Gravitational waves from Numerical relativity

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Why Numerical Relativity?

- Gravitational waves (GW) provide direct access to the strong-field regime of General Relativity.
- Compact-object systems are ideal strong-gravity laboratories:
  - Binary black holes (BBH) → vacuum strong-field dynamics.
  - Neutron star binaries, or NS-BH → matter effects.
  - Other systems, exotic objects...
- However, Einstein's equations (1915) are fully nonlinear, coupled PDEs.
  - Exact solutions are very limited, e.g. Schwarzschild (1915), Kerr (1963).
  - Analytic approaches rely on approximations, e.g. post-Newtonian, perturbation theory, effective one body...
  - These methods do not work well around the merger.
- NR provides the only way to get accurate solutions in the most extreme regimes.

History of Numerical Relativity: mathematical foundations

- Choquet-Bruhat 1952 → Einstein's vacuum equations are well-posed:
  - Existence and uniqueness of solutions.
  - Stability under small perturbations of initial data.
  - Used harmonic coordinates. $\nabla_\mu \nabla^\mu x^\nu = 0$

  → Mathematical foundation for dynamical simulations.

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

- Arnowitt, Deser, Misner 1962 → 3+1 decomposition (ADM formalism).
  - Spacetime is divided into spatial slices evolving with time.
  - Converts GR into a system of evolution + constraint equations.

  → Practical formulation for numerical evolution.

- Hahn, Lindquist 1964 → First numerical evolution of a BBH system.
  - Simulation of head-on collision of two black holes.
  - Coarse resolution, short evolution time.

  → Proof of principle that BBH spacetimes can be simulated.

History of Numerical Relativity: initial data

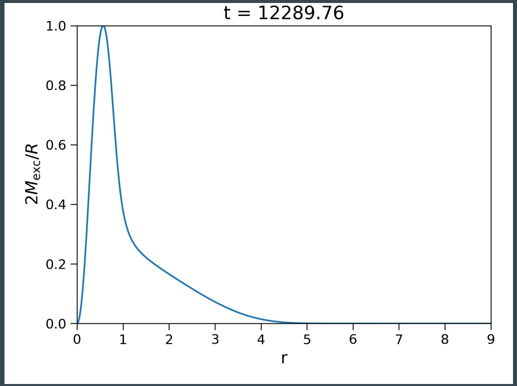
- Initial data must satisfy certain constraint equations on a spatial slice.
- Brill, Lindquist 1963 → Multiple non-spinning BHs at rest.
  - First dataset with multiple BHs.
- Bowen, York 1980 → Arbitrary BH linear momenta and spins.
  - Approach still used to date.
- Brandt, Brügmann 1997 → More convenient topology for BHs (puncture).
  - Foundation of modern moving-puncture BBH simulations.

History of Numerical Relativity: towards robust formulations

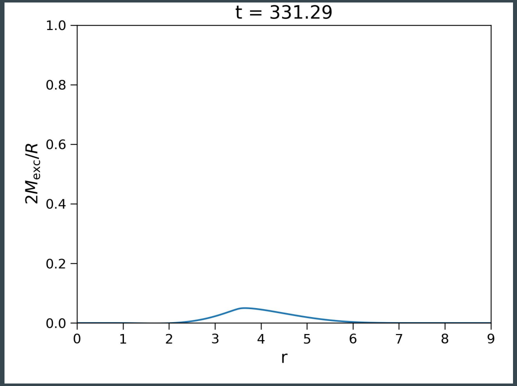
- Choptuik 1993 → critical collapse in spherical symmetry.
  - First large success for NR.
  - Discovery of universal scaling laws near BH threshold.
    → Demonstrated precision NR was possible in reduced symmetry.

- Nakamura, Oohara, Kojima 1987,
  Shibata, Nakamura 1995,
  Baumgarte, Shapiro 1998
  → BSSN (or BSSNOK) formalism.
  - Reformulation of ADM prescription.
  - Greatly improves numerical stability.
    → Enables long-term 3D evolutions of compact binaries.

Above threshold

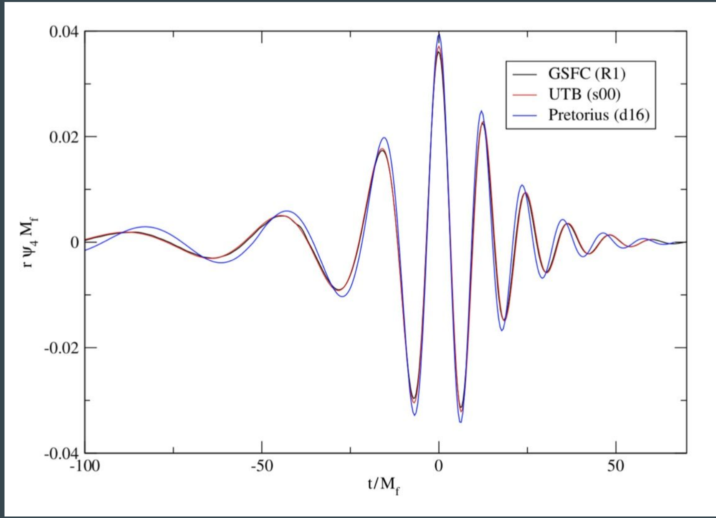


Below threshold



History of Numerical Relativity: accurate BBH simulations

- Pretorius 2005: first stable BBH merger simulation.
  - Fully 3D, inspiral-merger-ringdown phases.
  - Used a generalized harmonic formulation ($\nabla_\mu \nabla^\mu x^\nu = H^\nu$) + excision.
- Independent confirmations:
  - Campanelli et al. 2006 (Brownsville).
  - Baker et al. 2006 (NASA Goddard).
  - Used BSSN formulation + moving punctures.
- All approaches achieved stable mergers with excellent agreement.
- Major breakthrough:
  - NR techniques are accurate and robust.
  - Marks the start of the era of modern NR simulations.
  - LIGO GW detection in 2015 relies directly on this.

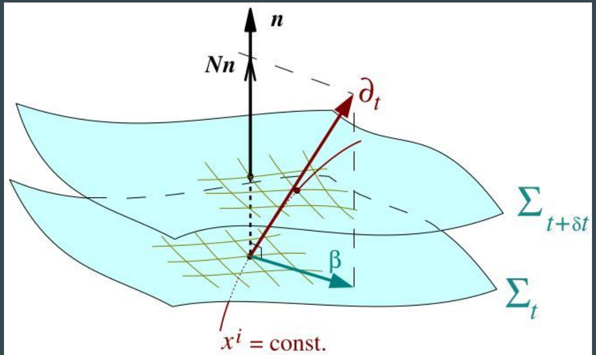


Baker et al. 2007

3+1 formalism: definitions

- The 4D spacetime is divided in a time succession of 3D spatial manifolds (slices) Σ_t: foliation.
  - Coordinates separate into time and spatial coordinates.
- The metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$



Gourgoulhon 2007

  - Lapse α (or N): controls rate of proper time flow between slices.
    - $\alpha \cdot dt$ is the proper time between Σ_t and $\Sigma_{t+\delta t}$.
  - Shift β^i: measures how x^i change from Σ_t to $\Sigma_{t+\delta t}$.
  - Spatial metric γ_{ij}: encodes intrinsic geometry of Σ_t.
  - α and β^i represent how we divide the spacetime, not how the spacetime changes → gauge freedom.
- Another relevant quantity: extrinsic curvature K_{ij}.
  - Measures how Σ_t is embedded in 4D spacetime.
  - Tracks how the shape of space changes with time.

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - D_i\beta_j - D_j\beta_i)$$

3+1 formalism: Einstein equations

- Einstein equations are then (shown in vacuum):
  - Two evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) - D_iD_j\alpha + \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k$$

  - Two constraint equations: Hamiltonian and momentum constraints.

$$\gamma^{ij}R_{ij} + K^2 - K_{ij}K^{ij} = 0$$

$$D_jK^j{}_i - D_iK = 0$$

- 3D Cauchy problem: γ_{ij} and K_{ij} are the dynamical fields.
  - The first two EEs give their time evolution.
  - The other two constraints must hold on every slice.
    - Mathematically preserved by the time evolution.

BSSN formalism

- The original ADM formalism is numerically unstable (weakly hyperbolic).
  - Violations of constraint equations grow rapidly.
- BSSN → splits variables to improve stability.
  - Separate trace of extrinsic curvature tensor,

$$A_{ij} = K_{ij} - \frac{1}{3}K\gamma_{ij}$$

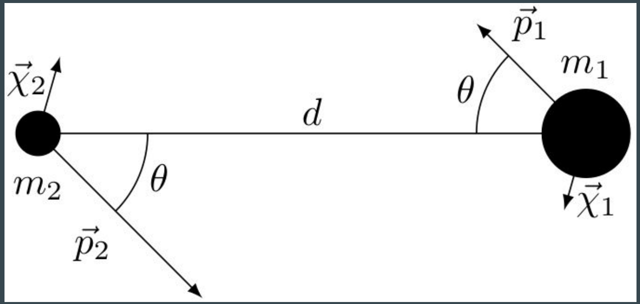
  - Separate spatial metric determinant through a conformal transformation,

$$\gamma_{ij} = \Psi^4\tilde{\gamma}_{ij} \quad , \quad A_{ij} = \Psi^4\tilde{A}_{ij} \quad , \quad \text{with} \quad \det(\tilde{\gamma}_{ij}) = 1$$

- Evolution equations rewritten for $\tilde{\gamma}_{ij}$, $\tilde{A}_{ij}$, Ψ and K.
  - Constraint equations are also rewritten.
- Improves numerical stability (strongly hyperbolic system).
  → Finally enabled long-term 3D BBH simulations.

Initial data: the problem

- To start the evolution, we need consistent initial conditions defined in our initial slice Σ_t:
  - Hamiltonian constraint (scalar).
  - Momentum constraint (vector, 3 components).
  - Nonlinear elliptic PDEs, not trivial to solve in 3D.
- Goal: generate initial 3D spatial metric γ_{ij} and extrinsic curvature K_{ij} representing a physically realistic system:
  - BH masses.
  - Linear momenta.
  - Spins.



Initial data: historical solutions

- Brill, Lindquist 1963: N static black holes.
  - Static solution: $K_{ij}=0$.
  - Conformally flat metric: $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$, with $\tilde{\gamma}_{ij}$ flat.
  - Momentum constraint trivially satisfied, Hamiltonian constraint reduces to a Laplace eq. $\Delta\Psi = 0$
  - Each BH generates a divergence at their center $\mathbf{r}_a$:

$$\Psi = 1 + \sum_{a=1}^{N} \frac{m_a}{2|\vec{r} - \vec{r}_a|}$$

- Bowen, York 1980: adding momenta and spins.
  - Analytic solutions $K_{ij}\neq 0$ for momentum constraints, corresponding to:
    - A BH with certain momentum P^i.
    - A BH with certain spin S^i.
  - Also in conformal flatness.
  - Hamiltonian constraint → Poisson-like equation for Ψ which must be solved numerically.

$$\Delta\Psi + \frac{1}{8}\Psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = 0$$

Initial data: historical solutions

- Brandt, Brügmann 1997: "puncture" method to handle singularities
  - Divides conformal factor into singular + smooth parts: $\Psi = b^{-1} + u$, $b^{-1} = \sum_{a=1}^{2} \frac{m_a}{2|\vec{r} - \vec{r}_a|}$
  - Poisson-like equation to be solved for u:

$$\Delta u + \frac{b^7}{8} \tilde{A}_{ij} \tilde{A}^{ij} (1 + bu)^{-7} = 0$$

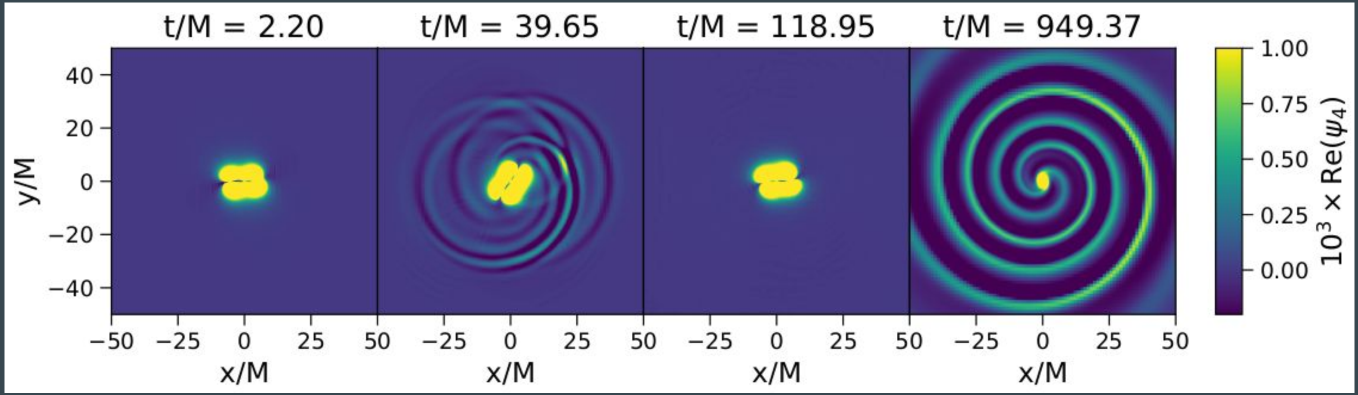
  - Stable numerical grids, with punctures initially fixed in space.
  - Initial approaches kept fixed punctures during dynamical evolution:
    - Strong gradients developed as the BHs got closer → numerical instabilities.
    - Coordinate system twisted as BH tried to orbit with fixed coordinate singularities.
  - Would need further developments for stable dynamical mergers.

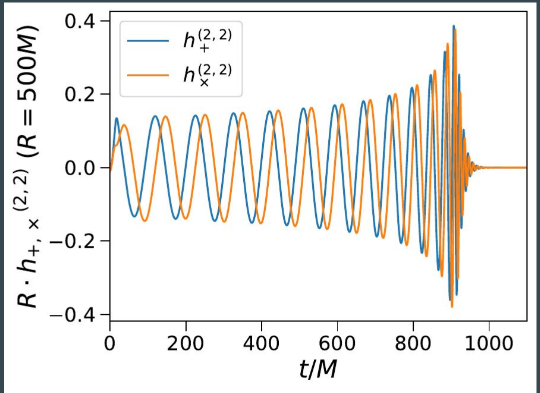
Dealing with singularities

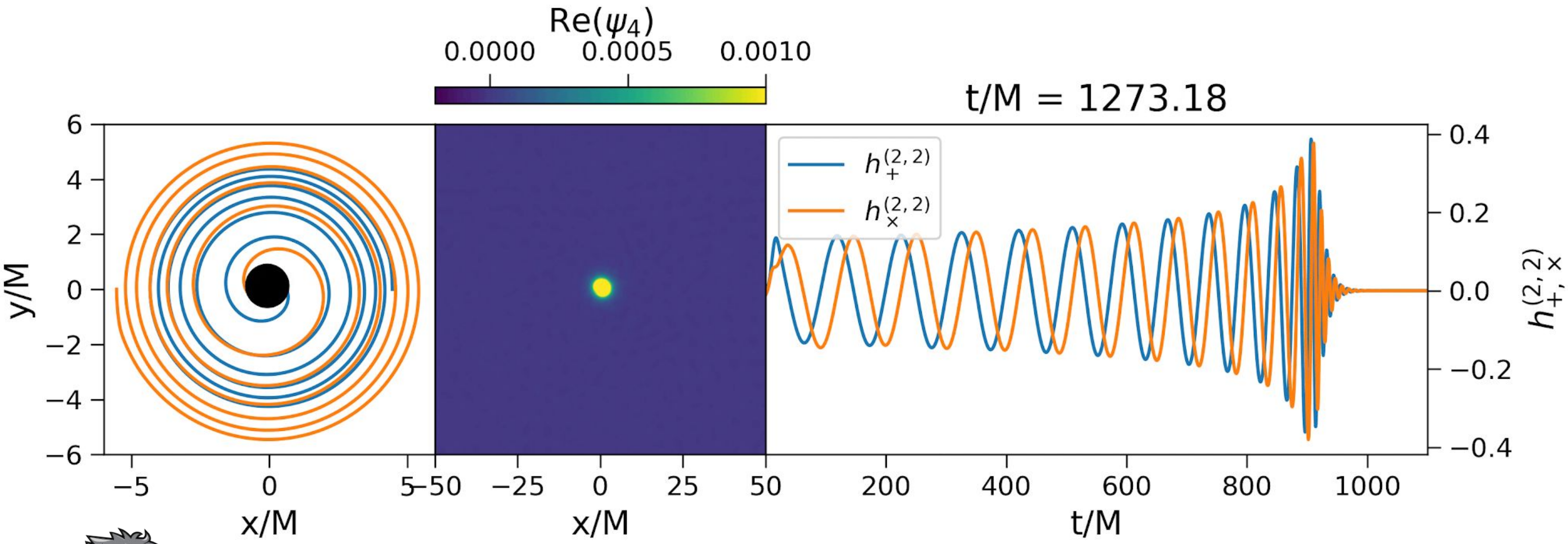
- Excision.
  - Main idea: BH singularities are hidden behind their horizons.
  - Their influence cannot be felt outside the BH → the BH interiors can safely be removed from the grid.
  - BH horizons must be known very precisely at each time → computationally expensive.
  - Used in Pretorius 2005, with a generalized harmonic formulation for the dynamical evolution, $\nabla_\mu \nabla^\mu x^\nu = H^\nu$
- Moving punctures.
  - Punctures on initial data move dynamically following the BHs.
  - Achieved by using suitable gauge conditions.
    - 1+log slicing: takes α→0 near punctures (K>>1), freezing time evolution. $\partial_t \alpha = -2\alpha K + \beta^j \partial_j \alpha$
    - Gamma-driver shift: spatial coordinates follow the BHs, prevents grid stretching. η damping parameter preventing oscillations. $\partial_t \beta^i = \frac{3}{4}\tilde{\Gamma}^i + \beta^j \partial_j \beta^i - \eta \beta^i$ $\tilde{\Gamma}^i = \partial_j \tilde{\gamma}^{ij}$
  - Used in Campanelli et al. 2006 (Brownsville) and Baker et al. 2006 (NASA Goddard), with BSSN.
  - Simple, robust, computationally efficient.
  - Became the standard for NR BBH simulations.

Initial data: junk radiation

- Initial data solve the constraints but are not exact inspiral solutions.
- Early evolution contains spurious gravitational waves → “junk radiation”.
- Amplitude decays as the system settles onto a consistent solution of full evolution equations.
- GWs are only reliable after junk radiation has left the considered domain.







Run with the Einstein Toolkit.

GW extraction

- Weyl tensor: trace-free part of Riemann tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}) + \frac{1}{6}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

- The Newman-Penrose formalism introduces a null tetrad l^μ, m^μ, $\bar{m}^\mu$, n^μ, such that

$$-l^\mu n_\mu = m^\mu \bar{m}_\mu = 1, \quad l^\mu m_\mu = l^\mu \bar{m}_\mu = n^\mu m_\mu = n^\mu \bar{m}_\mu = 0$$

  and defines the Weyl scalars $\Psi_0 = C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma, \Psi_1 = C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma, \Psi_2 = C_{\mu\nu\rho\sigma} l^\mu m^\nu \bar{m}^\rho n^\sigma$
  $\Psi_3 = C_{\mu\nu\rho\sigma} n^\mu l^\nu n^\rho \bar{m}^\sigma, \Psi_4 = C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma$

- Far away from the source, for a certain tetrad choice, $\Psi_4 \approx \ddot{h}_+ - i\ddot{h}_\times$
- GW can thus be recovered by integrating twice,

$$h_+(t,\vec{r}) - ih_\times(t,\vec{r}) = \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \Psi_4(t'',\vec{r})$$

GW extraction

$$h_+(t,\vec{r}) - i h_\times(t,\vec{r}) = \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \Psi_4(t'',\vec{r})$$

- In a simulation, t is finite. Also, early times are affected by junk radiation.
  - More extended approach: fixed-frequency integration, in Fourier space.
  - Involves setting a minimum frequency, getting rid of unphysical frequencies.
  - Usually done at post-processing: the simulation only outputs Ψ_4.
- We are usually not interested in the full 3D strain, only multipolar information.
  - Usual practice: extract multipoles of GW strain up to a certain max. l, at a certain (large-radius) sphere.

$$\Psi_4(t,\vec{r}) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \Psi_4^{(l,m)}(t,r)\ {}_{-2}Y_{lm}(\theta,\phi)$$

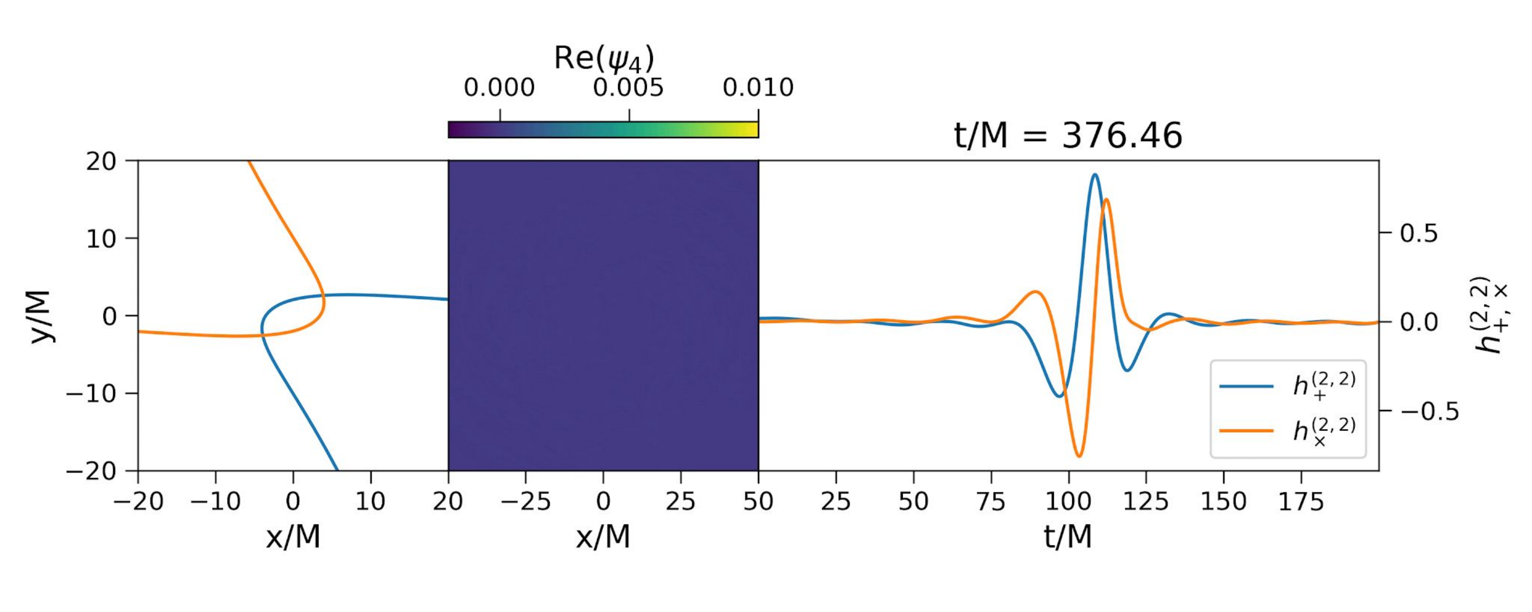
  - Once obtained, strain scales as 1/r.
    → with multipolar information, we can recover strain at a simulated detector at any distance, angle and orientation.
  - The simulation just needs to output several 1D Ψ_4 time series, reducing output size.

From NR to GW observations

- Since 2005, thousands of BBH simulations have been performed.
  - Creation of large public catalogs: SXS, RIT, BAM.
  - Cover wide ranges of mass ratios, momenta, spins and eccentricity.
- However, direct use of NR in GW data analysis is computationally impossible.
- Waveform models:
  - Surrogate models interpolate between NR waveforms.
    - Most reliable within their parameter space, but limited applicability.
  - Effective one body (EOB): combine PN and PM inspirals, motivated shapes for the merger, ringdown models from BH perturbation theory.
    - Need to be calibrated with NR.
  - Phenomenological inspiral-merger-ringdown: fit EOB, PN and NR to get a fast waveform generation.
- Fast waveform generation is essential for parameter estimation of observed GWs.
- All this enabled LIGO's first BBH detection in 2015, and subsequent LVK events.

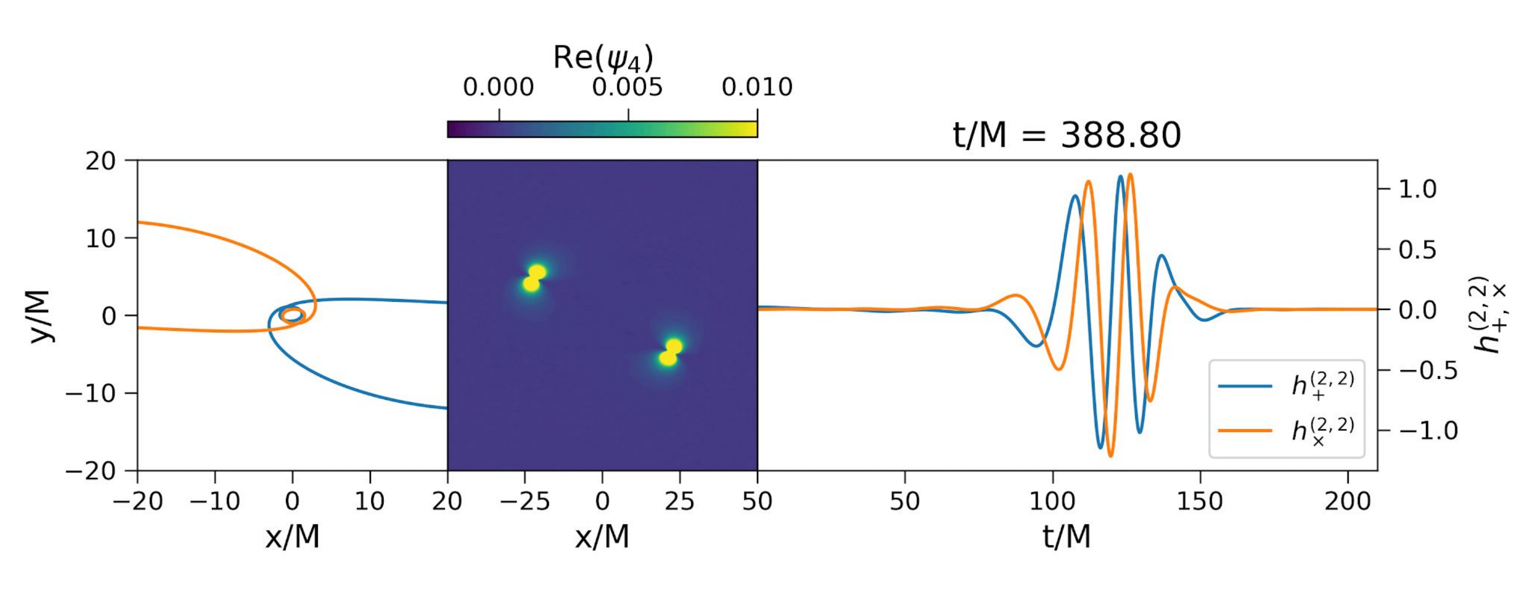
Other BBH interactions

- A pair of BHs can also get closer and leave after the interaction
  → hyperbolic encounters / scattering events.



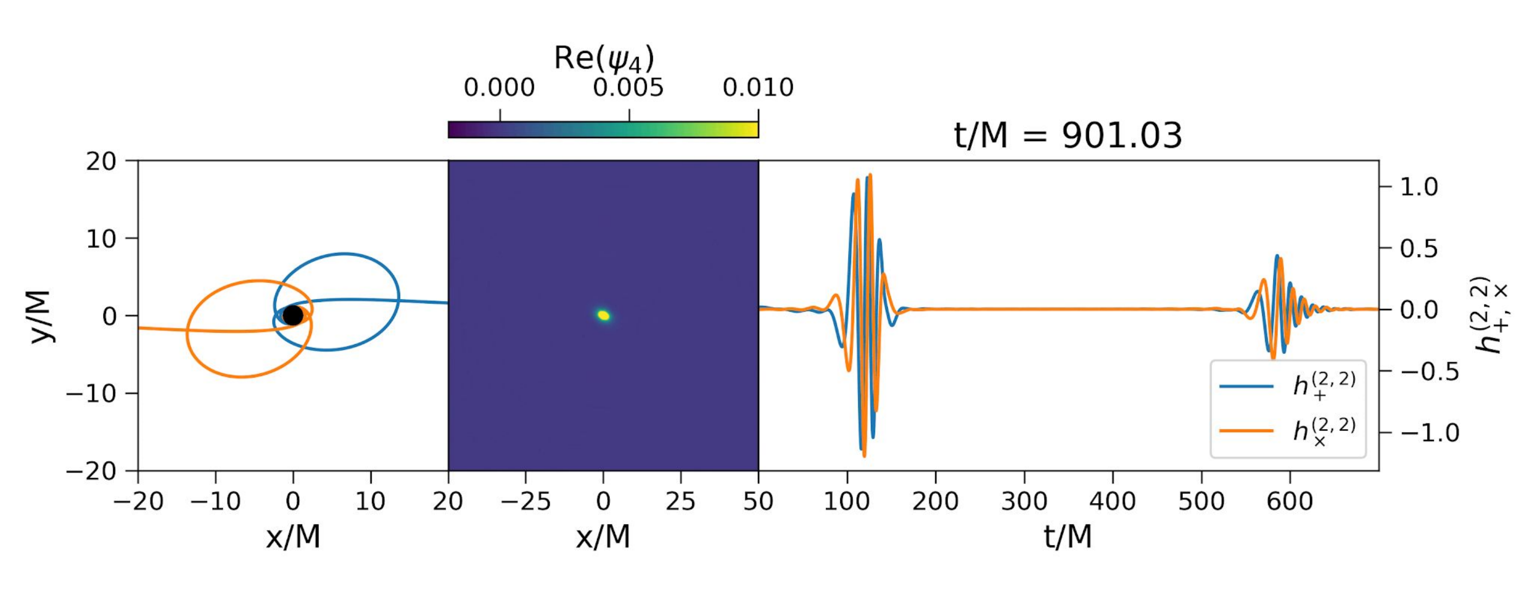
Other BBH interactions

- Orbits can bend more than in Keplerian dynamics
  → sometimes called "close hyperbolic encounters".



Other BBH interactions

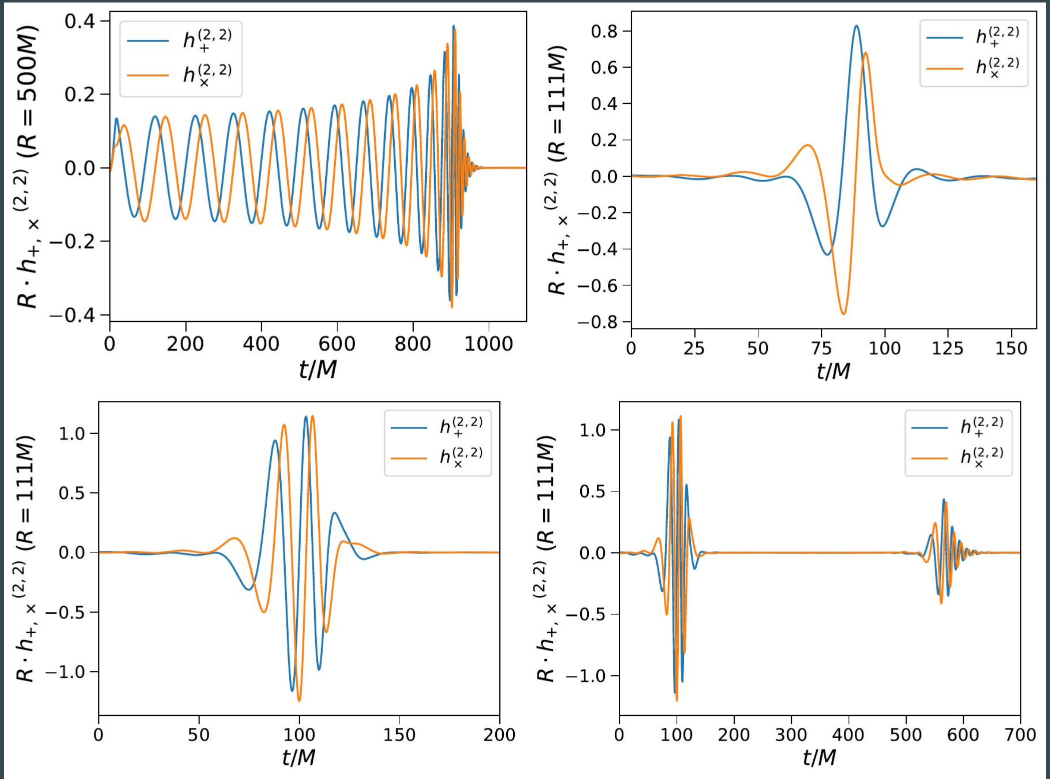
- Energy loss in a first interaction can bind both BHs
  → dynamical capture (see Rodríguez-Monteverde, SJ, García-Bellido 2025),



Other BBH interactions

BBH merger
(GW150914)

1st hyperbolic encounter
(d/M=100, q=1, p/M=0.49, θ=4.01º)

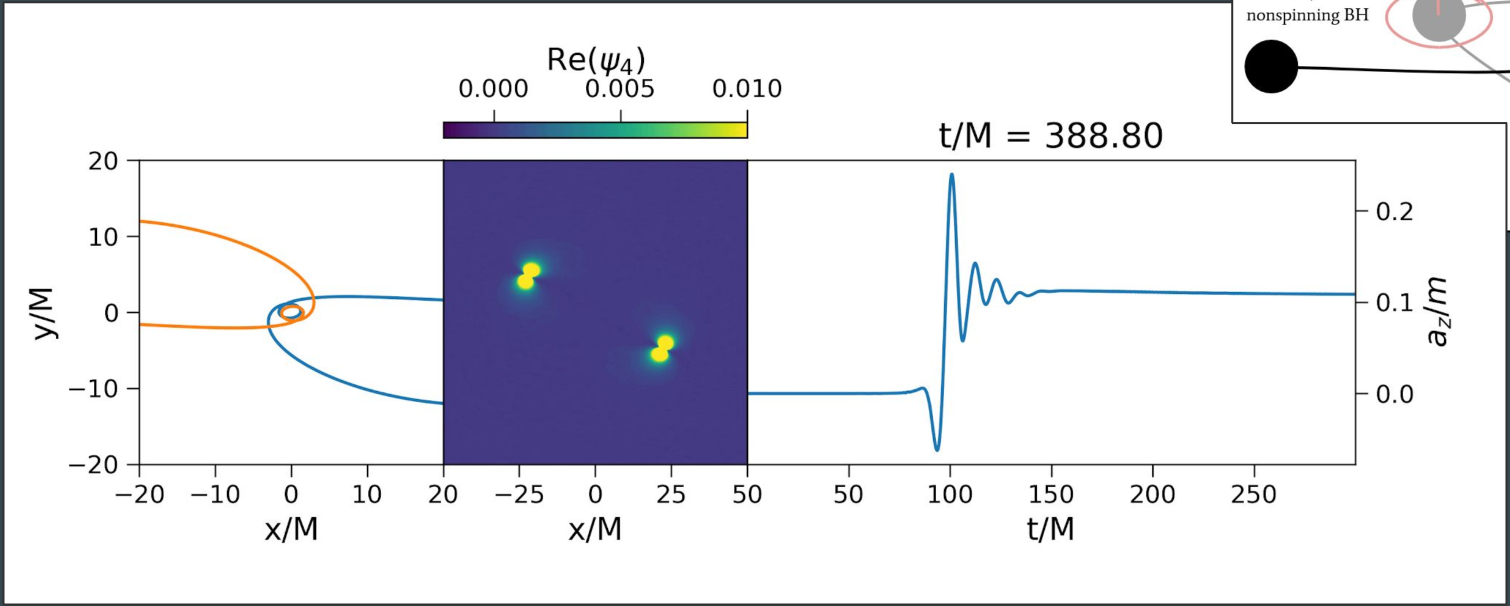


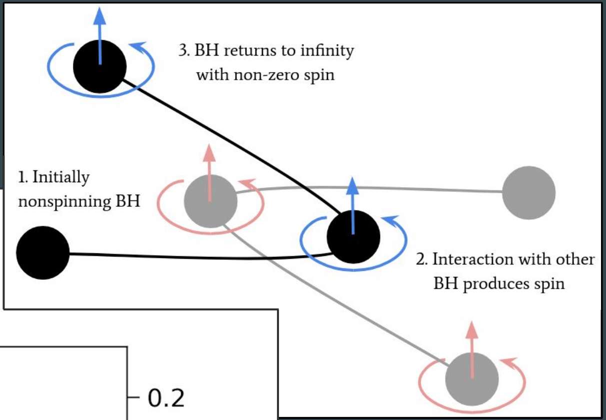
2nd hyperbolic encounter
(d/M=100, q=1, p/M=0.49, θ=3.12º)

Dynamical capture
(d/M=100, q=1, p/M=0.49, θ=3.11º)

Spin induction by hyperbolic interaction

- Strong-field interaction leads to other interesting phenomena → Spin induction. See SJ, García-Bellido 2021, Rodríguez-Monteverde, SJ, García-Bellido 2024.





Beyond standard BBH systems

- The history to simulate BBHs is a success, but many challenges remain:
  - Extreme mass-ratio binaries.
  - Precessing and nearly extremal spins.
  - Highly eccentric inspirals.
- Some efforts to model these hyperbolic/capture events:
  - Fontbuté et al. 2024: surrogate model for hyperbolic events.
  - Chiaramello et al. 2024: PN study using hyperbolic trajectories from EOB models.
  - Trenado et al. 2025: NR catalog of eccentric binaries and dynamical captures.
  - Far less developed field than standard BBH mergers.
- Including matter adds further complexity → neutron stars.

Neutron stars

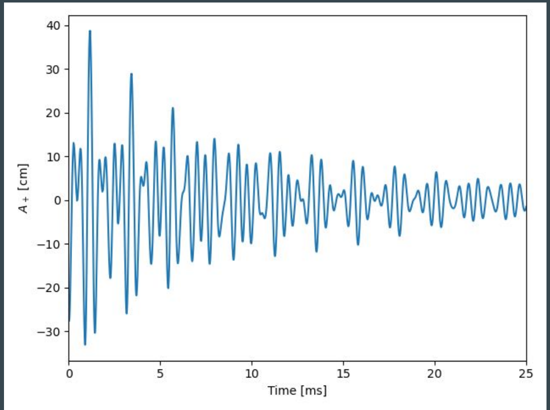
- BNS mergers probe matter effects absent in vacuum BBHs:
    - Tidal deformability.
    - Dense matter equation of state (EoS) signatures.
    - Post-merger oscillations.
- Fully relativistic hydrodynamic simulations exist.
    - High computational cost limits available simulations and waveforms.
- Post-merger phase:
    - Depending on progenitor masses, hypermassive NS remnant.
    - Short-lived (~10 s), would oscillate radiating GWs.
    - Rich mode spectrum sensitive to EoS.
    - Can be studied with NR by simulating perturbed, rotating NSs.

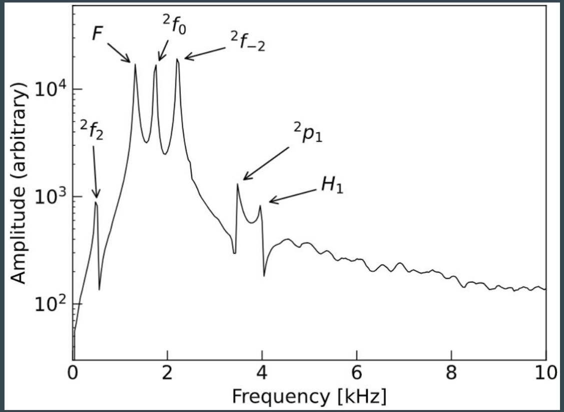
ROXAS (Relativistic Oscillations of non-aXisymmetric neutron stArS)

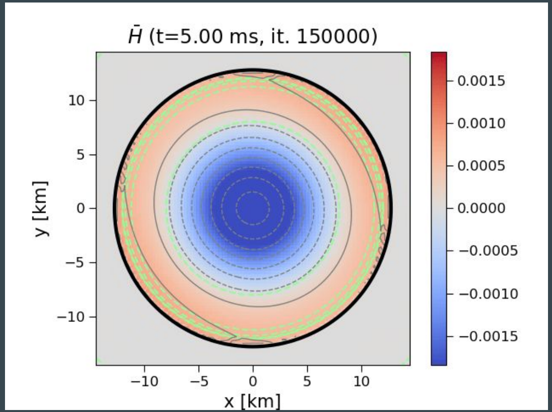
- Lightweight code to study isolated perturbed rotating NS. Presented in Servignat, Novak 2025.
- Based on LORENE (Langage Objet pour la RElativité NumériquE).
- Conformal flatness → no radiative degrees of freedom:
  - GW extracted from mass quadrupole changes → Einstein's quadrupole formula.

$$h_+(t,\vec{x}) = \frac{G}{c^4 r}(\ddot{I}_{11} - \ddot{I}_{22})(t_r),$$
$$h_\times(t,\vec{x}) = \frac{2G}{c^4 r}\ddot{I}_{12}(t_r).$$

- Allows mode studies across rotation rates, differential rotation profiles and EoS.







Summary and outlook

- Strong, collective efforts were needed to get the first stable BBH simulation.
- This made the first GW detections possible and the current era of GW observations.
- NR was essential in the process, and continues to be for waveform model calibration.
- Analytic waveform models are getting more sophisticated and complete.
  - Certain parameter space regions still not mature.
- The field has diversified since then: other interactions, neutron stars, exotic objects.
- Next-gen. detectors (Einstein Telescope, CE, LISA) → era of precision GW astronomy.
  - Higher sensitivity.
  - Access to new frequency bands.
  - Increased detection rates.

  → High-precision modeling and matter effects.
- Numerical Relativity will continue to play a key role.

Some references

- R. E. Eisenstein, *Numerical Relativity and the Discovery of Gravitational Waves* (2018).
- C. Palenzuela, *Introduction to Numerical Relativity* (2020).
- L. Lehner, F. Pretorius, *Numerical Relativity and Astrophysics* (2014).
- F. Löffler et al., *The Einstein Toolkit: A Community Computational Infrastructure for Relativistic Astrophysics* (2011).
- É. Gourgoulhon, *3+1 Formalism in General Relativity* (2012).
- T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer* (2010).
- (Initial data review) Slides by P. Grandclément at last year workshop.

- The Einstein Toolkit, https://einsteintoolkit.org/
- ROXAS, https://zenodo.org/records/14849547

Backup: Adaptive mesh refinement

- Wide spatial domain needed for simulations + GW propagation.
- High resolution required near BHs.
- Uniform grid would be too expensive.
- AMR: several nested grids, smaller and with higher resolution around the BHs.
- Grids can move with the BHs.
- Ensures accuracy and efficiency.