Langlands duality from modular duality

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Motivation

There is an interesting class of $N = 2$, $SU(2)$ gauge theories $\mathcal{G}_C$ associated to a Riemann surface C (Gaiotto), in particular $C \mapsto$ Lagrangian of $\mathcal{G}_C$. The theories $\mathcal{G}_C$ generalize theories studied by Seiberg and Witten.

Low energy theory described in terms of prepotential $\mathcal{F}_C(a)$, $a = (a_1, \dots, a_{3g-3+n})$, which can be calculated from spectral curve of the $SU(2)$-**Hitchin system** associated to C.

Gauge theory instanton calculus (Moore, Nekrasov, Shatashvili) defines a natural deformation $\mathcal{Z}_C(a;\epsilon_1,\epsilon_2)$ of $\mathcal{F}_C(a)$,

$$\mathcal{F}_C(a) = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \mathcal{Z}_C(a;\epsilon_1,\epsilon_2)\,.$$

Amazing observation (Alday, Gaiotto, Tachikawa):

$$\exp(\mathcal{Z}_C(a;\epsilon_1,\epsilon_2)) = \textbf{Conformal blocks of Liouville theory}.$$

What has Liouville theory to do with the Hitchin system?

Simplest example of a Hitchin system: Gaudin model

Case $C = \mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}$. DOF: $J_r \in \mathfrak{sl}(2)_{\mathbb{C}}$, $r = 1, \ldots, n$ modulo $SL(2, \mathbb{C})$. Consider

$$\theta(y) = \begin{pmatrix} J^0 & J^+ \\ J^- & -J^0 \end{pmatrix} = \sum_{r=1}^{n} \frac{J_r}{y - z_r},$$

In this case transfer matrix:

$$q = \mathrm{tr}(\theta^2) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right), \qquad H_r = \sum_{s \neq r} \frac{J_r^a J_s^b}{z_r - z_s} \eta_{ab}.$$

A Poisson bracket can be introduced as

$$\{ J_r^0 , J_s^{\pm} \} = \pm \delta_{rs} J_s^{\pm}, \qquad \{ J_r^+ , J_s^- \} = 2\delta_{rs} J_s^0.$$

The H_r commute w.r.t. the Poisson structure $\Rightarrow$ **Integrability**.

Quantum $SL(2,\mathbb{C})$-Gaudin model

Cconsider the tensor product of n principal series representations $\mathcal{P}_j$ of $SL(2,\mathbb{C})$. It corresponds to the tensor product of representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ generated by differential operators $\mathcal{J}_r^a$ acting on functions $\Psi(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n)$ as

$$\mathcal{J}_r^- = \partial_{x_r}, \quad \mathcal{J}_r^0 = x_r\partial_{x_r} - j_r, \quad \mathcal{J}_r^+ = -x_r^2\partial_{x_r} + 2j_r x_r,$$

and complex conjugate operators $\bar{\mathcal{J}}_r^a$. Casimir parameterized via j_r as $j_r(j_r+1)$. Let

$$\mathsf{H}_r \equiv \sum_{s\neq r} \frac{\mathcal{J}_{rs}}{z_r - z_s}, \qquad \bar{\mathsf{H}}_r \equiv \sum_{s\neq r} \frac{\bar{\mathcal{J}}_{rs}}{\bar{z}_r - \bar{z}_s},$$

where the differential operator $\mathcal{J}_{rs}$ is defined as

$$\mathcal{J}_{rs} := \eta_{aa'}\mathcal{J}_r^a\mathcal{J}_s^{a'} := \mathcal{J}_r^0\mathcal{J}_s^0 + \frac{1}{2}\left(\mathcal{J}_r^+\mathcal{J}_s^- + \mathcal{J}_r^-\mathcal{J}_s^+\right),$$

while $\bar{\mathcal{J}}_{rs}$ is the complex conjugate of $\mathcal{J}_{rs}$. The Gaudin Hamiltonians are mutually commuting,

$$[\,\mathsf{H}_r\,,\,\mathsf{H}_s\,] = 0\,, \qquad [\,\mathsf{H}_r\,,\,\bar{\mathsf{H}}_s\,] = 0\,, \qquad [\,\bar{\mathsf{H}}_r\,,\,\bar{\mathsf{H}}_s\,] = 0\,.$$

Separation of variables (SOV) for the Gaudin model I

Diagonalize J^- by means of the Fourier transformation

$$\tilde{\Psi}(\mu_1,\ldots,\mu_n) \,=\, \frac{1}{\pi^n}\int d^2x_1\ldots\int d^2x_n\,\prod_{r=1}^{n}|\mu_r|^{2j_r+2}e^{\mu_r x_r-\bar{\mu}_r\bar{x}_r}\Psi(x_1,\ldots,x_n)\,.$$

The generators J_r^a are mapped to the differential operators D_r^a,

$$\mathrm{D}_r^- = \mu_r, \quad \mathrm{D}_r^0 = \mu_r\partial_{\mu_r}, \quad \mathrm{D}_r^+ = \mu_r\partial_{\mu_r}^2 - \frac{j_r(j_r+1)}{\mu_r}\,,$$

Define variables to $y_1,\ldots,y_{n-2},u$ related to the variables $\mu_1,\ldots,\mu_n$ via

$$\sum_{i=1}^{n}\frac{\mu_i}{t-z_i} = u\frac{\prod_{j=1}^{n-2}(t-y_j)}{\prod_{i=1}^{n}(t-z_i)}\,.$$

Separation of variables (SOV) for the Gaudin model II

It was shown by `Sklyanin` that the eigenvalue equations $\mathsf{H}_r\Psi = E_r\Psi$ are transformed into **Baxter equations**

$$(\partial_{y_k}^2 + t(y_k))\chi(y_k) \,=\, 0\,, \qquad t(y) \,\equiv\, -\sum_{r=1}^{n}\left(\frac{j_r(j_r+1)}{(y_k-z_r)^2} - \frac{E_r}{y_k-z_r}\right)\,.$$

The dependence with respect to the variables y_k has completely separated.

Solutions to the Gaudin-eigenvalue equations $\mathsf{H}_r\Psi = E_r\Psi$ can therefore be constructed in factorized form

$$\Psi \,=\, \prod_{k=1}^{n-2}\chi_k(y_k;q)\,.$$

Generalization: Hitchin's integrable system

Phase space: $\mathcal{M}_{\mathrm{H}} \simeq$ Moduli space of Higgs pairs $(\mathcal{E}, \theta)$,

$$\left\{ \begin{array}{l} \mathcal{E} = (V, A^{0,1}) \text{ holomorphic } SU(2)\text{-bundle on } \mathcal{C} \\ \theta \in H^0(\mathcal{C}, \mathrm{End}(\mathcal{E}) \otimes \Omega^1_{\mathcal{C}}). \end{array} \right\},$$

modulo complex gauge transformations.

Associate to $(\mathcal{E}, \theta)$ the quadratic differential

$$q = \mathrm{tr}(\theta^2)\,.$$

Expanding q with respect to a basis $\{q_1, \ldots, q_{3g-3+n}\}$ of the $3g-3+n$-dimensional space of quadratic differentials,

$$q = \sum_{r=1}^{3g-3+n} H_r\, q_r\,,$$

defines functions H_r, $r = 1, \ldots, 3g-3+n$ on $\mathcal{M}_{\mathrm{H}}(C)$ called Hitchin's Hamiltonians.

(Pre-) Quantization of Hitchin's integrable system

The main results of Beilinson-Drinfeld on the **geometric Langlands correspondence** can roughly be reformulated in the language from integrable models:

- There exist differential operators $\mathcal{D}_r$, $r = 1, \dots, 3g-3+n$ which commute with each other and have H_r as their leading symbol (classical limit).

- To each solution of the eigenvalue equations $\mathcal{D}_r \Psi = E_r \Psi$ there corresponds an **oper**, a second order differential operator locally of the form $\epsilon^2 \partial_y^2 + t(y)$ ($\Rightarrow$ *Baxter equation*), which behaves under change of coordinates on C as

$$t(y) \mapsto (y'(w))^2 t(y(w)) - \frac{1}{2}\{y, w\}, \qquad \{y, w\} \equiv \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 .$$

- There exist natural operations $\mathfrak{H}(y)$ ("Hecke functors") on the spaces of solutions to the eigenvalue equations which act as

$$\mathfrak{H}(y) : \Psi \to \psi(y)\Psi, \qquad (\epsilon^2 \partial_y^2 + t(y))\psi(y) = 0 .$$

"Hecke functors" $\simeq$ Q-operators. (cf. A. Gerasimov's talk and O. Foda, to appear).

Liouville theory I

Liouville theory is a CFT, central charge $c = 1 + 6Q^2$, $Q := b + b^{-1}$. It is characterized by the correlation functions of n primary fields $e^{2\alpha_r\phi(z_r,\bar{z}_r)}$ denoted as

$$\left\langle\!\left\langle\, e^{2\alpha_n\phi(z_n,\bar{z}_n)} \cdots e^{2\alpha_1\phi(z_1,\bar{z}_1)} \,\right\rangle\!\right\rangle_{C_q}.$$

C_q is a family of Riemann surfaces parameterized by $q = (q_1, \dots, q_{3g-3+n})$.

The dimension Δ_r of the primary field $e^{2\alpha_r\phi(z_r,\bar{z}_r)}$ is $\Delta_r \equiv \Delta_{\alpha_r} := \alpha_r(Q - \alpha_r)$.

The correlation functions can be represented in a holomorphically factorized form

$$\left\langle\!\left\langle\, e^{2\alpha_n\phi(z_n,\bar{z}_n)} \cdots e^{2\alpha_1\phi(z_1,\bar{z}_1)} \,\right\rangle\!\right\rangle_{C_q} = \int d\mu(a)\, |\mathcal{F}_q^\sigma(a)|^2 .$$

The conformal blocks

$$\mathcal{F}_q^\sigma(a) \equiv \left\langle\, e^{2\alpha_n\phi(z_n)} \cdots e^{2\alpha_1\phi(z_1)} \,\right\rangle_a$$

are objects that are defined from the representation theory of the Virasoro algebra.

Liouville theory II

Consider insertions of degenerate fields like

$$\left\langle\, \mathcal{O}_{n,l} \,\right\rangle_a \equiv \left\langle\, e^{2\alpha_n\phi(z_n)} \cdots e^{2\alpha_1\phi(z_1)}\, e^{-\frac{1}{b}\phi(y_l)} \cdots e^{-\frac{1}{b}\phi(y_1)} \,\right\rangle_a$$

The conformal blocks satisfy the BPZ equations

$$\mathcal{D}^{\mathrm{BPZ}}_{y_k} \cdot \left\langle\, \mathcal{O}_{n,l} \,\right\rangle \,=\, 0\,, \qquad \forall\, k = 1, \ldots, l\,,$$

with differential operators $\mathcal{D}^{\mathrm{BPZ}}_{y_k}$ being for $g = 0$ given as

$$\mathcal{D}^{\mathrm{BPZ}}_{y_k} = b^2 \frac{\partial^2}{\partial y_k^2} + \sum_{r=1}^{n} \left(\frac{\Delta_r}{(y_k - z_r)^2} + \frac{1}{y_k - z_r}\frac{\partial}{\partial z_r} \right) - \sum_{\substack{k'=1 \\ k' \neq k}}^{l} \left(\frac{3b^{-2} + 2}{4(y_k - y_{k'})^2} - \frac{1}{y_k - y_{k'}}\frac{\partial}{\partial y_{k'}} \right).$$

Solutions to KZ equations from Liouville theory

Claim: (Stoyanovsky; Ribault, J.T.)

Ansatz $\tilde{\mathcal{G}}(\mu|z) := u\,\delta\big(\sum_{i=1}^{n}\mu_i\big)\,\Theta_n(y|z)\,\mathcal{F}(y|z)$, yields a solution to the KZ-equations

$$(k+2)\frac{\partial}{\partial z_r}\Phi(x|z) \;=\; \mathsf{H}_r\,\Phi(x|z)\,.$$

from any given solution $\mathcal{F}(y|z)$ to the BPZ-equations.

- $\Theta_n(y|z)$ is defined as

$$\Theta_n(y|z) \;=\; \prod_{r<s\leq n} z_{rs}^{\frac{1}{2b^2}} \prod_{k<l\leq n-2} y_{kl}^{\frac{1}{2b^2}} \prod_{r=1}^{n}\prod_{k=1}^{n-2}(z_r-y_k)^{-\frac{1}{2b^2}}. \tag{1}$$

- Variables $\mu_1,\dots,\mu_n$ related to $y_1,\dots,y_{n-2},u$ via $\displaystyle\sum_{r=1}^{n}\frac{\mu_r}{t-z_r} = u\frac{\prod_{k=1}^{n-2}(t-y_k)}{\prod_{r=1}^{n}(t-z_r)}$.

- Parameters are related via $b^2=-(k+2)^{-1}$, $\alpha_r\equiv\alpha(j_r):=b(j_r+1)+b^{-1}$.

Critical level limit

Consider $\tilde{\mathcal{G}}(\mu|z) := u\,\delta\big(\sum_{i=1}^n \mu_i\big)\,\Theta_n(y|z)\,\mathcal{F}(y|z)$, for $k \to -2 \Leftrightarrow b \to \infty$. On the one hand note that one may solve

$$(k+2)\frac{\partial}{\partial z_r}\mathcal{G}(x|z) \;=\; \mathsf{H}_r\,\mathcal{G}(x|z)\,.$$

in the form

$$\mathcal{G}(x|z) \;\sim\; \exp(-b^2 S(z))\Psi(\mu|z)(1+\mathcal{O}(b^{-2}))\,,$$

provided that $\Psi(x|z)$ is a solution to the Gaudin eigenvalue equations $\mathsf{H}_r\Psi = E_r\Psi$ with E_r given in terms of $S(z)$ by $E_r = -\partial_{z_r}S(z)$.

On the other hand: **Modular duality of Liouville theory** (invariance under $b \to b^{-1}$) $\Rightarrow$ limit $b \to \infty$ is equivalent to *classical* limit:

$$\mathcal{F}(y|z) \;\sim\; \exp(-b^2 S(z))\prod_{k=1}^{n-2}\psi_k(y_k)\,, \quad \text{where} \quad (\partial_y^2 + t(y)\psi_k(y) = 0.$$

$$\boxed{\textbf{Geometric Langlands correspondence}}$$

Quantization conditions from Yang's potential

It is not easy to define quantization conditions for the Hitchin systems.

Proposal (`Nekrasov-Shatashvili; cf. K.Kozlowski, J.T. for Toda-example`):

For algebraically integrable systems it is natural to formulate the quantization conditions in terms of Yang's potential $\mathcal{W}_C(a)$,

- Function $\mathcal{W}_C : \mathcal{B} \to \mathbb{C}$, $\mathcal{B}$: Space of eigenvalue of the Hamiltonians,
- $a = (a_1, \dots, a_N)$, a "special" system of coordinates on $\mathcal{B}$.

Quantization conditions can then be formulated as

$$\frac{\partial}{\partial a_r}\mathcal{W}_C(a) = 2\pi i n_r, \qquad r = 1, \dots, N\,.$$

For $SL(2,\mathbb{C})$-Gaudin/Hitchin it is also natural to consider **complex** quantization where

$$\mathrm{Re}(a_r) = \pi i m_r, \qquad \frac{\partial}{\partial a_r}\mathrm{Re}(\mathcal{W}_C(a)) = \pi i n_r, \qquad r = 1, \dots, N\,.$$

Yang's potential for Hitchin systems

Define $H_r(a,z)$ as the accessory parameters which gives the oper P the monodromy $\rho_P : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ such that

$$2\cosh\frac{a_r}{2} \;:=\; \mathrm{tr}(\rho_P(\gamma_r))\,,$$

for curves $\gamma_1,\dots,\gamma_{n-3}$ that constitute a cut system. We claim that the function $\mathcal{W}(a,z)$ which does the job can be defined by the equations

$$H_r(a,z) \;=\; -\frac{\partial}{\partial z_r}\mathcal{W}(a,z)\,,$$

with H_r defined by the expansion $q(y) = \sum_{r=1}^{n}\left(\frac{\delta_r}{(y-z_r)^2} + \frac{\mathsf{H}_r}{y-z_r}\right)$ for $g=0$. The integrability condition is

$$\frac{\partial}{\partial z_r}H_s \;=\; \frac{\partial}{\partial z_s}H_r\,.$$

This follows from $\mathcal{W}(a,z)$: **Semiclassical Liouville conformal block**. Note

$$\mathcal{W}(a,z) \;\equiv\; \mathcal{W}_{\epsilon_1}(a,z) \;=\; \lim_{\epsilon_2\to 0}\epsilon_2\mathcal{Z}_C(a;\epsilon_1,\epsilon_2)\,, \qquad \epsilon_1 \equiv \hbar\,.$$

This is part of a larger picture [arXiv 1005.2846]

$$\begin{tikzcd}
 & \text{Hitchin system} \arrow[dl, "(\mathrm{A})_{\epsilon_2}"'] \arrow[dr, "(\mathrm{B})_{\epsilon_1}"] & \\
\begin{array}{c}\text{Isomonodromic}\\ \text{deformations}\end{array} \arrow[dr, "(\mathrm{C})_{\epsilon_1}"'] & & \begin{array}{c}\text{quantized Hitchin}\\ \text{systems}\end{array} \arrow[dl, "(\mathrm{D})_{\epsilon_2}"] \\
 & \begin{array}{c}\text{Liouville theory}\\ \text{WZNW-model}\end{array} &
\end{tikzcd}$$

where the arrows may be schematically characterized as follows:

(A) Hyperkähler rotation within the Hitchin moduli space $\mathcal{M}_{\mathrm{H}}(C)$.

(B) Quantization [Beilinson-Drinfeld], [Nekrasov-Shatashvili]

(C) Quantization [arXiv 1005.2846]

(D) This arrow may be called *quantum* hyperkähler rotation.