SUSY Gauge Theories and Quantum Many Body Systems

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- N. Nekrasov, S. Sh. - '09-1, '09-2, '09-3
- A. Gerasimov, S. Sh. - '07, '08;
- A. Losev, N. Nekrasov, S. Sh. - '97, '98, '99
- G. Moore, N. Nekrasov, S. Sh. - '95, '97, '98

We start with YMH-theory, topological twist of 2d $\mathcal{N}=2$ massive gauge theory (four supercharges), pure $\mathcal{N}=2$ with massive adjoint matter, on Σ_g and the correspondence of MNS '97; GS '07, '08.

This topological field theory computes the intersection numbers on the moduli space $\mathcal{M}_g^H$ of Hitchin equations on Σ_g:

$$F_{z\bar{z}}(A) - [\Phi_z, \Phi_{\bar{z}}] = 0$$

$$\nabla_z(A)\Phi_{\bar{z}} = 0; \quad \nabla_{\bar{z}}(A)\Phi_z = 0$$

Symmetries - unitary gauge transformations and $U(1)$ action:

$$\Phi_z \to e^{i\alpha}\Phi_z; \qquad \Phi_{\bar{z}}, \to e^{-i\alpha}\Phi_{\bar{z}}$$

Here $F(A)$ is a curvature of unitary connection ∇_A (A - gauge field) and Φ is adjoint valued 1-form; we assume $G = U(N)$.

$\mathcal{M}_g^H$ is non-compact - intersection theory depends on one (equivariant) parameter c (regularization). $c = 0$ or ∞ - special.

The generating function of special, "chiral ring", operators O^i:

$$Z_{\Sigma_g}(t) = < e^{-t_i O^i} > = \sum_{n;\{i_1,\dots,i_n\}} \frac{t_{i_1} t_{i_2} \dots t_{i_n}}{n!} < O^{i_1} \dots O^{i_n} > =$$

$$= \sum_{n;\{i_1,\dots,i_n\}} \frac{t_{i_1} t_{i_2} \dots t_{i_n}}{n!} \int_{\mathcal{M}_g^H} w_{i_1} \wedge \dots \wedge w_{i_n} =$$

$$= \sum_{\sigma \in BA} D(\sigma)^{2-2g} e^{-\sum_{i=1}^N t_i p^i(\sigma)}$$

$$D(\sigma) = \mu(\sigma)^{-\frac{1}{2}} \prod_{i<j} (\sigma_i - \sigma_j)(1 + \frac{(\sigma_i - \sigma_j)^2}{c^2})^{\frac{1}{2}}$$

$$\mu(\sigma) = det || \frac{\partial^2 W(\sigma)}{\partial \sigma_i \partial \sigma_j} ||$$

$$\sigma \in BA : \qquad e^{2\pi i \sigma_j} \prod_{k \neq j} \frac{\sigma_k - \sigma_j - ic}{\sigma_k - \sigma_j + ic} = 1 \quad \Leftrightarrow \quad \exp\left(\frac{\partial W(\sigma)}{\partial \sigma^i}\right) = 1$$

where $p^i(\sigma)$ is i-th order symmetric polynomial of $(\sigma_1, \dots, \sigma_N)$.

Φ in Hitchin is a matter field (adjoint), no matter - $F(A) = 0$.

Adding new matter fields in gauge theory $\Leftrightarrow$ corrections to the right hand side of Hitchin equations $\Leftrightarrow$ other Bethe Eq.'s.

Topologically theory $\Leftrightarrow$ vacuum sector of Physical Theory. $g = 1$:

$$Z_{\Sigma_1}(t) = Tr(-1)^F e^{-\beta H} e^{-\sum_i t_i O^i} = Tr_{vac} e^{-\sum_i t_i O^i}$$

$$\{Q_A, Q_A^\dagger\} = \{Q_B, Q_B^\dagger\} = 4H$$

$$Q_A^2 = Q_B^2 = 0; \qquad H|vacuum >= 0$$

Simpler question - Q_A (Q_B)-cohomology:

$$Q_{A(B)}|\Psi >= 0; \quad |\Psi >\sim |\Psi > + Q_{A(B)}|... >$$

$|vacuum >$ is a "harmonic" representative in this cohomology.

If $|0>$ is some vacuum state and operator O_i is in Q-cohomology

$$\{Q, O_i\} = 0, \qquad O_i \sim O_i + \{Q, ...\}$$

$|i> = O_i|0>$ is also a vacuum state.

Operator-state correspondence would relate the complete basis for vacuum states $|i>$ to operators from cohomology O_i.

- These operators are independent of position up to Q-comm.

$$dO_i = \{Q, ...\}$$

- They form a commutative ring called (twisted) chiral ring:

$$O_i O_j|0> = c_{ij}^k O_k|0>; \quad \Rightarrow \quad O_i O_j = c_{ij}^k O_k + \{Q, ...\}$$

- SUSY vacua form the representation of chiral ring.

Basically, for every $\mathcal{N} = 2$ theory there is a quantum integrable system (assuming all good conditions - discrete specturm ...).

For YMH this quantum integrable system is (GS '07, '08) Yang's system of N-particles on S^1 with Hamiltonian ($x_i \sim x_i + 1$):

$$H_2 = -\sum_{i=1}^{N} \frac{\partial^2}{\partial {x_i}^2} + c \sum_{i \neq j} \delta(x_i - x_j)$$

This can be written in terms of Dunkle operators D_i:

$$D_i = -i \frac{\partial}{\partial x_i} + i \frac{c}{2} \sum_{j=i+1}^{N} (\epsilon(x_i - x_j) + 1) s_{ij}$$

Commuting H_k's & spectrum (N-particle sector of NLS):

$$H_k = \sum_{j=1}^{N} D_j^k; \qquad H_k \Psi(\lambda) = (\sum_{j=1}^{N} \lambda_j^k) \Psi_(\lambda)$$

with λ solving Bethe Equations. For each $(n_1 \geq n_2 \geq ... \geq n_N)$ - one solution $(\lambda_1, ..., \lambda_N)$; Yang-Yang '69, using $W(\lambda)$.

$G/G\ WZW$ generalization of YMH introduces extra parameter, level of KM algebra k, in BA - $s \to i\infty$ limit of XXZ.

Gauge theory data:

- Gauge group (for us it will be $U(N)$), or products for various N's
- Supermultiplets ("representations" of super-Poincare algebra)

  1. Gauge filed is in Vectormultiplet (Coulomb Branch); also has complex scalar σ, adjoint representation of gauge group
  2. Matter fields (Higgs Branch) form Chiral multiplets - some representation of gauge group $R = \oplus_{\mathbf{i}} M_{\mathbf{i}} \otimes R_{\mathbf{i}}$; $R_{\mathbf{i}}$ - irrep.

- Global (unbroken) symmetry group $H \subset \times_{\mathbf{i}} U(M_{\mathbf{i}})$
- Twisted masses $\tilde{m}_i$ - belong to the complexification of the Lie algebra of the maximal torus of H
- For each $U(1)$ component of gauge group - $t_b = \frac{\theta_b}{2\pi} + i r_b$

These data determines:

- Twisted effective superpotential $\tilde{\mathcal{W}}^{eff}(\sigma)$ (holomorphic) as function of eigenvalues of σ: $(\sigma_1, ..., \sigma_N)$ and all above parameters

General formula for $\tilde{\mathcal{W}}^{eff}(\sigma)$ ($\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$):

$$\tilde{\mathcal{W}}^{\text{eff}}(\sigma) =$$

$$= -\sum_{\mathbf{b}} 2\pi i t_{\mathbf{b}} tr_{\mathbf{b}}\sigma + tr_R\left(\sigma + \tilde{\mathbf{m}}\right)\left(\log\left(\sigma + \tilde{\mathbf{m}}\right) - 1\right) - 2\pi < \rho, \sigma >$$

Chiral ring operators can be chosen to be $O^k = tr\sigma^k$ and:

$$Z_{\Sigma_g}(t) = \sum_{\sigma\in BA} D(\sigma)^{2-2g} e^{-\sum_{i=1}^{N} t_i p^i(\sigma)}$$

where sum is over:

$$\frac{1}{2\pi i}\frac{\partial\tilde{\mathcal{W}}^{\text{eff}}(\sigma)}{\partial\sigma^i} = n_i$$

Or equivalently - SUSY vacua ($g = 1$) correspond to solution of:

$$\exp\left(\frac{\partial\tilde{\mathcal{W}}^{\text{eff}}(\sigma)}{\partial\sigma^i}\right) = 1$$

$D(\sigma)$ is known explicitly - is determined by same data.

For every quantum integrable system, solved by BA, there is a SUSY gauge theory with 4 supercharges (Q_A, Q_B, Q_A^+, Q_B^+) s.t.

a) exact Bethe eigenstates correspond to SUSY vacua

b) ring of commuting Hamiltonians $\Leftrightarrow$ (twisted) chiral ring

Converse is also true but it is not always easy to recognize the quantum integrable system.

SUSY vacuum equations in gauge theory $\Leftrightarrow$ Bethe equations

VEVs of chiral ring operators $\Leftrightarrow$ eigenvalues $=$ energies

Vacuum Ward Identity $\Leftrightarrow$ Baxter equation

- Vacua: "critical" pts of effective twisted superpotential $\tilde{\mathcal{W}}^{eff}(\sigma)$
- Bethe equations: spectrum, critical points of Yang function $Y(\lambda)$
- *The effective twisted superpotential corresponds to Yang function*

$$\boxed{\tilde{\mathcal{W}}^{eff}(\sigma) = Y(\lambda)}$$

$$\sigma_i = \lambda_i; \quad i = 1, ..., N; \quad G = U(N)$$

- VEV of chiral ring operators O_k $\Leftrightarrow$ eigenvalues of Hamiltonians:

$$\boxed{< \lambda|O_k|\lambda >= E_k(\lambda)}$$

$$H_k \Psi(\lambda) = E_k(\lambda)\Psi(\lambda)$$

$\tilde{W}^{eff}(\sigma)$ - effective twisted superpotential on Couloumb branch
$Y(\lambda)$ - Yang's function as a function of rapidities λ_i

Details worked out $\Leftrightarrow$ gauge theories identified (NS '09 -1,2):

- XXX spin chain - 2d gauge theory on Σ
- XXZ spin chain - 3d gauge theory on $\Sigma \times S^1$ (Higgs Branch infinite-dim., H contains translations along S^1 - KK: $\tilde{m}_n = n$)
- XYZ spin chain - 4d gauge theory on $\Sigma \times T^2$
- Arbitrary spin group, representation, impurities, limiting models

IN THIS TALK WE FOCUS ON (NS '09-3)

- Periodic Toda - 4d pure $\mathcal{N} = 2$ theory on $\Sigma \times R^2_\epsilon$
- Elliptic Calogero-Moser - 4d $\mathcal{N} = 2^*$ theory on $\Sigma \times R^2_\epsilon$

For global group H instead of translation along S^1 in KK we use rotation of R^2 with angle ϵ (complexified).

Four-dimensional $\mathcal{N} = 2$ Gauge Theory

Low energy effective theory of $U(N)$, $\mathcal{N} = 2$, gauge theory in 4d is abelian $U(1)^N$ gauge theory. In two derivative approximation it is described by one function $\mathcal{F}(\{a\};\Lambda)$, SW '94, KLTY '94, AF '94.

For any given set $\{a\} = (a_1, ..., a_N)$ we expect a vacuum; more precisely - vacua are labeled by symmetric polynomials of these:

$$\{u\}: \quad (u_1 = \sum_{i=1}^{N} a_i; \quad u_2 = \sum_{i=1}^{N} a_i^2, \quad ... \quad u_N = \sum_{i=1}^{N} a_i^N)$$

They correspond to the vacuum expectations of $tr\Phi^k$ where Φ is a complex scalar in the 4d vector multiplet. $\{u\}$ is called "u" plane.

Once $\mathcal{F}(\{a\};\Lambda)$ is found the Lagrangian is written by simple rules.

$\mathcal{F}(\{a\};\Lambda)$ is sum of perturbative $\mathcal{F}^{pert}$ (tree level and 1-loop only contribute) and non-pertubative $\mathcal{F}^{inst}$ terms; Λ counts instantons.

$\mathcal{N}=2^* \Rightarrow$ pure $\mathcal{N}=2$ theory plus massive adjoint matter. "Mass" m is some complex number.

Instanton counting parameter is $q=e^{i\tau}; \tau=i/g^2+\theta$. In $\mathcal{N}=2^*$ theory τ doesn't run.

Pure $\mathcal{N}=2$ theory is a limit when $m\to\infty$ - matter decouples.

Λ, instanton counting parameter in this limit is: $m^{2N}q=\Lambda^{2N}$; Λ is kept finite when $m\to\infty; q\to 0$:

$$\mathcal{F}^{pert}(a;\tau,m)=\frac{\tau}{2}\sum_{i=1}^{N}a_i^2+\frac{3N^2m^2}{2}+\frac{1}{4}\sum_{i,j=1}^{N}[(a_i-a_j)^2 log(a_i-a_j)-$$
$$-(a_i-a_j+m)^2 log(a_i-a_j+m)]$$

Non-perturbative part is of course an infinite sum of the type:

$$\mathcal{F}^{non-pert}(a;\tau,m)=\sum_{k=1}^{\infty}q^k\mathcal{F}_k(a;m),\quad q=e^{2\pi i\tau}$$

Classical Algebraic Integrable System

$\mathcal{F}(\{a\} : \Lambda)$ has nice interpretation in terms of classical ACIS -pToda, eCM,... (GKMMM '95, MW '95, DW '96, G '09,...):

• A complex algebraic manifold M of complex dimension $2r$

• Everywhere non-degenerate, closed holomorphic (2, 0)-form $\Omega_C^{2,0}$

• A holomorphic map $H : M \to C^r$, fibers $J_h = H^{-1}(h)$ are (polarized) abelian varieties (complex tori), $\{H_i, H_j\}_{\Omega_C^{2,0}} = 0$

Polarization is a Kahler form ω whose restriction on each fiber is integral class: $[w] \in H^2(J_h, Z) \cap H^{1,1}(J_h)$

Using the polarization introduce A and B-cycles ($< A_i, B^j >= \delta_i^j$, bases in $H_1(J_h, Z)$) define "action variables" on base via periods:

$$a_i = \int_{A_i} \Theta_C, \qquad a_D^i = \int_{B^i} \Theta_C, \qquad \Omega_C = d\Theta_C$$

In real case, $C^r \leftrightarrow R^r$, fibers are real tori T^r, exactly r real periods - usual action variables; angular variables - r angles of tori.

Since we get twice as many as the (complex) dimension of the base these variables must be related. Locally:

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a^i} \quad \Rightarrow \quad \theta = \sum_i a_D^i da_i = d\mathcal{F}(a)$$

The base can be supplied with the Rigid Special Geometry structure - locally a Lagrangian submanifold (holomorphic) in C^{2r}.

There is a notion of prepotential in Gauge Theory and in ACIS.

- *For every 4d $\mathcal{N} = 2$ gauge theory there is ACIS with same $\mathcal{F}(a)$.*

Important ACIS's - pToda and eCM, particular cases of more general ACIS - Hitchin integrable systems.

$\mathcal{N} = 2^*$ and Elliptic Calogero-Moser

$U(N)$ 4d $\mathcal{N} = 2^*$ theory has prepotential $\mathcal{F}(a_1, ..., a_N; \tau, m)$ which comes from Elliptic Calogero-Moser (eCM) ACIS.

eCM - N particles $q_1, q_2, ..., q_N$ on the circle of circumference β , $q_i \sim q_i + \beta$, which interact with the pair-wise potential:

$$H_2 = \sum_{i=1}^{N} p_i^2 + U(q); \qquad U(q) = m^2 \sum_{i<j} \mathcal{P}(q_i - q_j)$$

$$\mathcal{P}(x) = \sum_{n \in Z} \frac{1}{sinh^2(x + n\beta)} = u_0(x) + \sum_{k=1}^{\infty} q^k u_n(x)$$

$$q = e^{-2\beta}; \quad u_0 = \frac{1}{sinh^2 x} = \sum_k n e^{-kx}; \quad u_k(x) = 4 \sum_{d|k} d(e^{dx} + e^{-dx})$$

In order to describe Poisson commuting H_i's - introduce the Lax operator on phase space $T^*(C^\times)^N$ with $\Omega_C^{2,0} = \sum_i dp_i \wedge dq_i$:

$$\Phi_{ij}(z|p,q) = p_i\delta_{ij} + m\frac{\Theta(z+q_i-q_j)\Theta'(0)}{\Theta(q_i-q_j)\Theta(z)}(1-\delta_{ij})$$

$$\Theta(x) = -\sum_{k=Z+\frac{1}{2}} (-1)^k q^{\frac{k^2}{2}} e^{2kx}; \quad q = e^{2\pi i\tau}; \quad \tau = \frac{i\beta}{\pi}$$

Invariants of matrix $\Phi(z)$, coefficients of the polynomial $\det(x-\Phi(z))$, give H_i's. For example $tr\Phi(z)^2 = H_2 - \mathcal{P}(z)$.

Spectral curve: $\mathcal{C}_h \subset C \times C^\times$ is defined as zero locus of characteristic polynomial: $\det(x-\Phi(z)) = 0$.

$H^{-1}(h)$ is given by the product $C \times J_h$. The C-factor corresponds to the center-of-mass mode $\sum_i q_i$, while the compact factor $J_h = Jac(\bar{C}_h)$ is the Jacobian of the compactied curve C_h .

a_i, a_D^i are periods of differential $\lambda = \frac{1}{2\pi}xdz \ \Rightarrow \mathcal{F}(a)$

Note: periods of $\Theta = \sum_i p_i dq_i$ are same as periods of λ.

Quantization $\Leftrightarrow$ Deformation of SYM

Now we are going to do two things:

1) Quantize integrable system - Planck constant we denote by ϵ

2) ϵ-deform 4d $\mathcal{N}=2$ gauge theory - s.t. vacua $\Leftrightarrow$ eigenstates

1. Suppose we choose a_i^D as action variables - BS (we also need to choose half-dimensiona submanifold, real slice):

$$a_i^D = \epsilon \times n_i = \frac{\partial \mathcal{F}(a)}{\partial a_i} \quad \Rightarrow \quad \frac{\partial Y(a)}{\partial a_i} = n_i \quad s.t. \quad Y(a) = \frac{\mathcal{F}(a)}{\epsilon}$$

This semi-classical picture it is very suggestive to lead to the exact formula in the form of Bethe equation with some $Y(a;\epsilon)$:

$$\frac{\partial Y(a;\epsilon)}{\partial a_i} = n_i$$

Semiclassical formula suggests to look for quantization when:

$$Y(a;\epsilon) = \frac{\mathcal{F}(a) + O(\epsilon)}{\epsilon}$$

2. We need to ϵ-deform the original 4d $\mathcal{N} = 2^*$ theory in such way that the effective low energy theory becomes two-dimensional.

• 4d $\mathcal{N} = 2^*$ theory has continues spectrum of vacua - "u"-plane.

• ϵ-deformed theory - must have four supercharges and discrete spectrum of vacua given by critical points of $\tilde{W}(a;\epsilon) = Y(a;\epsilon)$

Thus the residue of a single pole in ϵ for $\tilde{W}(a;\epsilon)$ should be given by 4d superpotential $\mathcal{F}(a)$.

In fact we know such theory - 4d gauge theory on $R^2 \times R^2_\epsilon$.

$\mathcal{N} = 2$ gauge theory on $R^2 \times R^2_\epsilon$ is a deformation of $\mathcal{N} = 2$ theory on $R^2 \times R^2$ with one, equivariant, parameter ϵ which corresponds to the rotation of second R^2 around its origin.

- Denote corresponding vector field $V=\epsilon(x^2\partial_3 - x^3\partial_2)$.
- $z_1 = x_0 + ix_1, z_2 = x_2 + ix_3$. ϵ rotates z_2 by a phase.

$$L = \frac{1}{g_0^2}(-\frac{1}{2}trF\star F + Tr(D_A\phi - i_V F)\star(D_A\bar{\phi} - i_{\bar{V}}F) +$$

$$+\frac{1}{2}Tr([\phi,\bar{\phi}] + i_V D_A\bar{\phi} - i_{\bar{V}}D_A\phi)^2 + \frac{\theta_0}{2\pi}TrF\wedge F + fermions$$

Only 2d (first R^2) super-Poincare invariance is unbroken, four Q's.

$$\Phi(z_1,\bar{z}_1;z_2,\bar{z}_2) = \sum_{l;\bar{l}} \Phi_{l;\bar{l}}(z_1,\bar{z}_1) z_2^l \bar{z}_2^{\bar{l}} e^{-(|z_1|^2+|z_2|^2)}$$

All fields with non-zero $l,\bar{l}$ are massive ($+$ usual massive fields) and can be integrated out.

These are date for our theory in 2d on R^2 at the origin of transverse R^2_ϵ. Instead of shift symmetry along T^2 for KK we have rotation symmetry in transverse R^2_ϵ..

R^2 can be replaced by any 2d manifold Σ, for example $R \times S^1$.

Effective theory is 2d and is abelian, $U(1)^N$, $\mathcal{N}=2$ gauge theory on $\Sigma = R \times S^1$ with exactly computable twisted effective superpotential and vacuum equation:

$$\tilde{W}^{eff}(\{a\};\tau,m,\epsilon); \quad Vacua \Leftrightarrow \quad \frac{1}{2\pi i}\frac{\partial \tilde{W}^{\text{eff}}}{\partial a^i} = n_i$$

Twisted effective superpotential: One could rotate both R^2's: $R^4 \Rightarrow R^2_{\epsilon_1} \times R^2_{\epsilon_2}$.

This deformation gives effectively 0-dimensional theory with action:

$$\mathcal{A}(\{a\};\tau,m,\epsilon_1,\epsilon_2) = -log Z(\{a\};\tau,m,\epsilon_1,\epsilon_2)$$

where $Z(\{a\};\tau,m,\epsilon_1,\epsilon_2)$ is full partition function.

$$Z(\{a\}; q, m, \epsilon_1, \epsilon_2) = Z^{pert}(\{a\}; \tau, m, \epsilon_1, \epsilon_2) Z^{inst}(\{a\}; q, m, \epsilon_1, \epsilon_2)$$

Z^{inst} is an expansion in the powers of q where n-th order term is an integral over moduli space $\mathcal{M}_n$ of instanton number n.

(ϵ_1, ϵ_2) were introduced, MNS '97, to regularize these integrals over $\mathcal{M}_n$ since $\mathcal{M}_n$ is non-compact. We can take the formula from MNS '97-'98, LNS '97-'98, N '02 for Z^{inst}:

$$\sum_{k=0}^{\infty} \frac{q^k}{k!} \int_{R^k} \prod_{1 \le I < J \le k} \frac{R_+(\phi_{IJ})}{R_-(\phi_{IJ})} \prod_{I=1}^{k} Q(\phi_I) \frac{\epsilon\,(m+\epsilon_1)(m+\epsilon_2)}{\epsilon_1 \epsilon_2\, m(m+\epsilon)} \frac{\mathrm{d}\phi_I}{2\pi i}$$

$$\epsilon = \epsilon_1 + \epsilon_2; \quad \phi_{IJ} = \phi_I - \phi_J$$

$$R_+(x) = x^2(x^2 - \epsilon^2)(x^2 - (m+\epsilon_1)^2)(x^2 - (m+\epsilon_2)^2)$$

$$R_-(x) = (x^2 - \epsilon_1^2)(x^2 - \epsilon_2^2)(x^2 - m^2)(x^2 - (m+\epsilon)^2)$$

$$Q(x) = \frac{P(x-m)P(x+m+\epsilon)}{P(x)P(x+\epsilon)}; \quad P(x) = \prod_{l=1}^{N}(x - a_l)$$

• According to LNS '98 (ϵ_1, ϵ_2) correspond to $Q_{\epsilon_1,\epsilon_2} = Q + \epsilon_\mu J^\mu \rightarrow$ can test SW prepotential directly from instanton calculus.

• UV Lagrangian in Ω-background - N '02, interpreted in terms of boundary conditions/branes in NW '10

$Z(\{a\}; \tau, m, \epsilon_1, \epsilon_2)$ defines many important things, among others:

• Prepotential for theory with $\epsilon_1 = \epsilon_2 = 0$ - N '02

$$\mathcal{F}(\{a\}; \tau, m) = \lim_{\epsilon_1,\epsilon_2 \to 0} \epsilon_1 \epsilon_2 log Z(\{a\}; \tau, m, \epsilon_1, \epsilon_2)$$

• Superpotential $\tilde{W}^{eff}(\epsilon)$ for the theory with $\epsilon_2 = 0$ - NS '09-3:

$$\tilde{W}^{eff}(\{a\}; \tau, m, \epsilon) = \lim_{\epsilon_2 \to 0} \epsilon_2 log Z(\{a\}; \tau, m, \epsilon_1 = \epsilon, \epsilon_2)$$

and importantly $\tilde{W}^{eff}(\{a\}; \tau, m, \epsilon) = \frac{\mathcal{F}(\{a\};\tau,m)}{\epsilon} + ...$

What is exactly the eCM quantization problem for which this $\tilde{W}^{eff}$ gives the Yang's function and SUSY vacua - the exact spectrum?

- For eCM replace $p_i = \epsilon \frac{\partial}{\partial q_i}$, and q_i, m^2, ϵ - complex

- Write the eigenvalue problem for all Hamiltonians, parametrize eigenvalues $E_1, ..., E_N$ in terms of $a_1, ..., a_N$ - e. g. for H_2:

$$[\frac{\epsilon^2}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial {q_i}^2} + m(m+\epsilon) \sum_{i<j} \mathcal{P}(q_i - q_j; \beta)]\Psi(q) = E_2(a)\Psi(q)$$

$$\epsilon = -i\hbar, \quad m = i\hbar\nu \quad \Rightarrow \quad m(m+\epsilon) = -\hbar^2\nu(\nu-1)$$

- Look for solutions in affine Weyl chamber with asympthotics at $(q_i - q_j) \to 0$ of $\Psi \to (q_i - q_j)^{\nu}$, and extend outside this domain by symmetry condition with respect to shift in β.

• Spectrum is discrete and is determined by our superpotential:

$$\frac{\partial \tilde{W}^{eff}(\{a\};q,m,\epsilon)}{\partial a_i} = n_i; \quad E_2(a) = q\frac{d}{dq}\tilde{W}^{eff}(\{a\};q,m,\epsilon)$$

Checked in q-expansion for eCM knowing $\tilde{W}^{eff}$ for $\mathcal{N} = 2^*$.

In the limit when eCM becomes pToda (thus for pure $\mathcal{N} = 2$ theory) - one can borrow the Bethe Ansatz solution from Gu '81, S '85, GP '92, KL '99 and make more precise comparison.

NS '09-3 gave a precise identification of variables of gauge theory with that of KL '99 for which $\tilde{W}^{eff}(\{a\};q,m,\epsilon)$ gives Bethe equation and Yang-Yang function for pToda - very recently checked exactly in KT '10.

Most effective description of $\tilde{W}^{eff}(\{a\};q,m,\epsilon)$ given in NS '09-3 is through TBA-like construction :

$\tilde{W}^{eff} = \tilde{W}^{eff}_{pert} + \tilde{W}^{eff}_{inst}$. Perturbative part leads to Bethe Equation:

$$1 = \exp\left(\frac{\partial \tilde{W}^{eff}_{\text{pert}}}{\partial a_i}\right) = e^{\frac{\pi i \tau a_i}{\epsilon}} \prod_{j \neq i} S(a_i - a_j)$$

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\epsilon}\right)}{\Gamma\left(\frac{-m-x}{\epsilon}\right)} \frac{\Gamma\left(1 - \frac{x}{\epsilon}\right)}{\Gamma\left(1 + \frac{x}{\epsilon}\right)}$$

Non-perturbative part is determined through integral equation:

$$\chi(x) = \int_{\mathcal{C}} \mathrm{d}y \, G_0(x-y) \log\left(1 - q e^{-\chi(y)} Q(y)\right)$$

$$G_0(x) = \partial_x \log \frac{(x+\epsilon)(x+m)(x-m-\epsilon)}{(x-\epsilon)(x-m)(x+m+\epsilon)}$$

On solutions of this equation evaluate the functional:

$$\tilde{W}^{eff}_{inst} = \int_{\mathcal{C}} \mathrm{d}x \left[-\frac{\chi(x)}{2} \log\left(1 - q Q(x) e^{-\chi(x)}\right) + \mathrm{Li}_2\left(q Q(x) e^{-\chi(x)}\right) \right]$$