Macdonald Polynomials and Quantum Algebras

Jun'ichi Shiraishi
Univ. Tokyo

A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials
B.Feigin, K.Hashizume, A.Hoshino, J.S and S.Yanagida

Kernel function and quantum algebras
B.Feigin, A.Hoshino, J.Shibahara, J.S and S.Yanagida

1. Macdonald Operators

I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed. Oxford University Press, (1995).

Let q, t be independent indeterminates, and $\Lambda_{n,\mathbb{F}}$ be the ring of symmetric polynomials in $x = (x_1, x_2, \cdots, x_n)$ over $\mathbb{F} = \mathbb{Q}(q, t)$.

The Macdonald difference operator D_n^r acting on $\Lambda_{n,\mathbb{F}}$ is defined by

$$D_n^r = t^{r(r-1)/2} \sum_{\substack{I \subset \{1,2,\ldots,n\} \\ \#I = r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{k \in I} T_{q,x_k},$$

where T_{q,x_i} denotes the q-difference operator
$T_{q,x_i} f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$.

Proposition 1.1 (Macdonald)**.** For $\lambda \vdash n$ such that $\ell(\lambda) \leq n$, $P_\lambda(x; q, t) \in \Lambda_{n,\mathbb{F}}$ is uniquely characterized by

$$P_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu(x) \qquad (c_{\lambda\mu} \in \mathbb{F}),$$

$$D_n^r(q, t) P_\lambda(x; q, t) = e_r^{(n)}(t^n s_1^\lambda, t^n s_2^\lambda, \ldots, t^n s_n^\lambda) P_\lambda(x; q, t),$$

where $e_r^{(n)}(s_1 \ldots, s_n)$ denotes the elementary symmetric polynomial in n variables $(s_1, \ldots, s_n)$, and $s_i^\lambda := q^{\lambda_i} t^{-i}$.

Proposition 1.2. Let $n \in \mathbb{N}_+$. Set a difference operator $E_r^{(n)}$ acting on $\Lambda_{n,\mathbb{F}}$ by

$$E_r^{(n)} := \sum_{l=0}^{r} \frac{t^{-nr-\binom{r-l+1}{2}}}{(t^{-1};t^{-1})_{r-l}} D_n^l,$$

$$(x;q)_n = \prod_{k=0}^{n-1}(1-xq^k).$$

For any partition λ satisfying $\ell(\lambda) \le n$, we have

$$E_r^{(n)} P_\lambda(x;q,t) = e_r(s^\lambda) P_\lambda(x;q,t),$$

where $P_\lambda(x;q,t) \in \Lambda_{n,\mathbb{F}}$ denotes the Macdonald symmetric *polynomial*, $e_r(s) \in \Lambda_{\mathbb{F}}$ is the r-th elementary symmetric *function* defined in the infinite variables $s = s_1, s_2, s_3, \ldots$), and $s^\lambda = (t^{-1}q^{\lambda_1}, t^{-2}q^{\lambda_2}, \ldots)$. Hence the inductive limit $E_r := \varprojlim_n E_r^{(n)}$ exists.

$$\prod_{i\ge 1}(1+s_i u) = \sum_{r\ge 0} e_r(s) u^r,$$

$$\prod_{i=1}^{n}(1+s_i u) = \sum_{r=0}^{n} e_r^{(n)}(s) u^r.$$

Consider the Heisenberg Lie algebra $\mathfrak{h}$ over $\mathbb{F}$ with the generators a_n $(n \in \mathbb{Z})$ and the relations

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}\, a_0.$$

Let $\mathfrak{h}^{\geq 0}$ (resp. $\mathfrak{h}^{<0}$) be the subalgebra generated by a_n for $n \geq 0$ (resp. $n < 0$) . Consider the Fock representation $\mathcal{F} := \mathrm{Ind}_{\mathfrak{h}^{\geq 0}}^{\mathfrak{h}} \mathbb{F}$ of $\mathfrak{h}$. We also use the notation $a_{-\lambda} := a_{-\lambda_1} \cdots a_{-\lambda_l}$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$.

Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates and Λ be the ring of symmetric functions in x over $\mathbb{Z}$. As a $\mathbb{F}$-vector spaces, $\mathcal{F}$ is isomorphic to $\Lambda_{\mathbb{F}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$ via $\iota : \mathcal{F} \longrightarrow \Lambda_{\mathbb{F}}$ defined by $a_{-\lambda} \cdot 1 \mapsto p_\lambda$. Here $1 \in \mathcal{F}$ is the highest vector, $p_n = p_n(x) := \sum_{i \geq 1} x_i^n$ is the power sum function and $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}$. For $n > 0$ and $v \in \mathcal{F}$, we have

$$a_{-n} v = p_n v,$$
$$a_n v = n\frac{1-q^n}{1-t^n}\frac{\partial}{\partial p_n} v,$$

and $a_0 v = v$.

In what follows we identify $\mathcal{F} \simeq \Lambda_{\mathbb{F}}$ via this isomorphism ι.

2. Vertex Operator

Introduce the vertex operator

$$\begin{aligned}\eta(z) &= \sum_{n\in\mathbb{Z}} \eta_n z^{-n} \\ &= \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n} a_{-n} z^n\right) \exp\left(-\sum_{n>0} \frac{1-t^n}{n} a_n z^{-n}\right) \\ &=: \exp\big(-\sum_{n\neq 0} \frac{1-t^n}{n} a_n z^{-n}\big) :,\end{aligned}$$

where the symbol : : denotes the normal ordering with respect to the decomposition $\mathfrak{h} = \mathfrak{h}^{<0} \oplus \mathfrak{h}^{\geq 0}$, *i.e.* all the negative generators a_{-n} are moved to the left of the positive generators a_n.

H. Awata, Y. Matsuo, S. Odake and J.S, Phys. Lett. B **347** (1995), 49–55.

The Fourier modes η_n of $\eta(z)$ are well-defined operators acting on the Fock space $\mathcal{F}$ or on $\Lambda_{\mathbb{F}}$.

The operator E of Macdonald is written as
$\eta_0 = (t-1)E + 1$.

We note that the plethystic operator Δ' of Haiman and η_0 are identical.

M. Haiman, *Macdonald polynomials and geometry*, in *New Perspectives in Geometric Combinatorics*, MSRI Publications **37** (1999), 207–254.

Examples

$$P_{\emptyset} = 1,$$

$$P_{(1)} = p_1,$$

$$P_{(21)} = \frac{(q-1)(t+1)}{2(qt-1)} p_2 + \frac{(q+1)(t-1)}{2(qt-1)} p_1^2,$$

$$P_{(31)} = \frac{(q-1)(t+1)\left(t^2+1\right)}{4\left(qt^3-1\right)} p_4 - \frac{\left(t^2+t+1\right)(q-t)}{3\left(qt^3-1\right)} p_3 p_1$$

$$- \frac{(q-1)(t+1)\left(t^2+1\right)}{8\left(qt^3-1\right)} p_2^2$$

$$- \frac{\left(qt^3-qt^2-qt-q+t^3+t^2+t-1\right)}{4\left(qt^3-1\right)} p_2 p_1^2$$

$$+ \frac{(t-1)\left(3qt^2+2qt+q+t^2+2t+3\right)}{24\left(qt^3-1\right)} p_1^4$$

$$p_n = \sum_{i=1}^{\infty} x_i^n \qquad \text{(power sum function).}$$

$$\eta_0 \cdot P_{(532)} = (1-t^{-1})(q^5 + q^3 t^{-1} + q^2 t^{-2} + t^{-3} + t^{-4} + \cdots) P_{(532)}$$

$$= \Bigg(1 - (1-q)(1-t^{-1})\Big(1 + q + q^2 + q^3 + q^4$$

$$+ t^{-1} + qt^{-1} + q^2 t^{-1} + t^{-2} + qt^{-2}\Big)\Bigg) P_{(532)}$$

Pieri rule

$$
\begin{aligned}
P_{(1)}P_{\emptyset} &= P_{(1)},\\
P_{(1)}P_{(1)} &= P_{(2)} + \frac{(q-1)(t+1)}{qt-1}P_{(11)},\\
P_{(1)}P_{(21)} &= P_{(31)} + \frac{(q-1)(t+1)}{qt-1}P_{(22)}\\
&\quad + \frac{(q-1)(t+1)\left(q^2t-1\right)\left(qt^3-1\right)}{(qt-1)^2(qt+1)\left(qt^2-1\right)}P_{(211)}.
\end{aligned}
$$

Pieri rule for η_{-1}

$$
\begin{aligned}
\frac{1}{1-t^{-1}}\eta_{-1}\cdot P_{\emptyset} &= P_{(1)},\\
\frac{1}{1-t^{-1}}\eta_{-1}\cdot P_{(1)} &= qP_{(2)} + t^{-1}\frac{(q-1)(t+1)}{qt-1}P_{(11)},\\
\frac{1}{1-t^{-1}}\eta_{-1}\cdot P_{(21)} &= q^2P_{(31)} + qt^{-1}\frac{(q-1)(t+1)}{qt-1}P_{(22)}\\
&\quad + t^{-2}\frac{(q-1)(t+1)\left(q^2t-1\right)\left(qt^3-1\right)}{(qt-1)^2(qt+1)\left(qt^2-1\right)}P_{(211)}.
\end{aligned}
$$

3. Commutative family

It is easy to observe that

$$[\eta_0, \sum_{n>0} c_n \eta_{-n}\eta_n] = 0$$
$$\Longrightarrow c_n = (q/t)^n c_1.$$

Set

$$I_1 = \frac{1}{1-t^{-1}} \oint \frac{dz}{z} \eta(z),$$

$$I_2 = \frac{t^{-1}}{(1-t^{-1})(1-t^{-2})} \oint\oint \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{1-qz_2/z_1}{1-qt^{-1}z_2/z_1}\eta(z_1)\eta(z_2),$$

$$\vdots$$

$$I_n = \frac{t^{-n(n-1)/2}}{(1-t^{-1})\cdots(1-t^{-n})} \times \oint\cdots\oint \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\prod_{1\le i<j\le n}\frac{1-qz_j/z_i}{1-qt^{-1}z_j/z_k}\cdot\eta(z_1)\eta(z_2)\cdots\eta(z_n).$$

Proposition 3.1. We have

$$[I_m, I_n] = 0,$$
$$I_m P_\lambda = e_m(s^\lambda) P_\lambda.$$

J.S, *A Family of Integral Transformations and Basic Hypergeometric Series*, Comm. Math. Phys. **263** (2006), 439–460.

4. Feigin-Odesskii algebra

Set

$$q_1 = q^{-1}, \qquad q_2 = t, \qquad q_3 = qt^{-1}.$$

We have

$$q_1 q_2 q_3 = 1.$$

Definition 4.1 (Space $\mathcal{A}$)**.** For $n \in \mathbb{N}$, the vector space $\mathcal{A}_n = \mathcal{A}_n(q_1, q_2, q_3)$ is defined by the following conditions.

(i) $\mathcal{A}_0 := \mathbb{F}$. For $n \in \mathbb{N}_+$, $f(x_1, \dots, x_n) \in \mathcal{A}_n$ is a rational function with coefficients in $\mathbb{F}$, and symmetric with respect to the x_i's.

(ii) For $n \in \mathbb{N}$, $0 \leq k \leq n$ and $f \in \mathcal{A}_n$, the limits $\partial^{(\infty,k)} f$ and $\partial^{(0,k)} f$ both exist and coincide: $\partial^{(\infty,k)} f = \partial^{(0,k)} f$ (*degenerate* $\mathbb{CP}^1$ *condition*).

(iii) The poles of $f \in \mathcal{A}_n$ are located only on the diagonal $\{(x_1, \dots, x_n) \mid \exists (i,j), i \neq j, x_i = x_j\}$, and the orders of the poles are at most two.

(iv) For $n \geq 3$, $f \in \mathcal{A}_n$ satisfies the *wheel conditions*

$$f(x_1, q_1 x_1, q_1 q_2 x_1, x_4, \dots) = 0,$$

$$f(x_1, q_2 x_1, q_1 q_2 x_1, x_4, \dots) = 0.$$

Then set the graded vector space $\mathcal{A} = \mathcal{A}(q_1, q_2, q_3) := \bigoplus_{n \geq 0} \mathcal{A}_n$.

$$\begin{aligned}
\partial^{(0,k)} &: f \mapsto \frac{n!}{(n-k)!} \lim_{\xi \to 0} f(x_1, \dots, x_{n-k}, \xi x_{n-k+1}, \xi x_{n-k+2}, \dots, \xi x_n) \\
\partial^{(\infty,k)} &: f \mapsto \frac{n!}{(n-k)!} \lim_{\xi \to \infty} f(x_1, \dots, x_{n-k}, \xi x_{n-k+1}, \xi x_{n-k+2}, \dots, \xi x_n)
\end{aligned}$$

Definition 4.2 (Star product $*$)**.** For an m-variable symmetric rational function f and an n-variable symmetric rational function g, we define an $(m+n)$-variable symmetric rational function $f * g$ by

$$\begin{aligned}&(f * g)(x_1, \ldots, x_{m+n})\\&= \operatorname{Sym}\left[f(x_1, \ldots, x_m)g(x_{m+1}, \ldots, x_{m+n}) \prod_{\substack{1 \le \alpha \le m \\ m+1 \le \beta \le m+n}} \omega(x_\alpha, x_\beta)\right].\end{aligned}$$

Here $\omega(x, y)$ is the rational function

$$\omega(x, y) = \omega(x, y; q_1, q_2, q_3) = \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3},$$

and the symbol Sym denotes the symmetrizer

$$\operatorname{Sym}(f(x_1, \ldots, x_n)) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(f(x_1, \ldots, x_n)),$$

where $\mathfrak{S}_n$ is the n-th symmetric group acting on the indices of x_i's.

Theorem 4.3. The vector space $\mathcal{A}$ is closed with respect to the star product $*$, hence the pair $(\mathcal{A}, *)$ defines a unital associative algebra. The algebra $(\mathcal{A}, *)$ is commutative. The Poincaré series is $\sum_{n \ge 0} (\dim_{\mathbb{F}} \mathcal{A}_n) z^n = \prod_{m \ge 1} (1 - z^m)^{-1}$.

5. Gordon filtration

Definition 5.1 (Specialization map φ)**.** Let $p \in \mathbb{F} = \mathbb{Q}(q_1, q_2)$. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of n, we define a linear map

$$\begin{array}{rccc} \varphi_\lambda^{(p)} : & \mathcal{A}_n & \longrightarrow & \mathbb{F}(y_1, \dots, y_m) \\ & f(x_1, \dots, x_n) & \mapsto & f(y_1, py_1, \dots, p^{\lambda_1 - 1} y_1, \\ & & & y_2, py_2, \dots, p^{\lambda_2 - 1} y_2, \dots \\ & & & \dots, y_m, py_m \dots, p^{\lambda_m - 1} y_m). \end{array}$$

Definition 5.2 (Gordon filtration)**.** For q_i $(i = 1, 2, 3)$ and $\lambda \vdash n$, define $\mathcal{A}_{n,\lambda}^{(q_i)} \subset \mathcal{A}_n$ by

$$\mathcal{A}_{n,\lambda}^{(q_i)} := \bigcap_{\mu \not\leq \lambda} \ker \varphi_\mu^{(q_i)} \quad (\lambda < (n)), \qquad \mathcal{A}_{n,(n)}^{(q_i)} := \mathcal{A}_n.$$

Definition 5.3 (Element ϵ_n)**.** For $n > 1$ and $i = 1, 2, 3$, define

$$\epsilon_n = \epsilon_n(x; q_i) = \epsilon_n(x_1, \dots, x_n; q_i) := \prod_{1 \leq k < l \leq n} \frac{(x_k - q_i x_l)(x_k - q_i^{-1} x_l)}{(x_k - x_l)^2}.$$

We also set $\epsilon_1 = 1$. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of n, we write $\epsilon_\lambda := \epsilon_{\lambda_1} * \cdots * \epsilon_{\lambda_m}$ for simplicity.

Proposition 5.4 (Bottom of the Gordon filter)**.** $\mathcal{A}_{n,(1^n)}^{(q_i)}$ is the 1-dimensional subspace of $\mathcal{A}_n$ spanned by $\epsilon_n(x; q_i)$.

Theorem 5.5. Let $n \in \mathbb{N}$ and λ be a partition of n. Then the intersection of the subspaces $\mathcal{A}_{n,\lambda}^{(q_1)}$ and $\mathcal{A}_{n,\lambda'}^{(q_2)}$ is 1-dimensional:

$$\dim_{\mathbb{F}} \big(\mathcal{A}_{n,\lambda}^{(q_1)} \cap \mathcal{A}_{n,\lambda'}^{(q_2)} \big) = 1.$$

6. Intersection of Gordon filters

Set

$$\begin{aligned} E(y) &:= \prod_{i=1}^{\infty}(1+x_i y) = \sum_{n\geq 0} e_n(x) y^n, \\ G(y) &:= \prod_{i=1}^{\infty} \frac{(t x_i y; q)_\infty}{(x_i y; q)_\infty} = \sum_{n\geq 0} g_n(x;q,t) y^n, \\ (x;q)_\infty &:= \prod_{i\geq 0}(1-q^i x). \end{aligned}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ set $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$. Similarly we write $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ and $g_\lambda := g_{\lambda_1} g_{\lambda_2} \cdots$. It is known that $\{p_\lambda\}$, $\{m_\lambda\}$, $\{e_\lambda\}$ and $\{g_\lambda\}$ form bases of $\Lambda_{\mathbb{F}}$.

$$\begin{aligned} P_\lambda(x;q,t) &= \sum_{\mu \geq \lambda'} c_{\lambda\mu}^{e\to P}(q,t) e_\mu(x;q,t), \\ P_\lambda(x;q,t) &= \sum_{\mu \geq \lambda} c_{\lambda\mu}^{g\to P}(q,t) g_\mu(x;q,t). \end{aligned}$$

$$\begin{aligned} f_\lambda^{(q^{-1})}(z;q,t) &:= \frac{t^{-|\lambda|}}{(1-t^{-1})^{|\lambda|}|\lambda|!} \sum_{\mu\geq\lambda'} c_{\lambda\mu}^{e\to P}(q,t) \epsilon_\mu(z;q) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}, \\ f_\lambda^{(t)}(z;q,t) &:= \frac{(-1)^{|\lambda|}}{(1-q)^{|\lambda|}|\lambda|!} \sum_{\mu\geq\lambda} c_{\lambda\mu}^{g\to P}(q,t) \epsilon_\mu(z;t) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}. \end{aligned}$$

Theorem 6.1. We have

$$f_\lambda^{(q^{-1})}(z;q,t) = f_\lambda^{(t)}(z;q,t) \in \mathcal{A}_{n,\lambda}^{(q_1)} \cap \mathcal{A}_{n,\lambda'}^{(q_2)}.$$

7. Duallity

Definition 7.1. Let $f \in \mathcal{A}_n(q^{-1}, t, qt^{-1})$. Define a mapping $\mathcal{O}(\cdot) = \mathcal{O}(\cdot\,; q, t) : \mathcal{A} \to \mathrm{End}_{\mathbb{F}}(\mathcal{F})$ by

$$\mathcal{O}(f) = \mathcal{O}(f; q, t) := \left[\frac{f(z_1, \ldots, z_n)}{\prod_{1 \le i < j \le n} \omega(z_i, z_j; q^{-1}, t, qt^{-1})} \eta(z_1) \cdots \eta(z_n) \right]_1,$$

for $f \in \mathcal{A}_n$ and extending it by linearity.

The star product $*$ and the operation $\mathcal{O}(\cdot)$ are compatible in the following sense.

Proposition 7.2. For $f, g \in \mathcal{A}$, we have

$$\mathcal{O}(f * g) = \mathcal{O}(f)\mathcal{O}(g),$$

which indicates that $[\mathcal{O}(f), \mathcal{O}(g)] = 0$ from the commutativity of $\mathcal{A}$.

Definition 7.3. Define the commutative ring $\mathcal{M}$ of operators on $\Lambda_{\mathbb{F}}$ by $\mathcal{M} := \{\mathcal{O}(f) \mid f \in \mathcal{A}\}$.

Proposition 7.4. The mapping $\mathcal{O}(\cdot) : \mathcal{A} \to \mathcal{M}$ gives an isomorphism of commutative rings.

Set

$$\mathcal{O}_\mu = \mathcal{O}(f_\mu^{(q^{-1})}(z; q, t)) = \mathcal{O}(f_\mu^{(t)}(z; q, t)).$$

Theorem 7.5. We have

$$\mathcal{O}_\mu \cdot P_\lambda(x) = P_\mu(s^\lambda) P_\lambda(x),$$
$$s^\lambda = (q^{\lambda_1} t^{-1}, q^{\lambda_2} t^{-2}, \cdots).$$

8. Ding-Iohara algebra

We have

$$P_\lambda(x;q,t) = P_\lambda(x;q^{-1},t^{-1}) \Longrightarrow [E_m(q,t), E_n(q^{-1},t^{-1})] = 0.$$

Set

$$\eta(z) = :\exp\left(-\sum_{n\neq 0}\frac{1-t^n}{n}a_n z^{-n}\right):,$$

$$\xi(z) = :\exp\left(+\sum_{n\neq 0}\frac{1-t^n}{n}(t/q)^{|n|/2}a_n z^{-n}\right):,$$

$$\varphi^+(z) = \exp\left(-\sum_{n>0}\frac{1-t^n}{n}(1-t^nq^{-n})(t/q)^{-n/4}a_n z^{-n}\right),$$

$$\varphi^-(z) = \exp\left(+\sum_{n>0}\frac{1-t^{-n}}{n}(1-t^nq^{-n})(t/q)^{-n/4}a_{-n} z^{n}\right).$$

We have

$$\begin{aligned}&[\eta(z),\xi(w)]\\ &= \frac{(1-q)(1-t^{-1})}{1-qt^{-1}}\Big(\delta((t/q)^{-1/2}z/w)\varphi^+((t/q)^{1/4}w)\\ &\qquad - \delta((t/q)^{1/2}z/w)\varphi^-((t/q)^{-1/4}w)\Big),\end{aligned}$$

where $\delta(z) := \sum_{n\in\mathbb{Z}} z^n$ is the formal delta function.

Then $[\eta_0,\xi_0] = \frac{(1-q)(1-t^{-1})}{1-qt^{-1}}(\varphi_0^+ - \varphi_0^-) = 0$ holds since $\varphi_0^+ = \varphi_0^- = 1$.

Set

$$g(z) := \frac{G^+(z)}{G^-(z)}, \qquad G^\pm(z) := (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-q^{\mp 1}t^{\pm 1}z).$$

Note that $g(z)$ satisfies the Ding-Iohara requirement $g(z) = g(z^{-1})^{-1}$.

Definition 8.1 (Ding-Iohara)**.** Let $\mathcal{U} = \mathcal{U}(q,t)$ be a unital associative algebra generated by the Drinfeld currents $x^\pm(z) = \sum_{n\in\mathbb{Z}} x_n^\pm z^{-n}$, $\psi^\pm(z) = \sum_{\pm n\in\mathbb{N}} \psi_n^\pm z^{-n}$ and the central element $\gamma^{\pm 1/2}$, satisfying the defining relations

$$\begin{aligned}
&\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z),\\
&\psi^+(z)\psi^-(w) = \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^-(w)\psi^+(z),\\
&\psi^+(z)x^\pm(w) = g(\gamma^{\mp 1/2}w/z)^{\mp 1}x^\pm(w)\psi^+(z),\\
&\psi^-(z)x^\pm(w) = g(\gamma^{\mp 1/2}z/w)^{\pm 1}x^\pm(w)\psi^-(z),\\
&[x^+(z), x^-(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\Big(\delta(\gamma^{-1}z/w)\psi^+(\gamma^{1/2}w)\\
&\qquad\qquad - \delta(\gamma z/w)\psi^-(\gamma^{-1/2}w)\Big),\\
&x^\pm(z)x^\pm(w) = g(z/w)^{\pm 1}x^\pm(w)x^\pm(z).
\end{aligned}$$

Proposition 8.2 (Ding-Iohara)**.** The algebra $\mathcal{U}$ is a Hopf algebra.

$$\begin{aligned}
&\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}\otimes\gamma^{\pm 1/2},\\
&\Delta(x^+(z)) = x^+(z)\otimes 1 + \psi^-(\gamma_{(1)}^{1/2}z)\otimes x^+(\gamma_{(1)}z),\\
&\Delta(x^-(z)) = x^-(\gamma_{(2)}z)\otimes\psi^+(\gamma_{(2)}^{1/2}z) + 1\otimes x^-(z),\\
&\Delta(\psi^\pm(z)) = \psi^\pm(\gamma_{(2)}^{\pm 1/2}z)\otimes\psi^\pm(\gamma_{(1)}^{\mp 1/2}z),
\end{aligned}$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2}\otimes 1$ and $\gamma_{(2)}^{\pm 1/2} = 1\otimes\gamma^{\pm 1/2}$.
Counit ε. Antipode a.

9. Some representations

We call a representation of *level* k, if the central element γ is realized by the constant $(t/q)^{k/2}$.

Proposition 9.1 (level zero representation)**.** We have a representation $\pi(\cdot)$ of $\mathcal{U}(q,t)$ on $V := \mathbb{Q}(q^{1/2}, t^{1/2})[x, x^{-1}]$ by setting

$$\begin{aligned}
&\pi(\gamma^{\pm 1/2}) = 1,\\
&\pi(\psi^+(z)) = \frac{(1-q^{1/2}t^{-1}x/z)(1-q^{-1/2}tx/z)}{(1-q^{1/2}x/z)(1-q^{-1/2}x/z)},\\
&\pi(\psi^-(z)) = \frac{(1-q^{1/2}t^{-1}z/x)(1-q^{-1/2}tz/x)}{(1-q^{1/2}z/x)(1-q^{-1/2}z/x)},\\
&\pi(x^\pm(z)) = c^{\pm 1}(1-t^{\mp 1})\delta(q^{\mp 1/2}x/z)T_{q^{\mp 1},x},
\end{aligned}$$

where $c \in \mathbb{Q}(q^{1/2}, t^{1/2})^\times$. Here we have used the q-shift operator $T_{q^{\pm 1},x}f(x) = f(q^{\pm 1}x)$.

Proposition 9.2 (level one representation)**.** Let $\mathfrak{h}$ be the Heisenberg algebra generated by a_n, and $\mathcal{F}$ be the corresponding Fock space. Let $\eta(z), \xi(z), \varphi^+(z)$ and $\varphi^-(z)$ be the vertex operators. Then we have a representation $\rho(\cdot)$ of $\mathcal{U}(q,t)$ on $\mathcal{F}$ by setting

$$\begin{aligned}
&\rho(\gamma^{\pm 1/2}) = (t/q)^{\pm 1/4},\\
&\rho(\psi^\pm(z)) = \varphi^\pm(z),\\
&\rho(x^+(z)) = y\,\eta(z),\\
&\rho(x^-(z)) = y^{-1}\xi(z),
\end{aligned}$$

where $y \in \mathbb{Q}(q^{1/2}, t^{1/2})^\times$.

10. Level two representation

Consider the level two representation on the tensor space

$$\mathcal{F}_{y_1} \otimes \mathcal{F}_{y_2}.$$

A basis of this space is $\{P_\lambda \otimes P_\mu\}$. We have

$$\Delta(x^+(z)) = y_1\eta(z) \otimes 1 + y_2\varphi^{-1}((q/t)^{1/4}z) \otimes \eta((q/t)^{1/2}z).$$

There is an integrable structure associated with $\Delta(x_0^+)$. The action of $\Delta(x_0^+)$ is triangular on the basis $\{P_\lambda \otimes P_\mu\}$. Hence there exists a basis $\{P_{\lambda,\mu}\}$ of $\mathcal{F}_{y_1} \otimes \mathcal{F}_{y_2}$ consisting of the eigenfunctions of $\Delta(x_0^+)$:

$$\begin{aligned}
&P_{\lambda,\mu} = P_\lambda \otimes P_\mu + \text{lower terms},\\
&\Delta(x_0^+)P_{\lambda,\mu} = (1-t^{-1})\big(y_1 e_1(s^\lambda) + y_2 e_1(s^\mu)\big)P_{\lambda,\mu}.
\end{aligned}$$

$$\begin{aligned}
&P_{\emptyset,\emptyset} = 1 \otimes 1,\\
&P_{(1),\emptyset} = P_{(1)} \otimes 1,\\
&P_{\emptyset,(1)} = 1 \otimes P_{(1)} + \frac{y_2(t-q)}{q(y_1-y_2)}P_{(1)} \otimes 1,\\
&P_{(2),\emptyset} = P_{(2)} \otimes 1,\\
&P_{(11),\emptyset} = P_{(11)} \otimes 1,\\
&P_{(1),(1)} = P_{(1)} \otimes P_{(1)} + \frac{y_2(t-q)}{q(qy_1-y_2)}P_{(2)} \otimes 1\\
&\qquad + \frac{(q-1)t(t+1)y_2(q-t)}{q(qt-1)(ty_2-y_1)}P_{(11)} \otimes 1,
\end{aligned}$$

Pieri rule

$$\frac{\Delta(x^+_{-1})}{1-t^{-1}}P_{\emptyset,\emptyset} = \frac{y_1(qy_1-ty_2)}{q(y_1-y_2)}P_{(1),\emptyset} + y_2P_{\emptyset,(1)},$$

$$\frac{\Delta(x^+_{-1})}{1-t^{-1}}P_{(1),\emptyset} = \frac{y_1\left(q^2y_1-ty_2\right)}{qy_1-y_2}P_{(2),\emptyset} + \frac{(q-1)(t+1)y_1\left(qy_1-t^2y_2\right)}{qt(qt-1)(y_1-ty_2)}P_{(11),\emptyset} + y_2P_{(1),(1)},$$

$$\frac{\Delta(x^+_{-1})}{1-t^{-1}}P_{\emptyset,(1)} = \frac{y_1(qy_1-y_2)(y_1-ty_2)(ty_1-qy_2)}{q(y_1-y_2)(y_1-qy_2)(ty_1-y_2)}P_{(1),(1)} + qy_2P_{\emptyset,(2)} + \frac{(q-1)(t+1)y_2}{t(qt-1)}P_{\emptyset,(11)},$$

11. Level m representation

Consider the tensor

$$\mathcal{F}_{y_1} \otimes \mathcal{F}_{y_2} \otimes \cdots \otimes \mathcal{F}_{y_m}.$$

We have

$$\rho_y^{(m)}(x^+(z)) = \sum_{i=1}^{m} y_i \Lambda_i(z),$$

where

$$\Lambda_i(z) := \varphi^-(p^{-1/4}z) \otimes \varphi^-(p^{-3/4}z) \otimes \cdots$$

$$\cdots \otimes \varphi^-(p^{-(2i-3)/4}z) \otimes \eta(p^{-(i-1)/2}z) \otimes 1 \otimes \cdots \otimes 1.$$

Set

$$K_n(x, z; q, t) = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_n=1}^{m} x_{i_1} x_{i_2} \cdots x_{i_n} \prod_{1 \le \alpha < \beta \le n} \gamma_{i_\alpha, i_\beta}(z_\alpha, z_\beta; q, t),$$

where the function $\gamma_{i,j}(z, w; q, t)$ is given by

$$\gamma_{i,j}(z, w; q, t) := \begin{cases} \dfrac{(z - tw)(z - t^{-1}w)}{(z - w)^2} & i = j, \\ \dfrac{(z - q^{-1}w)(z - tw)(z - qt^{-1}w)}{(z - w)^3} & i < j, \\ \dfrac{(z - qw)(z - t^{-1}w)(z - q^{-1}tw)}{(z - w)^3} & i > j. \end{cases}$$

We have

$$\langle x^+(z_1) x^+(z_2) \cdots x^+(z_n) \rangle \sim K_n(y, z; q, t),$$

up to a rational function.

Proposition 11.1. We have:

(1) $K_n(y,z;q,t)$ is a symmetric polynomial in $y=(y_1,y_2,\cdots,y_m)$.

(2) $K_n(y,z;q,t)\in\mathcal{A}_n$ w.r.t $z=(z_1,z_2,\cdots,z_n)$.

Proposition 11.2. Expand as

$$K_n(y,z;q,t)=\sum_\lambda P_\lambda(y)F_\lambda(z),$$

then $F_\lambda(z)\in\mathcal{A}^{(q_1)}_{n,\lambda}$ and $\mathcal{A}^{(q_2)}_{n,\lambda'}$.

For a partition $\lambda=(\lambda_1,\ldots,\lambda_l)$ of n and $\zeta\in\mathbb{F}$, we define the map $\widetilde{\varphi}^{(\zeta)}_\lambda$ by

$$\begin{array}{rcccl} \widetilde{\varphi}^{(\zeta)}_\lambda & : & \mathbb{F}(z_1,\ldots,z_n) & \longrightarrow & \mathbb{F}(y) \\ & & f(z_1,\ldots,z_n) & \longmapsto & f(y,q^{-1}y,\ldots,q^{-(\lambda_1-1)}y, \\ & & & & \quad \zeta y,q^{-1}\zeta y,\ldots,q^{-(\lambda_2-1)}\zeta y, \\ & & & & \quad \ldots, \\ & & & & \quad \zeta^{l-1}y,q^{-1}\zeta^{l-1}y,\ldots,q^{-(\lambda_l-1)}\zeta^{l-1}y). \end{array}$$

Theorem 11.3. For partitions μ,λ of n, $\widetilde{\varphi}^{(\zeta)}_\lambda(F_\mu/F_\lambda)$ is regular at $\zeta=t$ and its value is $\delta_{\lambda,\mu}$.

Namely we have

$$\lim_{\zeta\to t}\widetilde{\varphi}^{(\zeta)}_\lambda\frac{K_n(y,z;q,t)}{F_\lambda(z;q,t)}=P_\lambda(y;q,t).$$