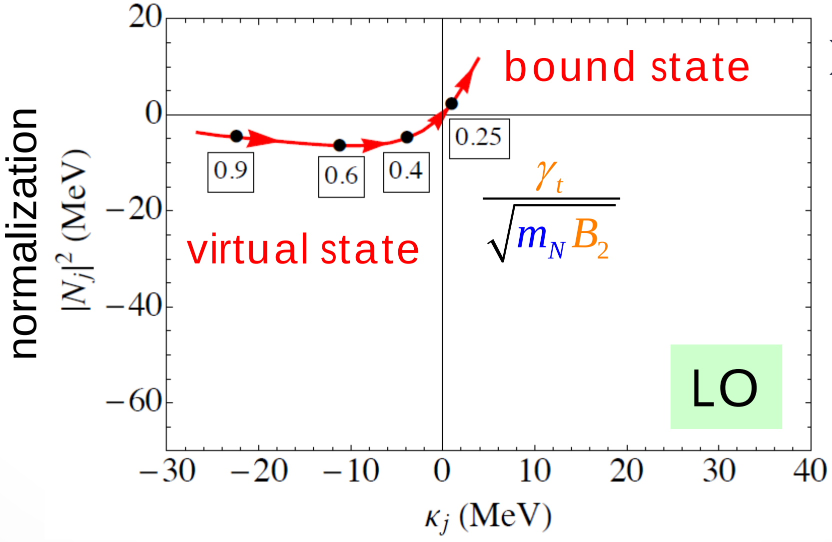
König, Grießhammer, Hammer + vK ’16



First-excited state of triton

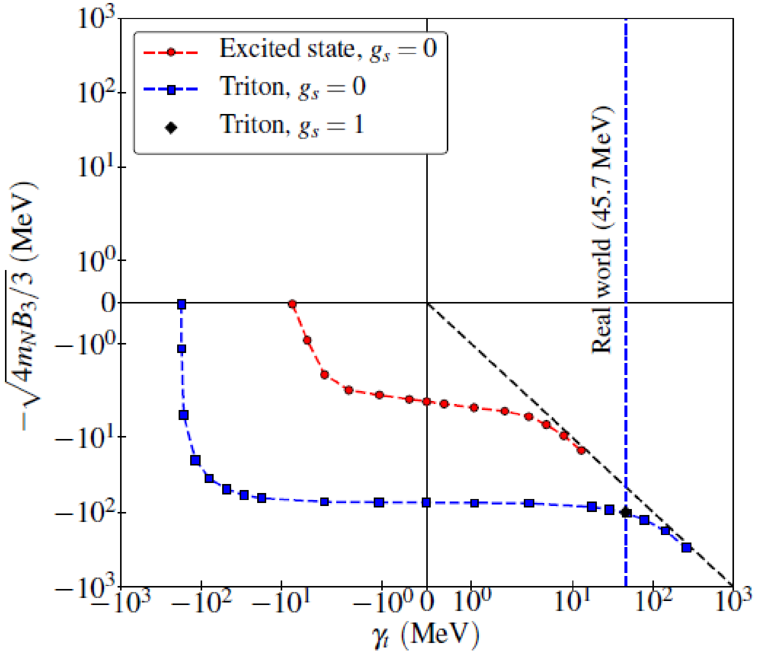
Rupak, Vaghani, Higa + vK ’18



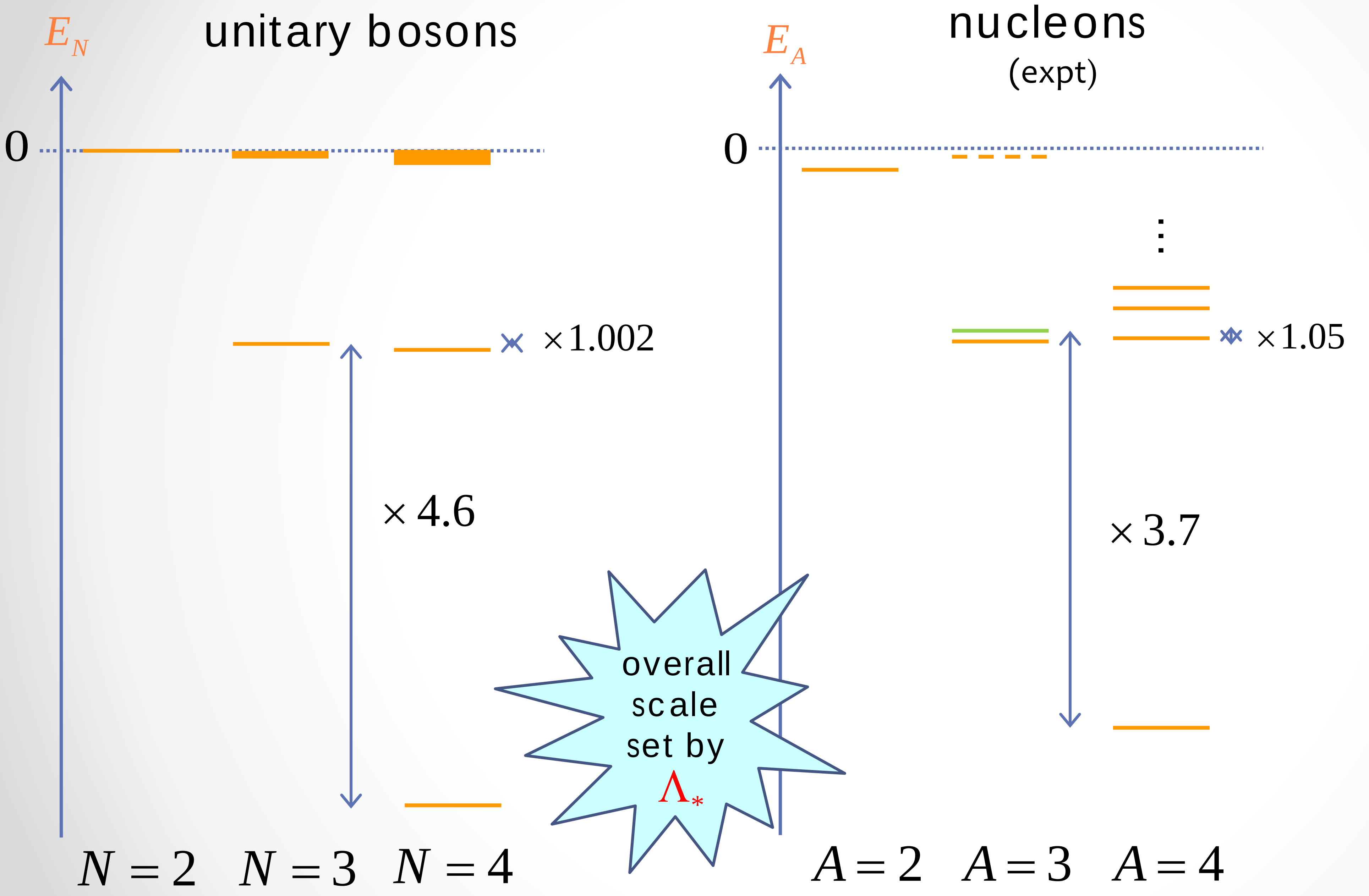


binding momentum

Rupak, Vaghani, Higa, vK, *Phys. Lett. B* **791** (2019) 414



deuteron binding momentum



$$\frac{3|B_2|}{2|B_3|} \simeq 0.4 < 1$$

subleading

virtual state in neutron-deuteron scattering

Rupak, Vaghani, Higa + vK '18

^{3}He-^{3}H isospin splitting obtained at NLO

König, Grießhammer, Hammer + vK '16

correction obtained at NLO

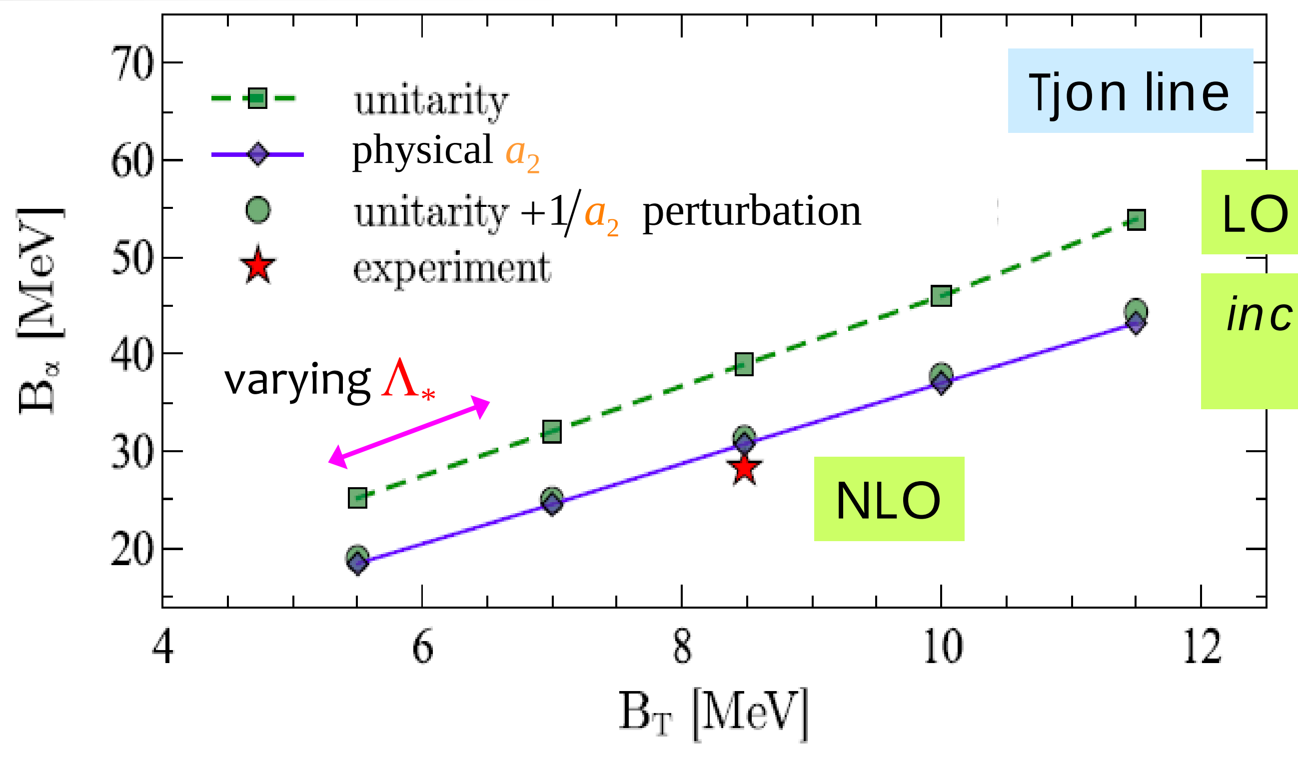
König, Grießhammer, Hammer + vK '17

König, Grießhammer, Hammer + vK ’16

Alpha particle

$A = 4$

König, Grießhammer, Hammer, vK, *Phys. Rev. Lett.* **118** (2017) 202501



LO

incomplete NLO

Ground state

perturbatively close to unitarity limit!

First-excited state

$$\frac{B_{\alpha*}^{(0)}}{B_t} = \frac{B_{\alpha*}^{(0)}}{B_h^{(0)}} \simeq 1.0023$$

Hammer, Platter ’07
...
Deltuva ’10

vs.

$$\left\{ \begin{array}{l} \dfrac{B_{\alpha*}}{B_h} = 1.05 \text{ MeV (exp)} \\ \dfrac{B_{\alpha*}}{B_t} = 0.95 \text{ MeV (exp)} \end{array} \right.$$

but needs full NLO

Similar for ^{4}He atoms   Wu, Lin, König + vK, in progress

More nucleons around unitarity

discrete scale invariance

Multiple towers of excited states?

Ground states

single scale ➡ $$\frac{B_A^{(\mathrm{LO})}(\Lambda_*)}{A} = \kappa_A \frac{B_3(\Lambda_*)}{3} \qquad \begin{cases} \kappa_2 \equiv 0 \\ \kappa_3 \equiv 1 \\ \kappa_4 \simeq 3.5 \\ \kappa_{A\geq 5} \simeq ? \end{cases}$$

= bosons

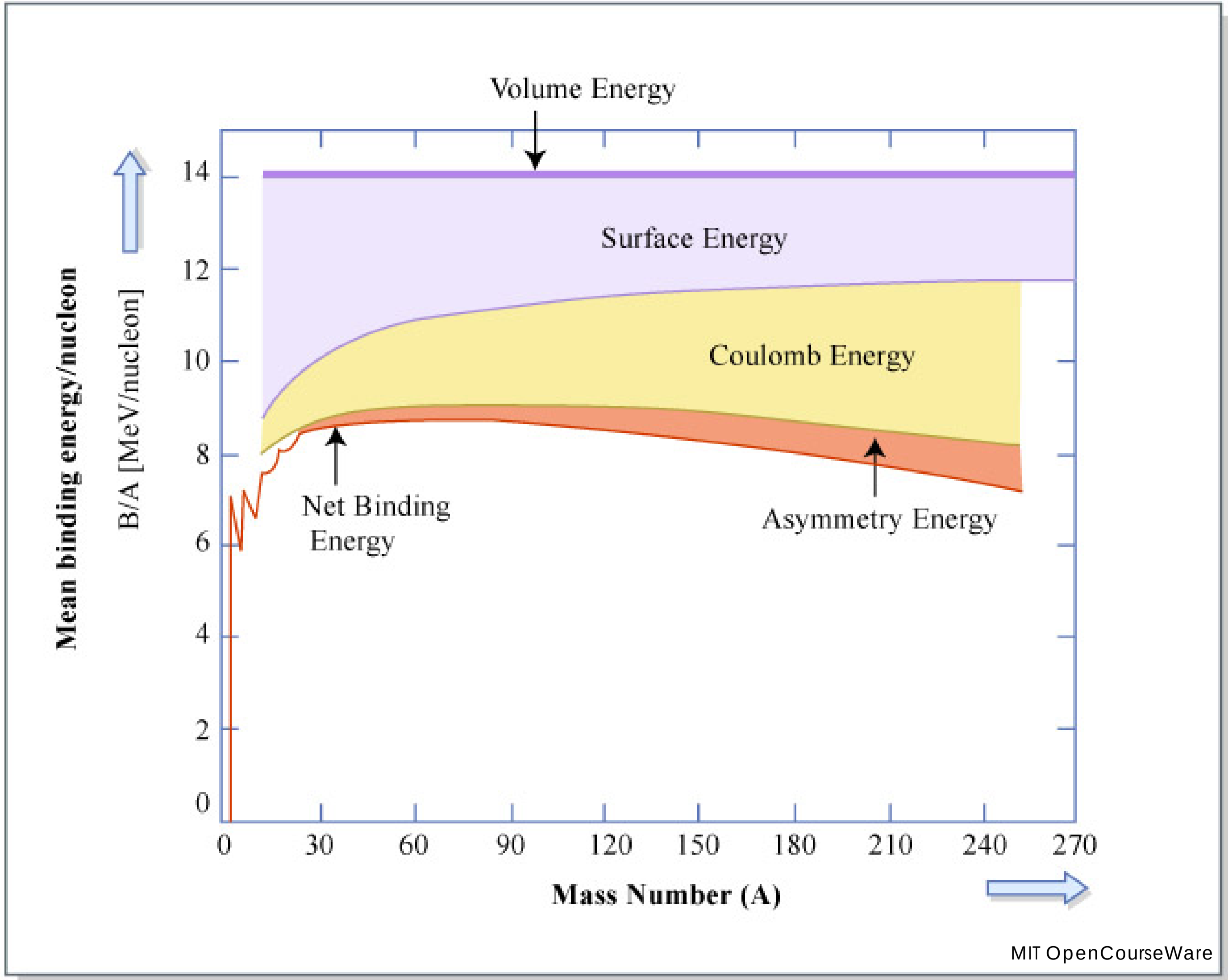
grows slower than bosons?

nuclear structure from a single parameter?

Semi-empirical mass formula

Von Weizsäcker '35

$$\kappa_A = \kappa_\infty \left[1 - \eta_S A^{-1/3} - \eta_C A^{-4/3} Z(Z-1) - \eta_A (1 - 2Z/A)^2 + \ldots \right]$$



$\kappa_\infty \approx 5.6$

$\eta_S \approx 1.2$

$\eta_C \approx 0.05$

$\eta_A \approx 1.5$

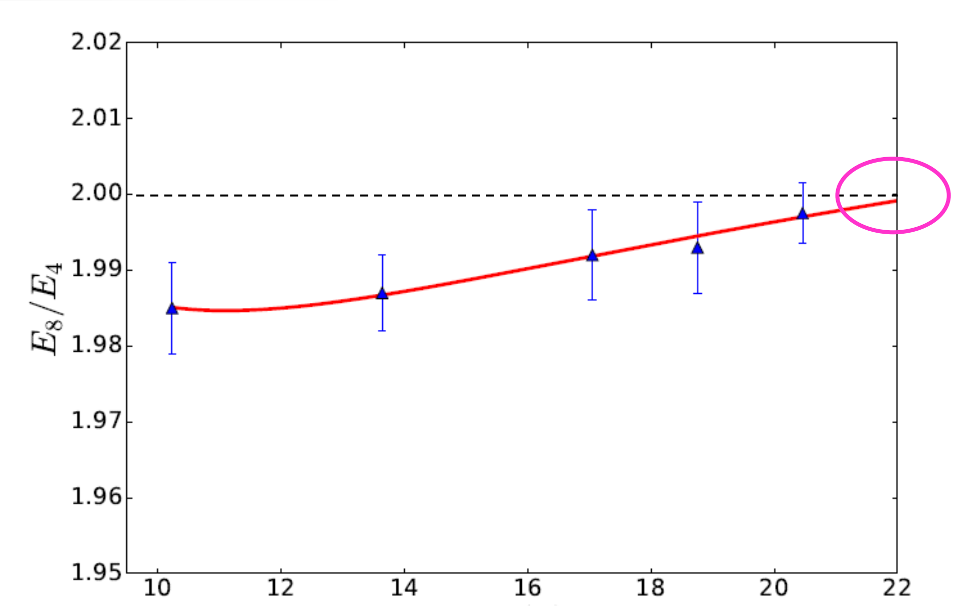
$$\frac{B_3(\Lambda_*)}{3} \simeq 2.8 \text{ MeV}$$

Dawkins, Carlson, vK + Gezerlis ’20



$A = 8$

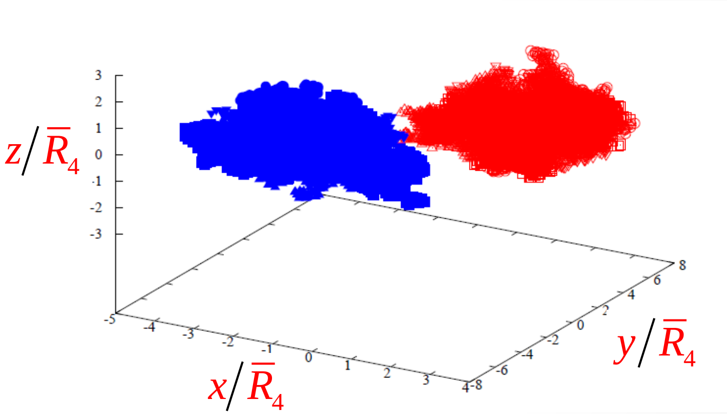
LO



$\bar{R}_4 \Lambda$

$$\bar{R}_4 \equiv (2mB_4)^{-1/2}$$

$\kappa_8 \simeq \kappa_4$?



consistent with ^{8}Be

Clustering a universal property of multi-component unitary fermions?

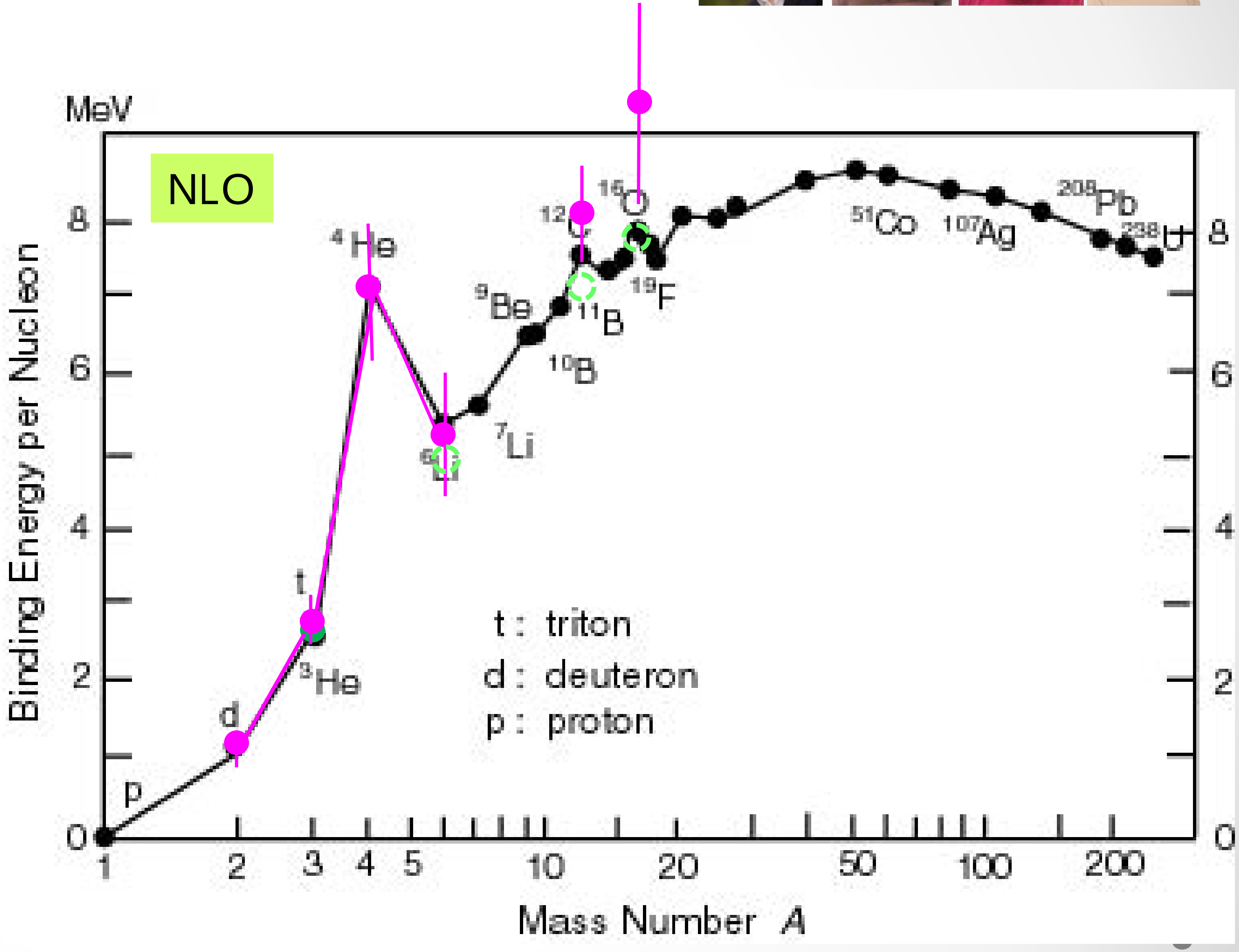
See also Schäfer, Contessi, Kirscher, Mareš ‘20

Contessi, Schäfer, Gnech, Lovato + vK '25

$$\frac{3B_A^{(1)}(\Lambda_*,\ldots)}{AB_3(\Lambda_*)} \equiv \kappa_A^{(1)}(\ldots)$$

A	$\kappa_A^{(1)}$	$\kappa_A^{(\mathrm{exp})}$
4	2.50	2.50
6	1.86	1.89
12	2.9	2.72
16	3.5	2.81

nuclear structure from one essential, three-body parameter!?

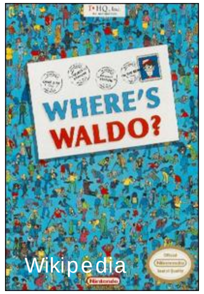


https://ne.phys.kyushu-u.ac.jp/seminar/MicroWorld3_E/3Part3_E/3P34_E/nuclear_fusion_E.htm

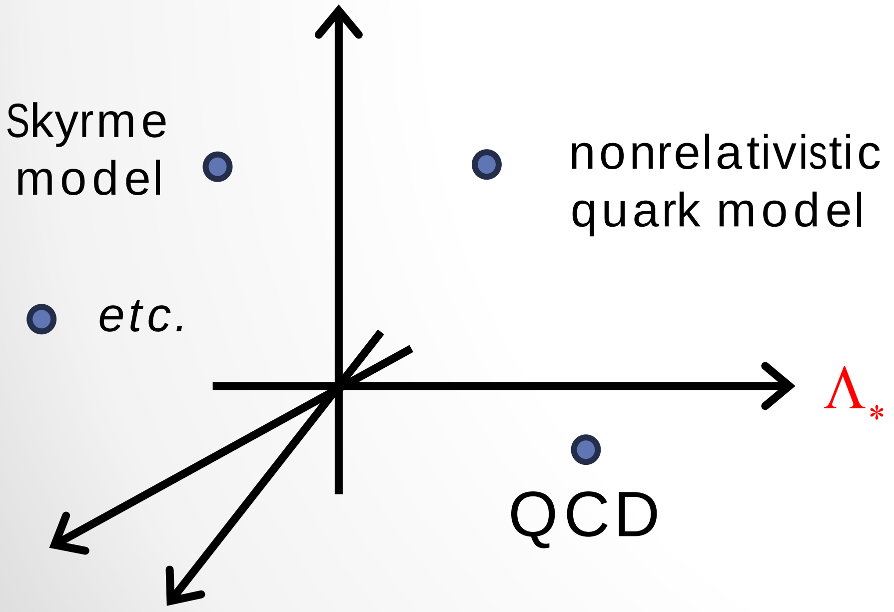
Universality but no chiral symmetry…

Where is QCD?

space of
EFT coefficients



$\aleph \equiv 1/a$



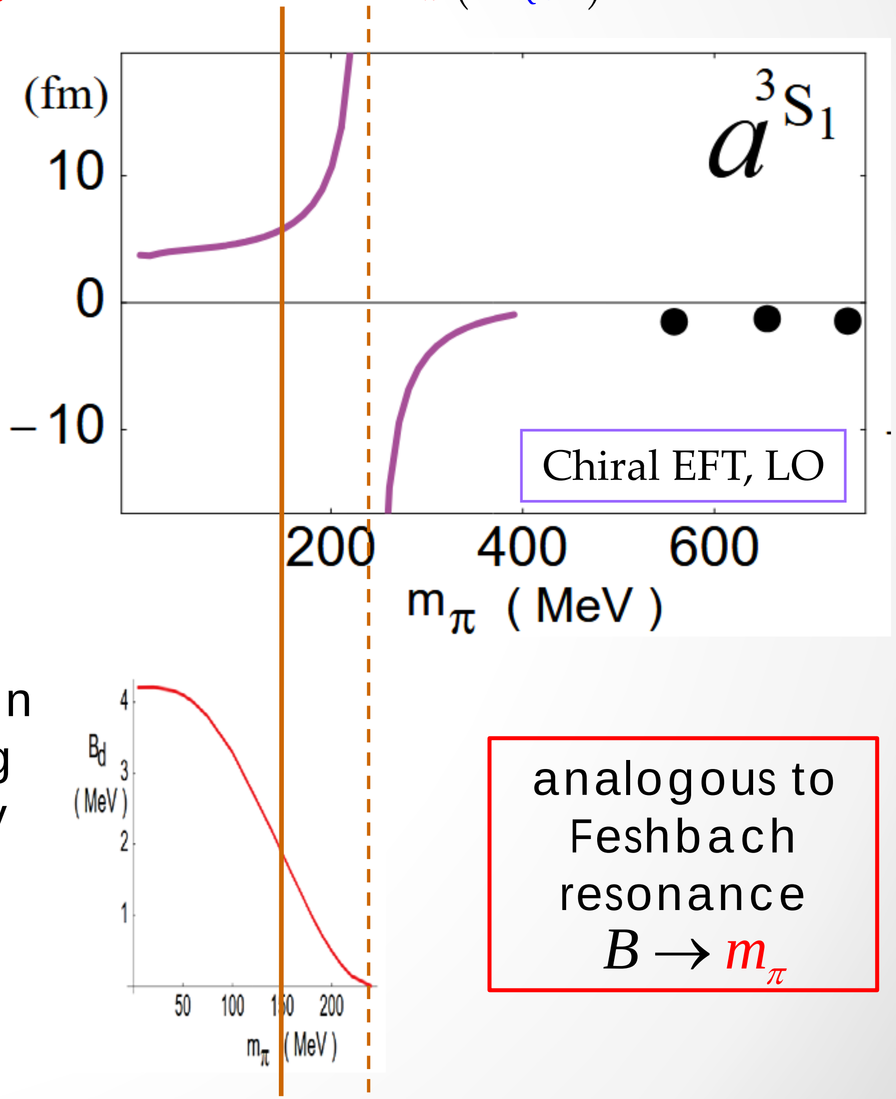
Unitarity: how?

Beane, Bedaque, Savage + vK '02
Beane + Savage '03
Epelbaum + Meißner '03
…

real world
$m_\pi \simeq 140 \text{ MeV}$

unitarity limit $|a| \to \infty$
$m_\pi^*(M_{QCD})$

A conjecture



deuteron
binding
energy

analogous to
Feshbach
resonance
$B \to m_\pi$

Beane, Bedaque, Savage, vK, *Nucl. Phys. A* **700** (2002) 377

Conclusion

Quantum systems near unitarity can be described by essentially *one* (three-body!) parameter Λ_*

Fractal-like structure emerges from discrete scale invariance

Bosons saturate and form a quantum liquid

Multi-component fermions tend to cluster

Can we explain nuclear matter with a *perturbative* expansion around unitarity?