Kerr magnetospheres and motion of charged particles: The role of symmetries

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Review talk and new results

In collaboration with I. El Mellah and E. Gourgoulhon (to appear)

Context and motivations

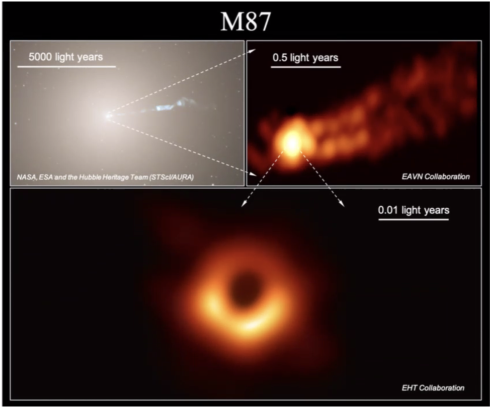
Black hole as astrophysical sources

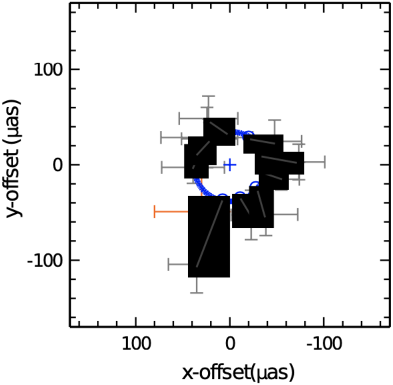
- Black hole are believed to play a key role in very high energy astrophysical phenomena
- Black holes accrete and eject matter, giving rise to a complicated spectrum in all frequency bands not yet understood
- Rotating black holes stand as the most important reservoir of extractible energy: natural accelerator of particles
- Compelling evidence that very high energy phenomena are triggered in the surrounding region
  $\rightarrow$ Relativistic jets of plasma from active galactic nuclei (AGN) and X-ray binaries systems,
  $\rightarrow$ γ-ray bursts

A new era: many on-going observations

- Various telescopes already allowed observations of relativistic jets from SMBH
- GRAVITY:
  $\rightarrow$ follow orbits of stars around Sagittarius A^*
  $\rightarrow$ detect non-thermal flare orbiting around Sagittarius A^* + polarization
  [GRAVITY collab '18]
- EHT / Black Hole Explorer (BHEX):
  $\rightarrow$ black hole image of the vicinity of the M87 and S (disk) [EHT collab '19 '22]
  $\rightarrow$ polarization data on the magnetosphere [EHT '21]
  $\rightarrow$ first photon subring (2032 ?)
- LIGO-Virgo-KAGRA: gravitational waves from binary black hole mergers (... LISA, ET)
  [LKV collab '15]

M87 jets and Sagittarius flare





Credit Gravity Collab

State-of-the-Art

Open questions

- Low quiescent luminosity
- Understanding the variability of the emission
- Nature of the non-thermal flares
- Mechanism for sustaining collimated jets (Pyonting flux, particles ejection)
- Strong evidence: Magnetic fields carried by the rotating black hole play a major role

Main points

- Rotating black hole are a powerful reservoir of accessible energy
- Understand how to extract their energy / accelerate matter around them
- Many energy extraction/acceleration mechanisms:
  $\rightarrow$ Penrose process
  $\rightarrow$ Blandford-Znajek (BZ) mechanism
  $\rightarrow$ Magnetic reconnection
- Force-free assumption for black hole magnetosphere and its limits :
  $\rightarrow$ Plasma intermediate entity: only sustained frozen magnetic lines
  $\rightarrow$ Force-free assumptions not valid at all times/everywhere (non-ideal electric field)
  $\rightarrow$ BZ extracts energy / Does not accelerate particles
- Dynamics of charged particles around magnetized rotating black hole harder to describe
  $\rightarrow$ need GR/PIC simulations assuming FFE-like plasma / or plasma injection
  $\rightarrow$ very limited analytic control so far

Two targets

- Find models for the magnetosphere of the Kerr black hole
- Study how charged particles move in this magnetosphere: solve electrogeodesics motion

Goals

Focus of the review talk

- Review the main points of extraction of energy from rotating black holes
- Review the use of symmetries to build minimal models of magnetospheres on Kerr
- Review the conditions to separate and solve the electrogeodesics in a magnetized Kerr
- New surprising result for black hole magnetosphere physics
  [**J. BA**, I. El Mellah, E. Gourgoulhon, to appear]

Rotating black hole as a reservoir of accessible energy

Generalities on the Kerr black hole

- The Kerr black hole is the unique stationary and axi-symmetric exact solution of GR which is asymptotically flat and contains an event black hole horizon –> No hair theorem
- Parameterized by the mass M and the spin $a = J/M$
- Kerr metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4aMr\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\,\mathrm{d}\varphi^2 \tag{1}$$

where the functions Σ and Δ read

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-) \tag{2}$$

- $r_\pm$ are the positions of the outer and inner black hole horizons

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{3}$$

- Area of the hole:

$$A_+ = 8\pi M^2\left(1 + \sqrt{1 - \frac{a^2}{M^2}}\right) \tag{4}$$

- The spin satisfies the condition

$$0 \leqslant a \leqslant M \tag{5}$$

- For $a = M$ (or $J = M^2$), one obtains the extremal Kerr black hole where $r_\pm = M$
- What is the maximal available energy one can extract from a Kerr black hole ?

Accessible rotational energy

- At a given mass, it is standard to consider:
  –> extremal Kerr black hole as the maximal energy state : $J = M^2$
  –> Schwarzschild black hole as the minimal energy state : $J = 0$
- Gain and loss of energy are constrained by the area theorem
- Area theorem: after any physical process from a initial state A_i to a final state A_f

$$A_i < A_f \tag{6}$$

- Recall the area of the hole:

$$A_+ = 8\pi M^2 \left(1 + \sqrt{1 - \frac{J^2}{M^4}}\right) \tag{7}$$

- M_i the initial mass of the extreme Kerr black hole: $M_i = M^2$ and $A_i = 8\pi M_i^2$
- M_f the final mass after all the rotational energy has been extracted: so $A_+ = 16\pi M_f^2$
- Then the area law teach us that

$$M_i \leqslant \sqrt{2} M_f \tag{8}$$

- The maximum efficiency of the process of energy extraction is

$$\epsilon = \frac{M_i - M_f}{M_i} = 1 - \frac{M_f}{M_i} = 1 - \frac{1}{\sqrt{2}} \simeq 0.29 \tag{9}$$

- The maximum rotational energy that can be extracted from a Kerr black hole is $29\% M_i$
- Black hole represent a huge reservoir of available energy ... applied to $M_i \sim 10^6 M_\odot$
- How do we extract this rotational energy ?

Different approaches

Many different mechanisms to extract the energy of black holes

- Original Penrose mechanism: minimal process to extract rotational energy [Penrose '69]
- Magnetized Penrose process: the Blandford-Znajek mechanism [Blandford-Znajek '77]
- Collisional Penrose mechanism:
  –> BSW effect for extremal spinning BH [Banados, Silk, West '09]
- Extraction of energy from the motion of boosted black hole [Penna '15]
- Magnetic field reconnections and negative energy plasmoid [Comisso, Asenjo '21]

Structure of the Kerr geometry

- Kerr metric

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right) \sin^2\theta \, d\varphi^2 \quad (10)$$

where the functions Σ and Δ read

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta \qquad \Delta(r) = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-) \quad (11)$$

- Horizons: $r_\pm$ are the positions of the outer and inner black hole horizons at which $\Delta = 0$

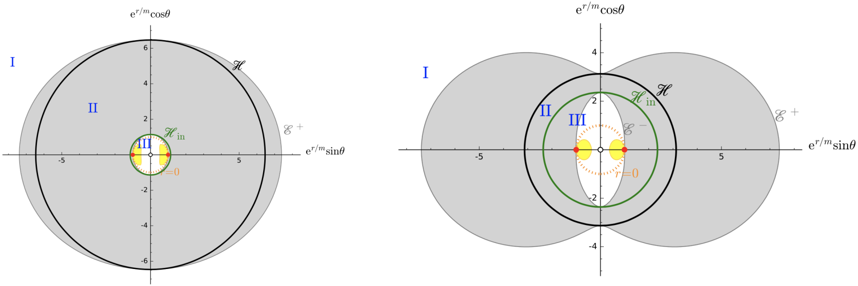
$$r_\pm = M \pm \sqrt{M^2 - a^2} \quad (12)$$

- Ergosphere: locus at which the timelike Killing vectors becomes null $\xi^\alpha\xi_\alpha = g_{tt} = 0$

$$r_e = M \pm \sqrt{M^2 - a^2\cos^2\theta} \quad (13)$$

- General structure

$$r_- < r_+ \leqslant r_e \quad (14)$$



- Structure of the Kerr geometry: Left $a = 0.5$ / Right $a = 0.9$
  [Credit E. Gourgoulhon's lectures]

Penrose process in short : Preliminaries

General properties of geodesic motion on Kerr

- Consider a test neutral particle of mass m with 4-velocity $u^\mu \partial_\mu$: timelike $u^\mu u_\mu = -1$
- Geodesic equation for free fall: non-linear coupled second order differential equations

$$a^\mu = u^\alpha \nabla_\alpha u^\mu = 0 \qquad \text{with} \qquad u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{15}$$

- The Carter's miracle: the geodesic motion is separable / integrable !
- Symmetry point of view: we have enough constant of motion to integrate the system

Hamiltonian point of view

- Hamiltonian for geodesic motion

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu \qquad \mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = -m^2 \qquad p_\mu = m u_\mu \tag{16}$$

- Kerr geometry is stationary and axisymmetric: two Killing vectors $\xi^\mu \partial_\mu$ and $\chi^\mu \partial_\mu$

$$\nabla_{(\mu} \xi_{\nu)} = 0 \qquad \nabla_{(\mu} \chi_{\nu)} \tag{17}$$

- Conserved energy and angular momentum:

$$E = -\xi^\mu p_\mu = -p_t \qquad L = \chi^\mu p_\mu = p_\varphi \tag{18}$$

- Two more conserved charges: the mass ... and the Carter's constant

$$m^2 = -\frac{1}{2} g^{\mu\nu} p_\mu p_\nu \qquad \mathcal{K} = K^{\mu\nu} p_\mu p_\nu \tag{19}$$

with $K^{\mu\nu}$ a Killing tensor

$$\nabla_{(\rho} K_{\mu\nu)} = 0 \tag{20}$$

Geodesic equations on Kerr

- Separable system in term of Mino time $\mathrm{d}\tau = \mathrm{d}\lambda/\Sigma$ with $\Sigma = r^2 + a^2\cos^2\theta$

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\Delta(r)}\left[(r^2+a^2)^2E - 2MarL\right] - E\sin^2\theta \tag{21}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \epsilon_r\sqrt{R(r)} \tag{22}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \epsilon_\theta\sqrt{\Theta(\theta)} \tag{23}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{L}{\sin^2\theta} + \frac{a}{\Delta(r)}(2MrE - aL) \tag{24}$$

with the radial and polar potentials given by

$$R(r) = \left[(r^2+a^2)E - aL\right]^2 - \Delta(r^2m^2 + \mathcal{K}) \tag{25}$$

$$\Theta(\theta) = \mathcal{K} - \left(\frac{L}{\sin\theta} - aE\sin\theta\right)^2 - m^2a^2\cos^2\theta \tag{26}$$

- A geodesic is characterized by the the three numbers $(E, L, \mathcal{K})$
- The radial and polar positions of equilibriums from the potentials $R(r)$ and $\Theta(\theta)$
- Miracle: each equation is separable !
- Can be fully integrated analytically w.r.t Mino time
- Many concrete applications: ISCO position (relevant for disc models) / photon ring science

Penrose process in short : Preliminaries

General conditions on timelike geodesic in Kerr

- The energy and the angular momentum satisfy several general inequalities in Kerr
- Descend from the fact that the 4-velocity $u^\mu \partial_\mu$ is necessarily timelike
- For any timelike Killing vector v of Kerr, one has

$$g(v, u) < 0 \tag{27}$$

Positivity of energy outside the ergoregion

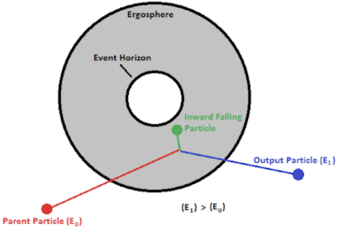
- Consider the Killing vector $\xi^\mu \partial_\mu = \partial_t$
  $\rightarrow$ timelike outside the ergoregion : $r > r_e = M + \sqrt{M^2 - a^2 \cos^2 \theta}$
  $\rightarrow$ for $r > r_e$

$$g(\xi, u) = u_t = \frac{p_t}{m} = -\frac{E}{m} < 0 \qquad \rightarrow \qquad E > 0 \tag{28}$$

Existence of negative energy state inside the ergoregion

- Consider the Killing vector $\xi^\mu \partial_\mu = \partial_t$
  $\rightarrow$ spacelike inside the ergoregion : $r_+ < r < r_e = M + \sqrt{M^2 - a^2 \cos^2 \theta}$
  $\rightarrow$ for $r_+ < r < r_e$, E can be positive or negative !
- Crucial novelty coming from the ergoregion: at the heart of Penrose mechanism

Penrose mechanism in short: Concretely ...



- Consider an ingoing particle P_0 with energy E_0 at infinity : $E_0 > 0$ and momenta p_0
- It enters in the ergosphere and splits into:
  $\rightarrow$ one particle P_1 escaping to infinity with momenta p_1
  $\rightarrow$ one particle P_2 falling in the hole with momenta p_2
- The conservation of the momenta at the ergosphere is

$$p_0 = p_1 + p_2 \tag{29}$$

- Gain of energy

$$\Delta E = E_1 - E_0 = (-\xi.p_1) - (-\xi.p_0) = \xi.(p_0 - p_1) = \xi.p_2 = -E_2 \tag{30}$$

- If P_2 accesses a negative energy state, i.e. $E_2 < 0$, then

$$\Delta E = -E_2 > 0 \tag{31}$$

- Extraction of energy allowed by the existence of negative energy state inside the ergosphere
- Question: What efficiency ?

Efficiency and limits of the Penrose process (PP)

Efficiency

- Studied in detail by Wald [Wald '74]

$$\epsilon_{PP} = \frac{E_1 - E_0}{E_0} = \frac{M}{2a}\left[\sqrt{2\left(1-\sqrt{1-\frac{a^2}{M^2}}\right)} - \frac{a}{M}\right] \tag{32}$$

- Maximal for maximally spinning BH : $a = M$ gives $\epsilon \sim 21\%$
- Inefficient for moderate spinning BH: $a = 0.5M$ gives $\epsilon < 2\%$

Other limitations

- Accessing to a negative energy state depends on (i) $(E, L, \mathcal{K})$, (ii) position w.r.t the horizon, (iii) direction of the momenta $\vec{p}$
- Very tiny region of the parameter space allows for the right conditions to extract energy
- Relative speed between the 2 fragements has to be $v \geqslant c/2$
- Not relevant for astrophysical models as it stands

Take away messages

- The rotational energy of black hole can be extracted by the simple Penrose process
- Access the negative energy states allowed by the ergosphere
- Astrophysically not relevant because highly inefficient

- Need to turn on electromagnetism !

From magnetized Penrose process to Blandford-Znajek mechanism

Historical account

Penrose process : 1969

- Penrose mechanism: minimal process to extract rotational energy [Penrose '69]
- Not efficient in vacuum ... but can be improve in electrovacuum

Magnetized collisional Penrose process : 1975

- Charged particles colliding in the ergosphere of a magnetized Kerr black hole [Ruffini, Wilson '75]
- Interaction with Maxwell field allows to overcome the previous limitations : Efficiency can reach 100%

Blandford-Znajek mechanism : 1977

- Convert mechanical spinning energy of the Kerr black hole into electromagnetic energy
- Existence of toroidal configuration allows to extract the energy via Poynting flux
- Based on slowly rotating configuration [Blandford, Znajek '77]
- Confirmed by modern GMHD simulations

Blandford-Znajek mechanism

Maxwell equations and Poynting flux in short

Basic equations

- The Maxwell equations read

$$\nabla_{[\mu}F_{\alpha\beta]} = 0 \qquad \nabla^{\mu}F_{\mu\nu} = j_{\nu} \qquad \nabla_{\mu}j^{\mu} = 0 \tag{33}$$

- Consider an observer with 4-velocity $u^{\mu}\partial_{\mu}$
- Electric current It can be decomposed w.r.t to the u-congruence as

$$j_{\mu} = \rho\, u_{\mu} + \mathcal{J}_{\mu} \qquad \mathcal{J}_{\mu}u^{\mu} = 0 \tag{34}$$

- Electric and magnetic fields w.r.t that observer

$$E_{\mu} = F_{\mu\nu}u^{\nu} \qquad B_{\mu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\nu\alpha}u^{\beta} \tag{35}$$

Energy of the Maxwell field

- Energy-momentum tensor

$$T_{\mu\nu} = -F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \qquad \nabla_{\mu}T^{\mu\nu} = -F^{\nu}{}_{\mu}j^{\mu} \qquad T^{\mu}{}_{\mu} = 0 \tag{36}$$

- Can be decomposed into a non-perfect radiation fluid

$$T_{\mu\nu} = \frac{1}{2}(E^2 + B^2)u_{\mu}u_{\nu} + \frac{1}{6}(E^2 + B^2)h_{\mu\nu} + 2Q_{(\mu}u_{\nu)} + \pi_{\mu\nu} \tag{37}$$

with

$$P = \frac{\rho}{3} = \frac{1}{6}(E^2 + B^2) \qquad Q_{\mu} = \epsilon_{\mu\nu\rho\sigma}E^{\nu}B^{\rho}u^{\sigma} \qquad \pi_{\mu\nu} = 2Ph_{\mu\nu} - E_{\mu}E_{\nu} - B_{\mu}B_{\nu} \tag{38}$$

Energy flux from toroidal electric and magnetic fields

- Consider a geometry g with timelike Killing vector $\xi^\mu \partial_\mu = \partial_t$
- Consider a sphere S of radius r with normal $s^\mu \partial_\mu = \partial_r$ and volume element $\mathrm{d}\Omega$
- For a given Maxwell solution, the flux of energy is given by

$$P = \lim_{r\to\infty} \oint_S \mathrm{d}\Omega \, T_{\mu\nu}\xi^\mu s^\nu = \lim_{r\to\infty} \oint_S \mathrm{d}\Omega \, T_{tr} \tag{39}$$

- Consider a stationary and axisymmetric spacetime:

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\varphi\varphi}\mathrm{d}\varphi^2 \tag{40}$$

one has

$$T_{tr} = -g^{\phi t} F_{t\phi} F_{tr} - g^{\theta\theta} F_{t\theta} F_{\theta r} \tag{41}$$

- Need toroidal electric or magnetic fields to have non-zero flux of energy, i.e.

$$E_\varphi \neq 0 \qquad \text{or} \qquad B_\varphi \neq 0 \tag{42}$$

Let see some concrete examples

Concrete examples

Non-spinning Michel solution:

- Consider flat spacetime $\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$
- Vacuum solution valid for flat (and Schwarzschild) spacetime: [Michel '73]

$$F^{\mathrm{M}} = \mathcal{G}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi \tag{43}$$

- Magnetic monopole: no-toroidal component so no flux

$$B^\mu\partial_\mu = \frac{2\mathcal{G}}{r^2}\partial_r \qquad E^\mu\partial_\mu = 0 \qquad \rightarrow \qquad T_{tr} = 0 \tag{44}$$

Rotating Michel solution

- Let the monopole slowly rotates: non-vacuum solution

$$F = \mathcal{G}\sin\theta\,\mathrm{d}\theta\wedge(\mathrm{d}\varphi - \Omega\mathrm{d}u) \qquad j^\mu\partial_\mu = -\frac{2\mathcal{G}\Omega\cos\theta}{r^2}\partial_r \tag{45}$$

- Electric and magnetic fields for $u^\mu\partial_\mu = \partial_t$

$$E_\theta = B_\varphi = -\frac{\mathcal{G}\Omega\sin\theta}{r} \qquad B_r = \frac{\mathcal{G}}{r^2} \tag{46}$$

- Non-vanishing Poynting flux:

$$\boxed{T_{tr} \propto \mathcal{G}^2\Omega^2 \qquad \rightarrow \qquad P^M = \frac{8\pi}{3}\mathcal{G}^2\Omega^2} \tag{47}$$

- Rotation allows for a toroidal magnetic field which triggers the Poynting flux
- Plasma is inertial: force-free

$$F_{\mu\nu}j^\nu = 0 \qquad \rightarrow \qquad \nabla_\mu T^{\mu\nu} = 0 \tag{48}$$

Blandford-Znajek monoppole and its flux

The force-free assumption

- BZ mechanism is based on a rigid magnetosphere rotating with the black hole
- The toroidal magnetic contribution induces a Poynting flux
- Stable rigid magnetosphere requires a dilute plasma not perturbing too much the field lines

$$B^2 \gg \rho_e c^2 \tag{49}$$

- The plasma is completely inertial and its energy negligible compared to EM

$$\nabla_\mu T^{\mu\nu} = -F_{\mu\nu} j^\nu = 0 \qquad \text{Force-Free Condition} \tag{50}$$

- Two consequences:

$$E_\mu B^\mu = 0 \qquad E^2 < B^2 \tag{51}$$

- Believed to be a good approximation of the plasma near black holes : high magnetizaton

BZ force-free magnetosphere

- Approximate solution of the force-free Maxwell equations in a slowly rotating Kerr black hole

$$F = \mathcal{G} \sin\theta \, \mathrm{d}\theta \wedge (\mathrm{d}\varphi - \Omega \mathrm{d}u) + \mathcal{O}(a^2) \qquad j^\mu \partial_\mu = -\frac{2\mathcal{G}\Omega\cos\theta}{r^2}\partial_r + \mathcal{O}(a^2) \tag{52}$$

- Regularity on the horizon : $\Omega = a/2$ at linear order (only difference with Michel !)
- Historical solution was presented at second order in spin a
- Many works to understand this solution beyond the perturbative scheme
- Poynting flux at fourth order in spin a:

$$P^{BZ} = \frac{\pi}{24}\frac{\mathcal{G}^2 a^2}{M^2} + \frac{(56 - 3\pi^2)\pi}{1080} a^4 + \mathcal{O}(a^5) \tag{53}$$

Open questions

Limitations

- Based on a force free solution on Kerr : not valid everywhere and all timescale
- BZ not tell you about the geometrical structure of the jet
- BZ cannot account for particles acceleration
- Need to break force-free assumptions (non-ideal electric field) to accelerate: need GR/PIC simulations
  [Parfey, Philippov, Cerutti '18]
- Only recently, analytical progress in describing the underlying charged particles motion
  [Mizumo, Gralla, Philippov '25]

To progress

- Study the trajectories of charged particles in the Kerr magnetosphere models
- Need to work out model for the Kerr magnetosphere

The Kerr magnetosphere : Building exact solutions from symmetries

Maxwell solution from Killing vectors

Wald construction [Wald '74]

- Consider a Killing vector $k^\mu \partial_\mu$ defined by

$$\nabla_{(\mu} k_{\nu)} = 0 \tag{54}$$

- It satisfies

$$\Box k_\mu = R_{\mu\nu} k^\nu \qquad \text{and} \qquad \nabla_\mu k^\mu = 0 \tag{55}$$

- Vacuum Maxwell equation in terms of the potential $A_\mu \mathrm{d}x^\mu$, one finds

$$\Box A_\mu = R_{\mu\nu} A^\nu \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{56}$$

- In the Lorentz gauge where $\nabla_\mu A^\mu = 0$, any Killing vector of a vacuum solution of GR provides an exact analytic solution for the vacuum Maxwell equation
- The Kerr metric being stationary and axisymmetric, it possesses two Killing vectors

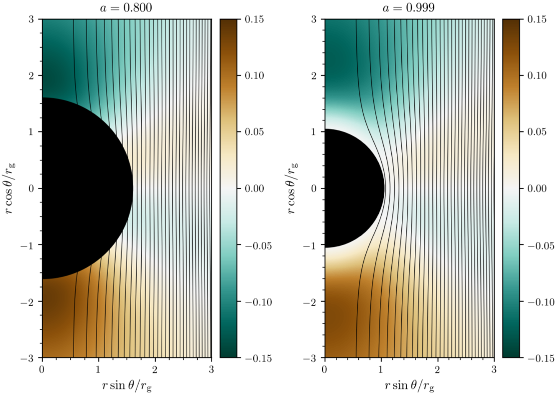
$$\xi^\mu \partial_\mu = \partial_t \qquad \chi^\mu \partial_\mu = \partial_\varphi \tag{57}$$

- Two parameters family of vacuum Maxwell solutions of the form

$$A_\mu^{(\alpha,\beta)} \mathrm{d}x^\mu = \alpha \xi_\mu \mathrm{d}x^\mu + \beta \chi_\mu \mathrm{d}x^\mu \tag{58}$$

- Two branches:
  $\rightarrow$ $\beta = 0$ gives asymptotically flat electrically charged solution
  $\rightarrow$ $\alpha = 0$ gives an asymptotically uniform electrically charged solution
- Wald solution: asymptotically uniform BUT electrically neutral if $\alpha = -2a\beta$
- Most studied solution so far ! Symmetries of the metric naturally carry the Maxwell solution

Magnetic field lines for Wald solution



[Credit: B. Crinquand thesis ’21]

Maxwell solution from Killing-Yano tensors

Killing-Yano tensors

- A geometry can exhibit generlized Killing objects : Killing-Yano tensor

$$\text{Killing vector} \qquad \nabla_{(\mu}\xi_{\nu)} = 0 \tag{59}$$

$$\text{Killing-Yano tensor} \qquad \nabla_{(\mu}Y_{\nu)\alpha} = 0 \qquad Y_{\mu\nu} = -Y_{\nu\mu} \tag{60}$$

- Consider the Penrose current [Penrose '82]

$$\boxed{F_{\mu\nu} = \mathcal{G}\, R_{\mu\nu\alpha\beta}Y^{\alpha\beta}} \tag{61}$$

with $\mathcal{G}$ a constant

- Use Bianchi identities and Killing-Yano properties

$$\nabla_\mu R^\mu{}_{\nu\rho\sigma} = 0 \qquad R_{\mu[\nu\rho\sigma]} = 0 \qquad \nabla_\mu Y_{\rho\sigma} = \nabla_{[\mu}Y_{\rho\sigma]} \tag{62}$$

- Provide an exact vacuum Maxwell solution

$$\nabla_\mu F^{\mu\nu} = 0 \qquad \nabla_\mu(*F)^{\mu\nu} = 0 \tag{63}$$

Let us apply this to Kerr !

The Kerr monopole

- The Kerr geometry enjoys such hidden symmetry:

$$Y = a\cos\theta\ \left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\right)\wedge\mathrm{d}r - r\sin\theta\mathrm{d}\theta\wedge\left(-a\mathrm{d}t + (r^2+a^2)\mathrm{d}\varphi\right) \tag{64}$$

- Penrose current gives

$$\begin{aligned} F^{\text{KY}} &= -\frac{8a\mathcal{G}r\cos\theta}{\Sigma^2}\mathrm{d}t\wedge\mathrm{d}r + \frac{4a\mathcal{G}(r^2-a^2\cos^2\theta)\sin\theta}{\Sigma^2}\mathrm{d}t\wedge\mathrm{d}\theta \\ &- \frac{8a^2\mathcal{G}r\sin^2\theta\cos\theta}{\Sigma^2}\mathrm{d}r\wedge\mathrm{d}\varphi - \frac{4\mathcal{G}(r^2+a^2)(r^2-a^2\cos^2\theta)\sin\theta}{\Sigma^2}\mathrm{d}\theta\wedge\mathrm{d}\varphi \end{aligned} \tag{65}$$

- Gauge-potential:

$$A^{\text{KY}} = \frac{4\mathcal{G}\cos\theta}{\Sigma}\left(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi\right) \tag{66}$$

A generalized spinning Michel monopole on Kerr

- Recovers the non-spinning Michel solution for $a=0$

$$\tilde{F}^{\text{KY}} = -2\mathcal{G}\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi = F^{\text{M}} \tag{67}$$

- Different from the spinning Michel solution in the slow rotating regime

$$F^{\text{KY}} \sim -2\mathcal{G}\sin\theta\mathrm{d}\theta\wedge\left(\mathrm{d}\varphi - \frac{a}{r^2}\mathrm{d}t\right) + \frac{4a\mathcal{G}\cos\theta}{r^3}\mathrm{d}t\wedge\mathrm{d}r + \mathcal{O}(a^2) \tag{68}$$

- Can be recovered by taking the dual of the solution: $F_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]}$ where $\xi^\mu\partial_\mu = \partial_t$
- What are its properties ?

A generalized spinning Michel monopole on Kerr

Electromagnetic configuration

- Introduce the Carter's observer located at some fix r and rotating with the BH: Carter tetrad

$$e^0 = \sqrt{\frac{\Delta}{\Sigma}}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\right) \tag{69}$$

$$e^1 = \sqrt{\frac{\Sigma}{\Delta}}\mathrm{d}r \tag{70}$$

$$e^2 = \sqrt{\Sigma}\mathrm{d}\theta \tag{71}$$

$$e^3 = \frac{\sin\theta}{\sqrt{\Sigma}}\left(-a\mathrm{d}t + (r^2+a^2)\mathrm{d}\varphi\right) \tag{72}$$

- Electric and magnetic fields w.r.t Carter observer are purely radial

$$\tilde{E}^{(\alpha,0)} = \frac{4a\mathcal{G}r\cos\theta}{\Sigma^2}e_1 \tag{73}$$

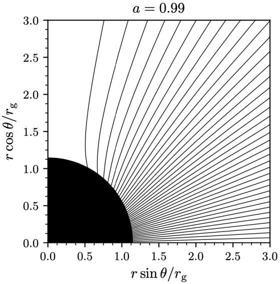
$$\tilde{B}^{(\alpha,0)} = -\frac{2\mathcal{G}(r^2 - a^2\cos^2\theta)}{\Sigma^2}e_1 \tag{74}$$

- Electric and magnetic charges:

$$Q_e = \frac{1}{4\pi}\oint_S \tilde{F}^{(1,0)} = 0 \qquad Q_m = \frac{1}{4\pi}\oint_S *\tilde{F}^{(1,0)} = \mathcal{G} \tag{75}$$

- Asymptotically flat magnetic monopole with purely radial electric and magnetic field lines for the Carter observer
- Exact Maxwell solution on the Kerr black hole for any spin a:
  $\rightarrow$ unique and selected by the Kerr symmetry !
- Serve as initial condition in ZELTRON simulations

Magnetic field lines for Kerr monopole solution



[Credit: B. Crinquand thesis '21]

Summary

Exact Maxwell solutions from symmetries

- Explicit and hidden Killing symmetries allow to build exact test Maxwell solutions on Kerr
- Wald construction: Killing vectors $\xi^\mu \partial_\mu \quad \to \quad F_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]}$
- New construction: Two Killing vectors $\xi^\mu \partial_\mu, \;\; \chi^\mu \partial_\mu \quad \to \quad F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\xi^\rho \chi^\sigma$
- Penrose construction: Killing-Yano tensor $F_{\mu\nu} = R_{\mu\nu\rho\sigma}Y^{\rho\sigma}$ (dual to a subsector of Wald)
- Carter construction: Killing-Yano tensor $F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}H^{\rho\sigma}$ with $H_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}Y^{\rho\sigma}$

Relevance of these solutions

- Exact solutions of magnetospheres on Kerr useful as initial conditions for simulation/perturbations
- Wald solution (asymptotically uniform) extensively studied numerically:
  $\to$ revealed new effects: Meissner effect, chaotic electrogeodesic motion, electric gaps
- Michel solution serves as a background for the Blandford-Znajek perturbative solution

$$F^{\mathrm{BZ}} = \text{Non-spinning Michel monopole} + \text{flux} \tag{76}$$

Open questions

- Geodesic motion (of neutral particles) is analytically solvable in the Kerr geometry [Carter '68]
- Under which condition can we analytically solve the motion of charged particle (electrogeodesic) in a magnetized Kerr black hole ?

Motion of charges particles in a magnetized Kerr black hole

Geodesic motion of neutral particles on Kerr

- Geodesic motion

$$u^\alpha \nabla_\alpha u^\mu = 0 \tag{77}$$

- Carter's miracle: geodesic motion is integrable $\rightarrow$ set of four first order separable equations

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\Delta(r)}\left[(r^2+a^2)^2 E - 2MarL\right] - E\sin^2\theta \tag{78}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{a}{\Delta(r)}(2MrE - aL) + \frac{L}{\sin^2\theta} \tag{79}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \epsilon_r\sqrt{R(r)} \tag{80}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \epsilon_\theta\sqrt{\Theta(\theta)} \tag{81}$$

- Equations for $\dot{r}$ and $\dot{\theta}$ are separable because of the existence of a hidden symmetry

$$\mathcal{K} = K^{\mu\nu} p_\mu p_\nu \tag{82}$$

First step for electrogeodesics

- Under which condition a magnetosphere preserves the Carter constant for electrogeodesic ?

Electrogeodesic phase space

- Consider a particle with mass μ and electric charge q with mass ratio: $\kappa = q/m$
- Electrogeodesic lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} u^\mu u^\nu + \kappa A_\mu u^\mu \qquad \text{with} \qquad u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \dot{x}^\mu \tag{83}$$

- Electrogeodesic equation

$$u^\alpha \nabla_\alpha u^\mu = \kappa F^\mu{}_\nu u^\nu \tag{84}$$

- Hamiltonian formulation: canonical momenta

$$P_\mu = \frac{\delta \mathcal{L}}{\delta \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu + \kappa A_\mu \tag{85}$$

- Canonical brackets are given

$$\{x^\mu, x^\nu\} = 0 \qquad \{x^\mu, P_\nu\} = \delta^\mu{}_\nu \qquad \{P_\mu, P_\nu\} = 0 \tag{86}$$

- The hamiltonian $\mathcal{H}$ takes the form

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} \left(P_\mu - \kappa A_\mu\right)\left(P_\nu - \kappa A_\nu\right) = -\frac{m^2}{2} \tag{87}$$

- Natural to introduce another set of canonical variables and in particular the dressed momenta

$$p_\mu = P_\mu - \kappa A_\mu = m u_\mu \tag{88}$$

- This dressed momenta satisfies the following brackets

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu \qquad \{p_\mu, p_\nu\} = \kappa F_{\mu\nu} \tag{89}$$

- In term of this momenta, the hamiltonian takes the usual form $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$

Condition to preserve the Carter-like constant

Geometrical condition on the magnetosphere

- Consider the Killing-Yano tensor and its associated Killing tensor in the Kerr geometry:

$$K_{\mu\nu} = Y_{\mu\alpha} Y^{\alpha}{}_{\nu} \qquad \nabla_{(\mu} Y_{\nu)\alpha} = \nabla_{(\mu} K_{\nu\alpha)} = 0 \tag{90}$$

- The Carter-like constant is preserved along the electrogeodesic flow if

$$\mathcal{K} = K^{\mu\nu} p_\mu p_\nu \qquad \to \qquad \{\mathcal{K}, \mathcal{H}\} = 0 \tag{91}$$

- Using the commutation relations, the condition reads

$$\{\mathcal{K}, \mathcal{H}\} = \left(F_{\mu}{}^{(\alpha} K^{\nu)\mu} \right) p_\alpha p_\nu = 0 \tag{92}$$

- Sufficient geometrical (and very elegant) condition [Visinescu '09]

$$\boxed{F_{\alpha}{}^{[\mu} Y^{\nu]\alpha} = 0} \tag{93}$$

- Provide a necessary but not sufficient conditions to analytically integrate the motion

Which magnetosphere satisfies this condition ?

- Wald asymptotically uniform : $F_{\mu\nu} = \nabla_{[\mu}(\xi_{\nu]} - 2a\chi_{\nu]})$ not integrable
- Killing-Maxwell-Carter system: $F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} Y^{\rho\sigma}$ satisfies the condition [Carter '87]
- Asymptotically flat Kerr monopole : $F_{\mu\nu} = R_{\mu\nu\rho\sigma} Y^{\rho\sigma}$ satisfies also the condition [J. **BA**, I. El Mellah, E. Gourgoulhon '25]
- Suggest that one can "hope" to analytically integrate electrogeodesic in the Kerr monopole

Separability of the $\dot{t}$ and $\dot{\varphi}$

- Consider the two constants of motion

$$\mathcal{K} = \mathcal{K}^{\mu\nu} p_\mu p_\nu \qquad -m^2 = g^{\mu\nu} p_\mu p_\nu \qquad p_\mu = P_\mu + \kappa A_\mu \tag{94}$$

- Challenge: new terms from the magnetosphere need to take a very special form

$$g^{\mu\nu} P_\mu P_\nu \qquad g^{\mu\nu} P_\mu A_\mu \qquad g^{\mu\nu} A_\mu A_\nu \tag{95}$$

- Can be shown that one can extract p_r and p_θ in a separable form provided that

$$\boxed{(r^2 + a^2) A_t + a A_\varphi = 0} \tag{96}$$

..... which happen to be true only for the Kerr monopole build from the KY tensor !

- Introducing the Mino time $\mathrm{d}\tau = \mathrm{d}\lambda / \Sigma(r(\lambda), \theta(\lambda))$, the $\dot{r}$ and $\dot{\theta}$ are gven by

$$\Sigma \frac{\mathrm{d}r}{\mathrm{d}\lambda} = \epsilon_r \sqrt{R(r)} \tag{97}$$

$$\Sigma \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \epsilon_\theta \sqrt{\Theta(\theta)} \tag{98}$$

with the radial and polar potentials given by

$$R(r) = \left[(r^2 + a^2) E_\kappa - a L_\kappa\right]^2 - \Delta(r)(r^2 m^2 + \mathcal{K}_\kappa) \tag{99}$$

$$\Theta(\theta) = \mathcal{K}_\kappa - m^2 a^2 \cos^2\theta - \left(\frac{L_\kappa - 2\kappa Q \cos\theta}{\sin\theta} - a E_\kappa \sin\theta \right)^2 \tag{100}$$

- The radial and polar electrogeodesic motion is integrable in the Kerr monopole !
  [J. **BA**, I. El Mellah, E. Gourgoulhon '25]

Next step: separability of the $\dot{t}$ and $\dot{\varphi}$

- Consider the Kerr monopole which has the form

$$A_\mu \mathrm{d}x^\mu = A_t(r,\theta)\mathrm{d}t + A_\varphi(r,\theta)\mathrm{d}\varphi \tag{101}$$

- Energy and angular momentum of the charged particle: $P_\mu = p_\mu + \kappa A_\mu$

$$-E_\kappa = \xi^\mu P_\mu = P_t = g_{tt}p^t + g_{t\varphi}p^\varphi + \kappa A_t \tag{102}$$

$$L_\kappa = \chi^\mu P_\mu = P_\phi = g_{\varphi\varphi}p^\varphi + g_{\varphi t}p^t + \kappa A_\phi \tag{103}$$

- Equations for $\dot{t}$ and $\dot{\varphi}$ obtained by inverting the above equations

$$\Sigma\frac{\mathrm{d}t}{\mathrm{d}\lambda} = -\Sigma\left[\frac{g_{t\varphi}L + g_{\varphi\varphi}E}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right] - \kappa\Sigma\left[\frac{g_{\varphi\varphi}A_t - g_{t\varphi}A_\varphi}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right] \tag{104}$$

$$\Sigma\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \Sigma\left[\frac{g_{tt}L + g_{t\varphi}E}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right] - \kappa\Sigma\left[\frac{g_{tt}A_\varphi - g_{t\varphi}A_t}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right] \tag{105}$$

- For $\kappa = 0$, we recover the geodesic equations (blue)
- For the Kerr monopole: new term only functions of θ ... again !

$$\Sigma\left[\frac{g_{\varphi\varphi}A_t - g_{t\varphi}A_\varphi}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right] = 4a\mathcal{Q}\cos\theta \qquad \Sigma\left[\frac{g_{tt}A_\varphi - g_{t\varphi}A_t}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right] = \frac{4\mathcal{Q}\cos\theta}{\sin^2\theta} \tag{106}$$

[J. **BA**, I. El Mellah, E. Gourgoulhon '25]

In conclusion ....

The separable electrogeodesic equations in the Kerr monopole

New results

- The radial and the polar equations $\dot{r}$ and $\dot{\theta}$ are separable: $F_\alpha{}^{[\mu} Y^{\nu]\alpha} = 0$
- It follows that the time and azimuthal equations $\dot{t}$ and $\dot{\varphi}$ are also separable !
- The electrogeodesics in the magnetized Kerr monopole is fully analytically (and miraculously) integrable
- Generalize the key result by Carter from 1968 for geodesic on Kerr

Separable electrogeodesic equations [J. **BA**, I. El Mellah, E. Gourgoulhon '25]

- Introducing the Mino time $\mathrm{d}\tau = \mathrm{d}\lambda/\Sigma(r(\lambda), \theta(\lambda))$, the electrogeodesic separable equations are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\Delta(r)}\left[(r^2 + a^2)^2 E_\kappa - 2MarL_\kappa\right] - E_\kappa \sin^2\theta + 2\kappa a Q\cos\theta \tag{107}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \epsilon_r \sqrt{R(r)} \tag{108}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \epsilon_\theta \sqrt{\Theta(\theta)} \tag{109}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{L_\kappa}{\sin^2\theta} + \frac{a}{\Delta(r)}(2MrE_\kappa - aL_\kappa) + \kappa\frac{4Q\cos\theta}{\sin^2\theta} \tag{110}$$

with the radial and polar potentials given by

$$R(r) = \left[(r^2 + a^2)E_\kappa - aL_\kappa\right]^2 - \Delta(r)(r^2 m^2 + \mathcal{K}_\kappa)$$

$$\Theta(\theta) = \mathcal{K}_\kappa - m^2 a^2 \cos^2\theta - \left(\frac{L_\kappa - 2\kappa Q\cos\theta}{\sin\theta} - aE_\kappa \sin\theta\right)^2$$

- Labelled by the constants $(E_\kappa, L_\kappa, \mathcal{K}_\kappa, m, \kappa Q)$

Radial and equilibrium positions in the Kerr monopole

- The radial equation did not change w.r.t the geodesic motion:
  $\rightarrow$ same (stable and unstable) equilibrium positions !
- Same position of the ISCO: application for the inner edge of the disc with charged particles

Acceleration of charged particles [J. **BA**, I. El Mellah, E. Gourgoulhon '25]

- Acceleration of charged particles in all spacetime in analytic form : $a^{\mu} = \kappa F^{\mu\nu} u_{\nu}$

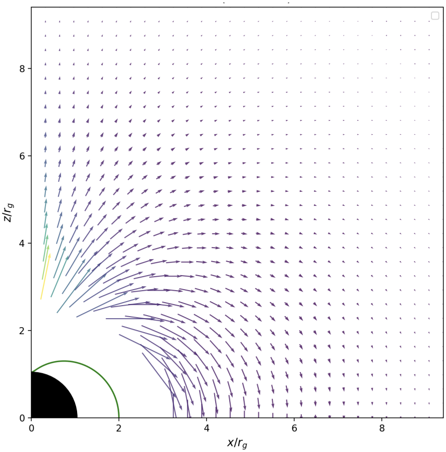
$$a^r = \frac{4\kappa a Q r\cos\theta}{\Sigma^4}\left[\left((r^2+a^2)^2 - 2a^2 M r\sin^2\theta\right)E_\kappa + a\left(a^2\sin^2\theta - 2Mr\right)L_\kappa \right.$$
$$\left. -\Delta\left[E_\kappa \sin^2\theta + a\left(L_\kappa + 2\kappa Q\cos\theta\right)\right]\right] \tag{111}$$

- At large distance

$$a^r \sim \frac{4\kappa a Q E_\kappa \cos\theta}{r^3} \tag{112}$$
$$a^\theta \sim \frac{4\kappa Q E_\kappa \sin\theta}{r^4} \tag{113}$$
$$a^\varphi \sim \frac{4\kappa Q}{r^3}\frac{\sqrt{\Theta(\theta)}}{\sin\theta} \tag{114}$$

- Radial acceleration maximal at the poles and triggered by rotation !
- Can we characterize the zones of escape? The geometrical form of the jet ?

Acceleration of charged particles in the Kerr magnetic monopole



Provide the minimal and simplest analytical model for jet launching by the Kerr black hole
Radial and azimuthal accelerations maximal at the poles: eject charged particles with a helicoidal motion
[J. **BA**, I. El Mellah, E. Gourgoulhon '25]

Conclusion

Conclusions and take away messages

General view

- Exact solutions for black hole magnetosphere are crucial to refine our understanding of energy extraction mechanisms and particle accelerations
- Complementary to numerical methods: benchmarks tests, initial conditions
- Reveal new subtle effects: extension of Penrose process beyond the ergosphere
- Exact magnetosphere models / electrogeodesics $\rightarrow$ intimate interlay with hidden symmetries

New results

- The motion of charged particles in the monopole magnetosphere of Kerr is fully integrable [J. **BA**, I. El Mellah, E. Gourgoulhon '25]
- Generalize the Carter's miracle known for Kerr geodesics
- Applications: analytic predictions of acceleration/ jet launching model/ modified Lense-Thirring effect/ equilibriums
- Can be countercheck by numerical simulations: ZELTRON
- General conditions on electrogeodesic: existence of negative state beyond the ergosphere

Open directions

- Characterize analytically the geometrical structure of the jets
- Perturbative treatment of the gravito-electric ringdown
- Exact versus perturbative Blandford-Znajek solution
- Study (analytically) plasmoid motion and magnetic reconnection using symmetries

Thank you

A new toroidal magnetic solution

Alternative construction with Killing vectors

- Consider the tensor

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\xi^\rho\chi^\sigma \tag{115}$$

- Explicitly, the Faraday tensor reads

$$F = \Sigma\,\mathrm{d}r \wedge \mathrm{d}\theta \tag{116}$$

- It is direct to show that this tensor satisfies the non-vacuum Maxwell equation

$$\nabla_\mu F^{\mu\nu} = J^\mu \qquad \nabla_\mu (*F)^{\mu\nu} = 0 \tag{117}$$

- Electric current is given by

$$J^\mu\partial_\mu = \frac{2}{\Sigma}\left[-\Delta\cos\theta\partial_r + (r-M)\sin\theta\partial_\theta\right] \tag{118}$$

- Provide a new toroidal magnetic solution w.r.t the Carter's observer

$$E = 0 \qquad B = \Delta\sin\theta\ e_3 \tag{119}$$

[J. **BA**, I. El Mellah, E. Gourgoulhon '25]