A Quantum Field Theory Model of Archimedean Geometry

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Based on a series of papers: AG, Dimitry Lebedev and Sergey Oblezin [GLO]

- *On q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions I,II,III*, Comm. Math. Phys. 294 (2010), 97–119, [math.RT/0803.0145]; Comm. Math. Phys. 294 (2010), 121–143, [math.RT/0803.0970]; [math.RT/0805.3754].
- *Archimedean L-factors and Topological Field Theories I*, [math.NT/0906.1065].
- *Archimedean L-factors and Topological Field Theories II*, [hep-th/0909.2016].
- *Parabolic Whittaker functions and Topological Field Theories I*, [math.AG/0057862].

1. Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1-p^{-s}}, \qquad \mathrm{Re}(s) > 1$$

Functional equation (B. Riemann)

$$\zeta^*(s) = \zeta^*(1-s), \qquad \zeta^*(s) = \zeta(s)\pi^{-s/2}\Gamma(\frac{s}{2})$$

Product formula for completed zeta-function

$$\zeta^*(s) = \prod_{p \in \mathcal{P} \cup \infty} \zeta_p(s)$$

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma(\frac{s}{2}), \qquad \zeta_p(s) = \frac{1}{1-p^{-s}}, \qquad p \neq \infty$$

2. Meaning of the product formula for $\zeta^*(s)$ after A. Weil

Exponential valuation (norm) $|\ | : K \to \mathbb{R}_+$

- $|x\,y| = |x|\,|y|$
- $|x| = 0 \leftrightarrow x = 0$
- $|x + y| \leq |x| + |y|$, *Archimedean*
  $|x + y| \leq max(|x|, |y|)$ *non − Archimedean*

Norms for $\mathbb{Z}$:

- non-Archimedean: for each prime p

$$|a|_p = p^{-n} \quad \text{iff} \quad a = p^n a_0, \qquad (p, a_0) = 1$$

- Archimedean:

$$|a|_\infty = |a|$$

3. Completions

Each norm map defines a completion of $\mathbb{Q}$:

$$|\ |_\infty \longrightarrow \mathbb{Q} \subset \mathbb{R}, \qquad |\ |_p \longrightarrow \mathbb{Q} \subset \mathbb{Q}_p$$

p-adic numbers:

$$a = a_0 + a_1 p + a_2 p^2 + \cdots \ \in \mathbb{Z}_p,$$
$$a = p^{-n}(a_0 + a_1 p + a_2 p^2 + \cdots) \ \in \mathbb{Q}_p$$

where $a_i \in \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$.

4. Rational numbers $\mathbb{Q}$ are rational functions on $\mathrm{Spec}(\mathbb{Z}) = \mathcal{P}$.

Adding the Archimedean norm $|\ |_\infty$ provides a "compactification" of $\mathrm{Spec}(\mathbb{Z})$

$$\overline{\mathrm{Spec}(\mathbb{Z})} = \mathcal{P} \cup (\infty)$$

Product formula

$$|a|_\infty \cdot \prod_p |a|_p = 1, \quad a \in \mathbb{Q}$$

is an analog of

$$\prod_{a \in \Sigma} \exp\big(t\, Res_{z=a} d \log f(z)\big) = 1, \qquad t \in \mathbb{C}$$

for rational functions on a **compact** surface Σ

5. Reformulation in terms of norms

$$\zeta^*(s) = \prod_{p \in \overline{\mathrm{Spec}(\mathbb{Z})}} \zeta_p(s)$$

where local zeta-functions are given by

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2}), \qquad \zeta_{\mathbb{Q}_p}(s) = \frac{1}{1-p^{-s}}, \qquad p \neq \infty$$

Additional part $\pi^{-s/2}\Gamma(\frac{s}{2})$ *is related with real numbers.*

Interpretation of local *zeta*-functions? Look at a generalization - *L*-functions (ζ-functions with non-trivial coefficients).

6. L-function is given by a product of local factor for each prime p

$$L(s|\{A_p\}) = \prod_p{}' L_p(s, A_p) = \prod_p{}' \det{}_V (1 - A_p\, p^{-s})^{-1},$$

where $A_p \in GL(V)$. Under some conditions on $\{A_p\}$ completed L-function

$$\Lambda(s|\{A_p\}, \alpha_\infty) = L(s|\{A_p\}) L_\infty(s, \alpha_\infty),$$

$$L_\infty(s, \alpha_\infty) = \det{}_V\ \pi^{-\frac{s-\alpha_\infty}{2}}\, \Gamma\Big(\frac{s - \alpha_\infty}{2}\Big), \qquad \alpha_\infty \in Mat(V),$$

satisfies a functional equation

$$\Lambda(1 - s|\{A_p\}, \alpha_\infty) = \epsilon(s) \Lambda(s|\{A_p\}, \alpha_\infty),$$

where ϵ-factor is of the form $\epsilon(s) = A\, B^s$.

7. Arithmetic Langlands duality is an equivalence of two ways to produce the data $(\{A_p\}, \alpha_\infty)$

I. Infinite-dimensional irreducible spherical representations of Lie groups $G(\mathbb{A})$ over ring of adeles = characters of spherical Hecke algebras $\mathcal{H}(G(\mathbb{A}), K)$.

II. Homomorphism of the Weil group $W_{\mathbb{Q}} \to {}^L G$.

Picking an finite-dimensional representation $\phi : {}^L G \to GL(V)$ we obtain $(\{A_p\}, \alpha_\infty)$.

${}^L G$ is an extension of the dual group $G^\vee$ by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (A_ℓ, B_ℓ, C_ℓ, D_ℓ are dual to A_ℓ, C_ℓ, B_ℓ, D_ℓ).

The Weil group $W_{\mathbb{Q}}$ is a version of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

8. Local data in non-Archimedean case I (Rep. theory)

Representations are realized in a space of functions on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. Commutative Hecke algebra

$$\mathcal{H}_p = \mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p)) \sim Fun(G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p))$$

of compactly supported $G(\mathbb{Z}_p)$-biinvariant functions acts from the right. For spherical representations the corresponding representation of $\mathcal{H}_p$ is one-dimensional i.e. is given by a multiplicative character of $\mathcal{H}_p$.

$\mathcal{H}_p$ is isomorphic to a representation ring of a dual complex Lie group $G^\vee$. Character of $\mathcal{H}_p$ are parametrized by conjugacy classes $g_p^{(1)}$ in $G^\vee$. Given a homomorphism $\phi : G^\vee \to GL(V)$ and a conjugacy class $g_p^{(1)}$ in $G^\vee$ we take

$$A_p = \phi(g_p^{(1)})$$

9. Reformulation

There is a generating function $\mathcal{Q}_p(s)$ of elements of $\mathcal{H}_P$ acting in an irreducible representation $\mathcal{V}$ of $G(\mathbb{Q}_p)$ by multiplication on the corresponding local non-Archimedean L-factor

$$\mathcal{Q}_p(s)\,\Psi = L_p(s)\,\Psi, \qquad \Psi \in \mathcal{V}$$

In representation theory local non-Archimedean L-factors (invariants of representations) naturally arise in an integral form (i.e. as periods) generalizing the identity

$$\frac{1}{1-p^{-s}} = \int_{\mathbb{Q}_p} d\mu_p(x)\psi(x)|x|_p^s$$

where $\psi(x+y) = \psi(x)\psi(y)$ is an additive character.

10. Local data in non-Archimedean case II(Arithmetic)

Homomorphisms of the Weil group $W_{\mathbb{Q}_p} \to {}^L G$. $W_{\mathbb{Q}_p}$ is close to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and ${}^L G$ is close to dual group $G^\vee$. Naively we should look at "homomorphism" $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to G^\vee$.

Consider homomorphisms of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ factored through the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ of the residue field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is generated by Frobenius homomorphism $Fr_p : x \to x^p$.

The image of Fr_p is a conjugacy class $g_p^{(2)}$ in $G^\vee$. Given a representation $\phi : G^\vee \to GL(V)$ we take

$$A_p = \phi(g_p^{(2)}), \qquad L_p(s, A_p) = \prod_p{}' \det{}_V (1 - A_p\, p^{-s})^{-1},$$

11. General duality relation: Period=Trace

Period representation is characteristic to Representation theory side (Construction I). Local non-Archimedean L-factors in this construction are naturally integrals e.g.

$$\frac{1}{1-p^{-s}} = \int_{\mathbb{Q}_p} d\mu_p(x)\psi(x)|x|_p^s$$

Trace representation is characteristic to Arithmetic side (Construction II). Local non-Archimedean L-factors in this construction are naturally traces e.g.

$$\frac{1}{\det{}_V(1-A)} = \mathrm{Tr}_{\oplus S^* V} A$$

12. Whittaker function

Most clearly the duality *Period=Trace* can be seen on the level of Whittaker functions.

Whittaker function is a matrix element of an infinite-dimensional representation $\pi_\lambda : G \to End(\mathcal{V})$ of a Lie group G

$$\Psi_\lambda(g) =< \psi_L| \, \pi_\lambda(g) \, |\phi\rangle, \qquad g \in G$$

such that

$$\Psi_\lambda(ngk) = \chi_L(n) \, \Psi_\lambda(g), \qquad n \in N, \quad k \in K$$

K - maximal compact subgroup,
N - maximal unipotent subgroup,
χ_L - character of N.

13. Properties of Whittaker functions

1. Whittaker function $\Psi_\lambda(g)$ reduces to a function $\Psi_\lambda(a)$ on a factor $A = N\backslash G/K$ (in split case A is a diagonal subgroup).

2. Irreducibility of the representation $\pi_\lambda : G \to End(\mathcal{V})$ leads to a system of difference/differential equations on $\Psi_\lambda(g)$

$$\mathcal{H}_r \Psi_\lambda(a) = c_r(\lambda)\Psi_\lambda(a)$$

This is a quantum integrable system of Toda type (open Toda chain for $G(\mathbb{R})$).

3. Whittaker functions have integral representations arising from explicit realizations of the pairing in representation $\pi_\lambda : G \to End(\mathcal{V})$ i.e. Whittaker function is naturally a period.

14. Shintani-Casselman-Shalika formula (Whittaker function as a trace)

Whittaker function for $G(\mathbb{Q}_p)$ can be expressed as a character of a finite-dimensional representation of the Langlands dual group $G^\vee$ (non-Archimedean Langlands duality on the level of Whittaker functions).

$G(\mathbb{Q}_p) = GL(\ell+1, \mathbb{Q}_p)$ and $\mathcal{V}_{\gamma_1,\ldots,\gamma_{\ell+1}}$ is a representation induced from a character $\chi^B_{(p^{\gamma_1},\ldots,p^{\gamma_{\ell+1}})}(g) = \prod_{j=1}^{\ell+1} |g_{jj}|^{\gamma_j}$ of the Borel subgroup $B \subset GL(\ell+1, \mathbb{Q}_p)$.

$V_{(n_1,n_2,\ldots,n_{\ell+1})}$ is a f.d.i. representation of $GL(\ell+1, \mathbb{C})$ corresponding to a partition $(n_1 \geq n_2 \geq \ldots \geq n_{\ell+1})$

$$\Psi_{(\gamma_1,\ldots,\gamma_{\ell+1})}(\mathrm{diag}(p^{n_1},\ldots,p^{n_{\ell+1}})) = \mathrm{Tr}_{V_{(n_1,\ldots,n_{\ell+1})}}\,\mathrm{diag}(p^{\gamma_1},\ldots,p^{\gamma_{\ell+1}})$$

15. Local data in Archimedean case I (Rep. theory)

For Archimedean place the Hecke algebra is usually replaced by the ring of invariant differential operators on $G(\mathbb{R})$. Recently the Archimedean analog $\mathcal{Q}_\infty(s)$ of the generating function $\mathcal{Q}_p(s)$ of elements of local non-Archimedean Hecke algebra $\mathcal{H}(G,K)$ was constructed [GLO] such that

$$\mathcal{Q}_\infty(s)\,\Psi = L_\infty(s,\alpha_\infty)\,\Psi, \qquad \Psi\in\mathcal{V}$$

where the eigenvalue is the local Archimedean L-factor

$$L_\infty(s,\alpha_\infty)=\det{}_V\,\pi^{-\frac{s-\alpha_\infty}{2}}\,\Gamma\Big(\frac{s-\alpha_\infty}{2}\Big),$$

The construction is based on the Baxter operator and is close to the constructions due to M. Gaudin and V. Pasquier.

Thus, all works as in non-Archimedean case.

16. Local data in Archimedean case II (Arithmetic)

The Weil group $W_{\mathbb{R}}$ is generated by $\mathbb{C}^*$ and an element j :

$$jxj^{-1} = \overline{x}, \qquad j^2 = -1 \in \mathbb{C}^*,$$

The datum to construct a local Archimedean L-factor is basically a homomorphism $\mathbb{C}^* \to G^\vee$.

Weil group is much larger then the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$*!!*

$\mathbb{C}^*$ looks like a "missing part" of the Archimedean Galois group e.g. there is a canonical action of $\mathbb{C}^*$ on the complexified cohomology of compact non-singular algebraic varieties over $\mathbb{C}$ providing Hodge decomposition. This action is similar to an action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on the étale cohomology of schemes over $\overline{\mathbb{F}}_p$.

17. Main problem - to understand missing part of Archimedean Galois group

There is a representation of local Archimedean L-factors as a period via Euler integral representation of Γ-function

$$\Gamma(s) = \int_{-\infty}^{+\infty} dx\, e^{xs}\, e^{-e^{x}}$$

What is an Archimedean analog of the representation of local L-factor as a trace?

There is a representation of Archimedean Whittaker functions as a period via realization of matrix element as an integral pairing of functions on G-homogeneous spaces.

What is an Archimedean analog of the Shintani-Casselman -Shalika formula?

18. q-Interpolation of Archimedean and non-Archimedean constructions

Gamma-functions:

$$\Gamma_q(s,t) = \frac{(q;q)_\infty}{(t^s;q)_\infty}, \qquad (a,q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1$$

"Classical" limit:

$$\pi^{-1/2}(\pi^{-1}\ln q)^{(s-1)/2}\Gamma_q(s,t) \to \pi^{-s/2}\Gamma(s/2), \qquad q\to 1,\ t=q^{1/2}$$

i.e. local Archimedean zeta-function.

"p-adic" limit:

$$\Gamma_q(s,t) \to \frac{1}{1-p^{-s}}, \qquad q\to 0,\ t\to p^{-1}$$

i.e. local non-Archimedean zeta-function.

19. q-deformed Whittaker function

Replace universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ by a quantum deformation $\mathcal{U}_q(\mathfrak{g})$ (or affine Kac-Moody algebra with $q = \exp(2\pi\imath/k + h^\vee))$. The resulting q-deformed Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_1,\dots,p_{\ell+1})$ is a common eigenfunction of a system of mutually commuting *difference* equations

$$\mathcal{H}_2(p_1,\dots,p_{\ell+1}) = T_1 + \sum_{i=1}^{\ell}(1 - q^{p_{i+1}-p_i+1})\,T_{i+1},$$

where $T_i f(p_j) = f(p_j + \delta_{ij})$. The resulting quantum integrable system is q-deformed Toda chain.

20. Theorem[GLO] Function given by

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_1,\dots,p_{\ell+1}) \;=\; \sum_{p_{k,i}\in\mathcal{P}^{(\ell+1)}} \; \prod_{k=1}^{\ell+1} \; z_k^{\sum_{i=1}^{k} p_{k,i}-\sum_{i=1}^{k-1} p_{k-1,i}}$$

$$\times \frac{\prod\limits_{k=2}^{\ell}\prod\limits_{i=1}^{k-1}(p_{k,i+1}-p_{k,i})_q!}{\prod\limits_{k=1}^{\ell}\prod\limits_{i=1}^{k}(p_{k+1,i+1}-p_{k,i})_q!\;(p_{k,i}-p_{k+1,i})_q!}$$

for $p_1 \leq \cdots \leq p_{\ell+1}$ and zero otherwise is a solution of the eigenfunction problem for q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. Here $(n)_q! = (1-q)...(1-q^n)$ and $\mathcal{P}^{(\ell+1)}$ is a set of Gelfand-Zetlin tableaux with a fixed top raw

$$(p_{\ell+1,1},\cdots,p_{\ell+1,\ell+1}) := (p_1,\cdots,p_{\ell+1})$$

21. q-version of Shintani-Casselman-Shalika formula

Theorem[GLO] The common eigenfunction of q-deformed Toda chain allows the following trace representation:

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(p_1,\dots,p_{\ell+1}) \,=\, \mathrm{Tr}_{V_{p_1,\dots,p_{\ell+1}}}\, q^d \prod_{i=1}^{\ell+1} q^{\lambda_i\, E_{i,i}},$$

where $V_{p_1,\dots,p_{\ell+1}}$ is a $\mathbb{C}^* \times GL(\ell+1,\mathbb{C})$-module, $E_{i,i}$, $i = 1,\dots \ell+1$ are Cartan generators of $\mathfrak{gl}_{\ell+1} = \mathrm{Lie}(GL(\ell+1,\mathbb{C}))$ and d is a generator of $\mathrm{Lie}(\mathbb{C}^*)$.

This q-version of SCS formula reduces to the classical non-Archimedean SCS formula for $q \to 0$ and, in the limit $q \to 1$, produces a substitute for SCS-formula in the Archimedean case.

Topological Field Theory as a proper framework for Archimedean geometry

Local Archimedean L-factors allow two constructions as correlation functions in two-dimensional topological sigma models:

- $S^1 \times U_{\ell+1}$-equivariant Type A topological linear sigma model on disk D.
- S^1-equivariant Type B topological Landau-Ginzburg model on D with a superpotential W.

23. Local Archimedean Langlands duality=Mirror duality in TFT

Two TFT representations correspond to two types of representations for local L-factors:

- In Type A TFT local Archimedean L-factors are given by equivariant symplectic volumes of spaces of holomorphic maps of D into $V=\mathbb{C}^{\ell+1}$ (Arithmetic side).
- In Type B TFT local Archimedean L-factors are given by periods of holomorphic forms over middle-dimensional cycles (Representation theory side)

This is an Archimedean analog of "Period=Trace" relation for non-Archimedean case (trace is replaced by its classical limit - equivariant volume of the underlying symplectic manifold).

24. Setup

World-sheet: $D = \{z|\ |z| \leq 1\}$ with a flat metric

$$h = \frac{1}{2}(dz d\bar{z} + d\bar{z}\, dz) = (dr)^2 + r^2 (d\sigma)^2, \qquad z = re^{\imath\sigma}$$

Lie group S^1 acts by rotations on D.
Target space: $\mathbb{C}^{\ell+1}$ supplied with the Kähler form and the Kähler metric

$$\omega = \frac{\imath}{2} \sum_{j=1}^{\ell+1} d\varphi^j \wedge d\bar{\varphi}^j, \qquad g = \frac{1}{2} \sum_{j=1}^{\ell+1} (d\varphi^j \otimes d\bar{\varphi}^j + d\bar{\varphi}^j \otimes d\varphi^j).$$

Lie group $U_{\ell+1}$ acts on $\mathbb{C}^{\ell+1}$ via standard representation.

25. Type A topological sigma model=Fields+BRST+Action

K and $\bar{K}$ - canonical and anti-canonical bundles on world-sheet.
$T_{\mathbb{C}}X = T^{1,0} \oplus T^{0,1}$ - decomposition of the complexified tangent bundle of target space $X = \mathbb{C}^{\ell+1}$.

Commuting fields:

φ, $\bar{\varphi}$- describe maps $\Phi : D \to X$.
F, $\bar{F}$ - sections of $K \otimes \Phi^*(T^{0,1})$, $\bar{K} \otimes \Phi^*(T^{1,0})$.

Anticommuting fields:

χ, $\bar{\chi}$ - sections of $\Phi^*(\Pi T^{1,0})$, $\Phi^*(\Pi T^{0,1})$
ψ, $\bar{\psi}$ - sections of $K \otimes \Phi^*(\Pi T^{0,1})$, $\bar{K} \otimes \Phi^*(\Pi T^{1,0})$.
Metrics g and h induce a Hermitian paring $\langle \, , \, \rangle$

$$\langle \chi, \chi \rangle = \sum_{j=1}^{\ell+1} g_{i\bar{j}} \, \bar{\chi}^{\bar{j}} \, \chi^i, \qquad \langle F, F \rangle = \sum_{j=1}^{\ell+1} h^{z\bar{z}} g_{i\bar{j}} \bar{F}^{\bar{j}}_z \, F^i_{\bar{z}}.$$

26. $S^1 \times U_{\ell+1}$-equivariant BRST transformations

$$\delta_G \varphi = \chi, \qquad \delta_G \chi = -(\imath \Lambda \varphi + \hbar\, \mathcal{L}_{v_0} \varphi)$$

$$\delta_G \psi = F, \qquad \delta_G F = -(\imath \Lambda \psi + \hbar\, \mathcal{L}_{v_0} \psi)$$

Λ - an element of $\mathrm{Lie}(U_{\ell+1})$
$v_0 = \frac{\partial}{\partial \sigma}$ is a generator of $\mathrm{Lie}(S^1)$ and $\mathcal{L}_{v_0} = d\, i_{v_0} + i_{v_0}\, d$ is the Lie derivative
Equivariant BRST operator satisfies

$$\delta_G^2 = -(\text{inf. symmetry transformation})$$

27. Action functional

$$S_D = \int_D d^2z\, \delta_G(\imath\langle\psi, \overline{\partial}\varphi\rangle + \imath\langle\bar{\psi}, \partial\bar{\varphi}\rangle) =$$

$$\imath \int_D d^2z\, \Big(\langle F, \overline{\partial}\varphi\rangle + \langle \bar{F}, \partial\bar{\varphi}\rangle + \langle\bar{\psi}, \partial\bar{\chi}\rangle + \langle\psi, \overline{\partial}\chi\rangle\Big),$$

δ_G-invariant observable:

$$\mathcal{O} = \frac{\imath}{\pi}\int_0^{2\pi} d\sigma\, (-\langle\chi(e^{\imath\sigma}), \chi(e^{\imath\sigma})\rangle + \langle\varphi(e^{\imath\sigma}), (\imath\Lambda + \hbar\mathcal{L}_{v_0})\varphi(e^{\imath\sigma})\rangle)$$

28. Theorem A [GLO] In $S^1 \times U_{\ell+1}$-equivariant Type A topological linear sigma model with the target space $V = \mathbb{C}^{\ell+1}$ one has the following representation for correlation function of $\exp(\mathcal{O})$:

$$\left\langle e^{\mathcal{O}} \right\rangle_D = \hbar^{-\frac{\ell+1}{2}} \det{}_V \left(\frac{\pi}{\hbar}\right)^{-\Lambda/\hbar} \Gamma(\Lambda/\hbar),$$

By taking $\hbar = 1$ and changing the variables $\Lambda \to (s \cdot \mathrm{id} - \Lambda)/2$ the correlation function turns into local Archimedean L-factor.

Left hand side is an integral over space of symplectic space of holomorphic maps $D \to \mathbb{C}^{\ell+1}$ and is given by inverse infinite-dimensional determinant.

29. Type B topological Landau-Ginzburg theory

Type B topological sigma model associated with a pair $(\mathbb{C}^{\ell+1}, W)$, $W \in H^0(\mathbb{C}^{\ell+1}, \mathcal{O})$.

Commuting fields:
ϕ, $\bar{\phi}$ - describe maps $\Phi: D \to \mathbb{C}^{\ell+1}$
$\bar{G}$, G - sections of $\Phi^*(T^{0,1})$, $K \otimes \bar{K} \otimes \Phi^*(T^{1,0})$

Anticommuting fields:
η, θ - sections of $\Phi^*(\Pi T^{0,1})$
ρ - sections of $(K \oplus \bar{K}) \otimes \Phi^*(\Pi T^{1,0})$

30. Real structure

Topological linear sigma model allows a non-standard real structure

$$(\phi^i)^\dagger = \phi^i, \qquad (\bar{\phi}^i)^\dagger = -\bar{\phi}^i, \qquad (\theta_i)^\dagger = -\theta_i,$$

$$(\bar{\eta}^i)^\dagger = -\bar{\eta}^i, \qquad (\rho^i)^\dagger = \rho^i, \qquad (G^i)^\dagger = G^i, \qquad (\bar{G}^i)^\dagger = -\bar{G}^i.$$

Remark. This real structure is imposed by the condition on Type B topological sigma model to be a mirror dual to the Type A topological sigma model discussed previously.

31. S^1-equivariant BRST transformations

$$\delta_{S^1}\phi^i_- = \eta^i, \qquad \delta_{S^1}\eta^i = \hbar\iota_{v_0} d\phi^i_-,$$
$$\delta_{S^1}\theta^i = G^i_-, \qquad \delta_{S^1} G^i_- = \hbar\iota_{v_0} d\theta^i,$$
$$\delta_{S^1}\rho^i = -d\phi^i_+ - \hbar\iota_{v_0} G^i_+, \qquad \delta_{S^1}\phi^i_+ = \hbar\iota_{v_0}\rho^i, \qquad \delta_{S^1} G^i_+ = d\,\rho^i.$$

δ_{S^1}-**invariant observable**:

$$\mathcal{O} = \prod_{i=1}^{\ell+1} \delta(\phi^i_-(0))\,\eta^i(0)$$

32. Action functional

$$S = -\imath \sum_{j=1}^{\ell+1} \int_D \Big((d\phi_+^j + \hbar \iota_{v_0} G_+^j) \wedge *d\phi_-^j + \rho^j \wedge *d\eta^j - \theta_j d\rho^j$$

$$+ G_+^j G_-^j \Big) + \sum_{i,j=1}^{\ell+1} \int_D d^2 z \sqrt{h} \left(- \frac{\partial^2 W_-(\phi_-)}{\partial \phi_-^i \partial \phi_-^j} \eta^i \theta^j - \imath \frac{\partial W_-(\phi_-)}{\partial \phi_-^i} \, G_-^i \right)$$

$$+ \sum_{i,j=1}^{\ell+1} \int_D \left(-\frac{1}{2} \frac{\partial^2 W_+(\phi_+)}{\partial \phi_+^i \partial \phi_+^j} \rho^i \wedge \rho^j + \frac{\partial W_+(\phi_+)}{\partial \phi_+^i} \, G_+^i \right)$$

$$- \frac{1}{\hbar} \int_{S^1 = \partial D} d\sigma W_+(\phi_+).$$

where W_+ and W_- are arbitrary independent regular functions on $\mathbb{R}^{\ell+1}$.

33. Theorem B [GLO] The correlation function of $\exp\mathcal{O}$ in the type B topological S^1-equivariant Landau-Ginzburg sigma model with

$$W_+(\phi_+) = \sum_{j=1}^{\ell+1}(\lambda_j\phi_+^j - e^{\phi_+^j}), \qquad W_-(\phi_-) = 0,$$

is given by

$$\langle\mathcal{O}_*\rangle = \int_{\mathbb{R}^{\ell+1}} \prod_{j=1}^{\ell+1} dt^j \quad e^{\frac{1}{\hbar}\sum_{j=1}^{\ell+1}(\lambda_j t^j - e^{t^j})} = \prod_{j=1}^{\ell+1} \hbar^{\frac{\lambda_j}{\hbar}}\,\Gamma\left(\frac{\lambda_j}{\hbar}\right).$$

This coincides with the correlation function calculated in Type A TFT. The reason - considered Type A and Type B TFT are mirror dual.

34. Direct derivation of mirror symmetry

Two dual integral representations for Γ-function:

- Type A TFT - equivariant volume of the space $\mathcal{M}(D,\mathbb{C})$ of holomorphic maps $D \to \mathbb{C}$ (essentially infinite-dimensional)
- Type B TFT - Euler finite-dimensional integral representation

$$\Gamma(s) = \int_{-\infty}^{+\infty} dx\, e^{xs}\, e^{-e^{x}}$$

Proof of the mirror symmetry by directly calculating an integrand of the Euler integral representation in Type A TFT.

35. $S^1 \times U(1)$-equivariant volume of the space of holomorphic maps $\mathcal{M}(D, \mathbb{C})$ (correlation function in Type A TFT):

$$Z(\lambda, \hbar) = \int_{\Pi\mathcal{M}(D,\mathbb{C})} e^{\lambda H_{U(1)} + \hbar H_{S^1} + \Omega}$$

$S^1 \times U(1)$-invariant symplectic form on the space $\mathcal{M}(D, \mathbb{C})$:

$$\Omega = \frac{\imath}{4\pi} \int_0^{2\pi} \delta\varphi(\sigma) \wedge \delta\bar{\varphi}(\sigma)\, d\sigma, \qquad \varphi(\sigma) = \varphi(z)|_{\partial D = S^1}$$

Hamiltonian momenta for $S^1 \times U(1)$

$$H_{S^1} = -\frac{\imath}{4\pi} \int_0^{2\pi} \bar{\varphi}(\sigma)\partial_\sigma \varphi(\sigma)\; d\sigma, \qquad H_{U(1)} = \frac{1}{4\pi} \int_0^{2\pi} |\varphi(\sigma)|^2\; d\sigma.$$

36. Rewrite the integral as follows:

$$Z(\lambda, \hbar) = \int_{-\infty}^{+\infty} dt\, e^{\lambda t}\, Z_t(\hbar),$$

$$Z_t(\hbar) = \int_{\mathcal{M}(D,\mathbb{C})} e^{\hbar H_{S^1} + \Omega}\ \delta(t - H_{U(1)}).$$

$Z_t(\hbar)$ is an equivariant symplectic volume of $\mathbb{P}\mathcal{M}(D, \mathbb{C})$

$$Z_t(\hbar) = 2\pi \int_{\mathbb{P}\mathcal{M}(D,\mathbb{C})} e^{\hbar \tilde{H}_{S^1} + \tilde{\Omega}(t)},$$

To derive the Euler integral representation of Γ-function one shall prove:

$$Z_t(\hbar) \sim e^{e^{-\hbar t}} \qquad ????$$

37. Duistermaat-Heckman formula

Let (M,Ω) be a $2N$-dimensional symplectic manifold with the Hamiltonian action of S^1 having only isolated fixed points. Let H_{S^1} be the corresponding momentum. The tangent space $T_{p_k}M$ to a fixed point $p_k \in M^{S^1}$ has a natural action of S^1. Let v be a generator of $\mathrm{Lie}(S^1)$ and let $\hat{v}$ be its action on $T_{p_k}M$

$$\int_M e^{\hbar H_{S^1}+\Omega} = \sum_{p_k \in M^{S^1}} \frac{e^{\hbar H_{S^1}(p_k)}}{\det_{T_{p_k}M} \hbar\hat{v}/2\pi}$$

38. Fixed points of S^1 acting on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$ are given in homogeneous coordinates by

$$\varphi^{(n)}(z) = \varphi_n z^n, \qquad \varphi_n \in \mathbb{C}^* \quad n \in \mathbb{Z}_{\geq 0}.$$

The tangent space to $\mathcal{M}(D,\mathbb{C})$ at an S^1-fixed point $\varphi^{(n)}$ has natural linear coordinates φ_m/φ_n, $m \in \mathbb{Z}_{\geq 0}, m \neq n$ where $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k$.
Action of $\mathrm{Lie}(S^1)$ on the tangent space at the fixed point is given by a multiplication of each φ_m/φ_n on $(m-n)$. The regularized denominator in the right hand side of the Duistermaat-Heckman formula is given by

$$\frac{1}{\left[\prod_{m\in\mathbb{Z}_{\geq 0}, m\neq n}(m-n)\right]} \sim \frac{(-1)^n}{n!}$$

39. Difference of H_{S^1} at two fixed points

$$H_{S^1}(\varphi^{(n)}) - H_{S^1}(\varphi^{(0)}) = nt$$

Formal application of the Duistermaat-Heckman formula gives

$$Z_t(\hbar) \sim \sum_{n=0}^{\infty} (-1)^n \frac{e^{nt\hbar}}{n!} = e^{-e^{\hbar t}}$$

This proves mirror symmetry in this particular case (we recover a superpotential of the dual theory by explicitly summing instantons).

40. TFT version of Shintani-Casselman-Shalika formula

Two constructions of Archimedean Whittaker functions as correlation functions:

- $S^1 \times G$-equivariant Type A topological sigma model on disk D with the target space G/B
- S^1-equivariant Type B topological Landau-Ginzburg model on D with a superpotential W on an open part of $G^\vee/B^\vee$

Quantum integrable systems: realization of eigenfunctions as correlation functions in TFT on a disk D.

Generalization to partial flags G/P, P parabolic subgroup, produces *new* Toda type quantum integrable systems.

41. q-deformation via three-dimensional TFT

q-deformation naturally arises in TFT if one considers three-dimensional theories on $S^1 \times D$. This provides q-deformed expressions for L-factors and Whittaker functions.

Equivariant volumes of symplectic manifolds of holomorphic maps of D upgrade to partition functions of the corresponding quantum mechanical systems (quantization of X is a classical geometry of LX) producing naturally q-versions of Shintani-Casselman-Shalika formulas.

In the limit of shrinking S^1 (such that $q \to 1$) one recovers the Archimedean expressions.

42. Other approaches and related constructions

- P. Vojta: visualization of algebraic numbers using Nevanlinna's theory of holomorphic function value distributions.
- C. Denninger: Archimedean analog of Barsotti-Tate rings, Γ-function as an infinite-dimensional determinant.
- B. Mazur: Arithmetical topology (developed further by Kapranov and Reznikov).
- A. Beilinson and V. Drinfeld: relation with geometric Langlands correspondence (also a four-dimensional QFT version due to E. Witten et al).

43. Conclusions

- Archimedean geometry arises as symplectic geometry of the infinite-dimensional spaces of holomorphic maps of two-dimensional disks.
- S^1-equivariant topological sigma model is a way to describe topological sigma model coupled with topological gravity. Thus topological string theory is behind the geometry over $\mathbb{R}$.
- The dichotomy between **holomorphic periods** and **infinite-dimensional symplectic volumes/traces** is a guiding principle to construct Archimedean analogs of all standard notions of algebraic geometry.