Exact solvability of the 1D polynomial Schrödinger equation

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Polynomial 1D stationary Schrödinger problem

(an **O**rdinary **D**ifferential **E**quation)

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} \quad + \quad \left[\quad V(q) \quad + \quad \lambda \quad \right]\right)\psi(q) = 0$$

$$V(q) \updownarrow \{+q^N + v_1 q^{N-1} + \cdots + v_{N-1} q\} \qquad \lambda \updownarrow \{-E\}$$

Notations: $\vec{v} = (v_1, \ldots, v_{N-1})$ $\boxed{\text{Degree} = N}$

- Traditional view (in any dimension):

  - $N = 2$ exactly solvable (harmonic oscillator),
  - $N \neq 2$ are not:
    - Airy equation for $N = 1$ (more transcendental than $N = 2$)
    - anharmonic oscillator for $N \geq 3$ (workhorse for perturbative, semiclassical ... methods).

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$$\qquad\qquad V(q) \updownarrow \qquad\qquad \lambda \updownarrow$$

$$\{+\,q^N + v_1\,q^{N-1} + \cdots + v_{N-1}\,q\} \qquad \{-E\}$$

Notations: $\vec{v} = (v_1, \cdots, v_{N-1})$ $\boxed{\text{Degree} = N}$

- Recent view (in 1 dimension):

AN EXACTLY SOLVABLE PROBLEM IN ANY DEGREE

by an **exact WKB** method (backed by **zeta-regularization**), through **exact Bohr–Sommerfeld** quantization formulae (selfconsistent: $\approx$ **Bethe Ansatz**)

Pioneers ($\sim$ 1975) : Balian–Bloch, Dingle, Leray, Sibuya, Zinn-Justin.

Polynomial 1D stationary Schrödinger problem

- Initial equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} \quad + \quad \left[\quad \underset{\{+q^N + v_1 q^{N-1} + \cdots + v_{N-1} q\}}{\underset{\updownarrow}{V(q)}} \quad + \quad \underset{\{-E\}}{\underset{\updownarrow}{\lambda}} \quad \right] \right) \psi(q) = 0$$

Notations: $\vec{v} = (v_1, \ldots, v_{N-1})$ $\boxed{\text{Degree} = N}$

- **Conjugate** equations:

$$V^{[\ell]}(q) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\ell\varphi} V(\mathrm{e}^{-\mathrm{i}\ell\varphi/2} q), \qquad \lambda^{[\ell]} \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\ell\varphi} \lambda$$

for $\ell = 0, 1, \ldots, L-1 \pmod{L}$ with $\boxed{\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}}$

Number of distinct conjugates : $L = \begin{cases} N+2 & \text{generically} \\ \frac{N}{2}+1 & \text{for } \textit{even polynomials } V(q) \end{cases}$

Semiclassical tools (I): Spectral functions (parity-split)

Assume confining potential $V(|q|) \quad \Longrightarrow \quad$ discrete E-spectrum $\mathcal{E} = \{E_k\}_{k=0,1,2,\dots}$

- $E \to +\infty$ expansions:

  Classical action: $\displaystyle\oint_{\{p^2+V(q)=E\}} \frac{p\,\mathrm{d}q}{2\pi} \sim b_\mu E^\mu$, $\quad \boxed{\mu \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{N}}$ (*growth order*)

  Semiclassical quantization condition (*Bohr–Sommerfeld expansion*):

$$\underbrace{b_\mu E^\mu \left[+ b_{\mu-1/N} E^{\mu-1/N} + b_{\mu-2/N} E^{\mu-2/N} + \cdots \right]}_{\widehat{F}(E) \quad \text{(formal, divergent)}} \sim k + \tfrac{1}{2} \quad \text{for integer } k \to +\infty.$$

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- **Exact** Bohr–Sommerfeld quantization condition ?

$$F_{\text{exact}}(E) = k + \tfrac{1}{2} \quad \longrightarrow \quad \{E_k\}, \qquad k = 0, 1, 2, \dots$$

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- **Exact** Bohr–Sommerfeld quantization condition ?

$$F_{\text{exact}}(E) = k + \tfrac{1}{2} \quad \rightleftarrows \quad \{E_k\}, \qquad k = 0, 1, 2, \ldots$$

Yes, but **selfconsistent** $\approx$ **Bethe Ansatz**.

Semiclassical tools (I): Spectral functions (parity-split)

Semiclassical quantization condition (*Bohr–Sommerfeld expansion*):

$$\sum_\alpha b_\alpha {E_k}^\alpha \sim k + \frac{1}{2} \quad \text{for integer } k \to +\infty \quad \left(\alpha = \mu,\ \mu - \frac{1}{N},\ \mu - \frac{2}{N}, \cdots\right)$$

$$\Downarrow$$

- **(Generalized) zeta functions:**

$$Z^{\pm}(s,\lambda) \stackrel{\text{def}}{=} \sum_{k \ \substack{\text{even} \\ \text{odd}}} (E_k + \lambda)^{-s} \qquad \qquad (\text{convergent for } \operatorname{Re} s > \mu)$$

$$\text{and} \quad Z \equiv Z^+ + Z^- \quad (\textit{full}), \qquad Z^{\mathrm{P}} \equiv Z^+ - Z^- \quad (\textit{skew}),$$

meromorphic in the whole s-plane, regular at $s = 0$.

- **Spectral determinants** (zeta-regularized), entire functions of λ:

$$D^{\pm}(\lambda) \equiv D(\lambda \mid \mathcal{E}_{\pm}) \stackrel{\text{def}}{=} \exp[-\partial_s Z^{\pm}(s,\lambda)]_{s=0}$$

$$\text{and} \quad D \equiv D^+ D^- \quad (\textit{full}), \qquad D^{\mathrm{P}} \equiv D^+/D^- \quad (\textit{skew},\ \text{meromorphic})$$

Semiclassical quantization condition (*Bohr–Sommerfeld expansion*):

$$\sum_\alpha b_\alpha {E_k}^\alpha \sim k + \frac{1}{2} \quad \text{for integer } k \to +\infty \quad \left(\alpha = \mu,\ \mu - \frac{1}{N},\ \mu - \frac{2}{N}, \cdots\right)$$

$$\Downarrow$$

For instance, $D(\lambda) \equiv \det(\hat{H}+\lambda) =$ "$\prod_k (\lambda + E_k)$":

$D(\lambda)$ is an **entire** function in λ, of finite order $= \mu$, and whose logarithm has

- a **structure equation**:

$$\log D(\lambda) \equiv \lim_{K \to +\infty} \left\{ \sum_{k<K} \log(E_k + \lambda) + \tfrac{1}{2} \log(E_K + \lambda) - \sum_{\{\alpha>0\}} \underbrace{b_\alpha {E_K}^\alpha \left[\log E_K - \frac{1}{\alpha}\right]}_{\textit{counterterms}} \right\},$$

- and a **canonical** large-λ (*generalized Stirling*) expansion, of order μ :

$$\log D(\lambda) \sim \sum_\alpha a_\alpha \{\lambda^\alpha\}, \quad \{\lambda^\alpha\} \stackrel{\text{def}}{=} \lambda^\alpha\ (\alpha \notin \mathbb{N}),\ \{\lambda^1\} \stackrel{\text{def}}{=} \lambda(\log\lambda - 1),\ \{\lambda^0\} \stackrel{\text{def}}{=} \log\lambda\,;$$

banned: pure λ^n $(n \in \mathbb{N})$ terms, including **additive constants** $(\propto \lambda^0)$.

Semiclassical tools (II): recessive WKB solutions (cf. Sibuya)

Exact solution $\psi_\lambda(q)$, recessive for $q \to +\infty$ (*canonical* WKB specification):

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(q')\, \mathrm{d}q' , \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q)+\lambda)^{1/2} \text{ (classical momentum)}$$

$$\int_q^{+\infty} \Pi_\lambda(q')\, \mathrm{d}q' : \textit{improper} \text{ action integral} \qquad (\Pi_\lambda(q') \sim {q'}^{N/2}).$$

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Trick:
$$= \Bigg[\underbrace{\int_q^{+\infty} (V(q') + \lambda)^{1/2-s}\,\mathrm{d}q'}_{I_q(s,\lambda)} \Bigg]_{s \rightsquigarrow 0} \qquad \text{(convergent for } \operatorname{Re} s > \mu)$$

(analytical continuation in s): fine if $I_q(s,\lambda)$ is **regular** at $s = 0$, which is often true.
Still, in full generality,

$$(V(q)+\lambda)^{1/2\,-s} \;\sim\; \sum_\rho \beta_\rho(s)\, q^{\rho - Ns} \qquad \left(\rho = \frac{N}{2},\ \frac{N}{2}-1, \ldots\right) \qquad (q \to +\infty)$$

$$\Rightarrow \quad \underbrace{\int_q^{+\infty} (V(q')+\lambda)^{1/2\,-s}\, \mathrm{d}q'}_{I_q(s,\lambda)} \;\sim\; -\sum_\rho \beta_\rho(s) \frac{q^{\rho+1-Ns}}{\rho+1-Ns} \qquad \text{(singular expansion)}$$

thus $I_q(s,\lambda)$ has at most a simple pole at $s=0$, of residue $\boxed{\dfrac{1}{N}\,\beta_{-1}(s=0)}$

$$\beta_{-1}(s) \text{ (\textbf{``residual'' polynomial})} \begin{cases} \equiv 0 \ : & \textbf{N}\text{ormal case} \\ \not\equiv 0 \ : & \textbf{A}\text{nomaly case} \end{cases}$$

Semiclassical interpretation of improper action integral

QUANTUM

zeta function

$$Z(s,\lambda) = \mathrm{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+V(q)+\lambda\right)^{-s} = \sum_k (E_k+\lambda)^{-s}$$

determinant

$$D(\lambda) \overset{\textit{formally}}{=} \text{“}\prod_k (\lambda+E_k)\text{”}$$

$$D(\lambda) \overset{\mathrm{def}}{=} \exp\left\{-\partial_s Z(s,\lambda)|_{s=0}\right\}$$

Semiclassical interpretation of improper action integral

QUANTUM $\longleftrightarrow$ CLASSICAL

correspondence

zeta function

zeta function?

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formally

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zeta function

$$Z_{\mathrm{cl}}(s,\lambda) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^2} \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} \left(p^2+V(q)+\lambda\right)^{-s}$$

determinant

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determinant

$$D_{\mathrm{cl}}(\lambda) = \exp\ \text{“}\overset{\textit{formally}}{\int_{-\infty}^{+\infty}(V(q)+\lambda)^{1/2}\,\mathrm{d}q}\text{”}$$

$$D_{\mathrm{cl}}(\lambda) \stackrel{\mathrm{def}}{=} \exp\{-\partial_s Z_{\mathrm{cl}}(s,\lambda)|_{s=0}\}$$

CANONICAL normalization of recessive solution

$$\log D_{\rm cl}(\lambda) = \int_{-\infty}^{+\infty} \Pi_\lambda(q)\,{\rm d}q \;=\; \int_{-\infty}^{+\infty} (V(q)+\lambda)^{1/2}\,{\rm d}q\ ?$$

($N > 2$ for simplicity.) Then,

$$\frac{\rm d}{{\rm d}\lambda}\log D_{\rm cl}(\lambda) \;=\; \int_{-\infty}^{+\infty} \tfrac{1}{2}(V(q)+\lambda)^{-1/2}\,{\rm d}q$$

&

$$\log D_{\rm cl}(\lambda) \sim \text{CANONICAL} \quad \text{for } \lambda \to +\infty$$

(= no pure λ^0 terms in large-λ expansion)

fully specify improper action integral, giving $\quad [I_q(s,\lambda) = \int_q^{+\infty} (V(q')+\lambda)^{1/2-s}\,{\rm d}q']$

$$\boxed{\int_q^{+\infty} \Pi_\lambda(q')\,{\rm d}q' \stackrel{\rm def}{=} {\rm FP}_{s=0}\, I_q(s,\lambda) \;+ 2(1-\log 2)\,\beta_{-1}(0)/N.}$$

Still *additive*: $\displaystyle\int_q^{+\infty} \Pi_\lambda(q')\,{\rm d}q' = \int_q^{q''} \Pi_\lambda(q')\,{\rm d}q' + \int_{q''}^{+\infty} \Pi_\lambda(q')\,{\rm d}q'$ for finite q, q''

(as $\displaystyle\int_q^{q''} \Pi_\lambda(q')\,{\rm d}q \sim (q''-q)\,\lambda^{1/2} + O(\lambda^{-1/2})$ is canonical).

Simplest examples:

$$\int_0^{+\infty} (q^4+vq^2)^{1/2}\, \mathrm{d}q \;=\; -\tfrac{1}{3}\, v^{3/2} \qquad \mathbf{N}$$

$$\int_0^{+\infty} (q^N+\lambda)^{1/2}\, \mathrm{d}q \;=\; -(2\sqrt{\pi})^{-1}\, \Gamma(\tfrac{1}{2}+\mu)\, \Gamma(-\mu)\, \lambda^{\mu} \quad (N \neq 2) \quad \boxed{\mu \equiv \tfrac{1}{2}+\tfrac{1}{N}} \qquad \mathbf{N}$$

$$\int_0^{+\infty} (q^2+\lambda)^{1/2}\, \mathrm{d}q \;=\; -\tfrac{1}{4}\, \lambda(\log\lambda - 1) \quad (N = 2) \qquad \mathbf{A}$$

Even quartic oscillator ($v, \lambda \geq 0$ for simplicity):

$$\int_0^{+\infty} (q^4+vq^2+\lambda)^{1/2}\, \mathrm{d}q \;=$$

$$(v \geq 2\sqrt{\lambda}): \quad = \tfrac{1}{3}(v+2\sqrt{\lambda})^{1/2}\big[2\sqrt{\lambda}\, K(k) - vE(k)\big], \quad k = \left(\frac{v-2\sqrt{\lambda}}{v+2\sqrt{\lambda}}\right)^{1/2};$$

$$(v \leq 2\sqrt{\lambda}): \quad = \tfrac{1}{3}\, \lambda^{1/4}\big[(2\sqrt{\lambda}+v)K(\tilde{k}) - 2vE(\tilde{k})\big], \quad \tilde{k} = \frac{(2\sqrt{\lambda}-v)^{1/2}}{2\,\lambda^{1/4}}$$

($K(k),\ E(k)$: complete elliptic integrals).

Semiclassical interpretation of improper action integral

QUANTUM $\longleftrightarrow$ CLASSICAL

correspondence

determinant | determinant

$$D(\lambda) \overset{formally}{=} \text{“}\prod_k (\lambda + E_k)\text{”}$$

$$D(\lambda) \overset{\text{def}}{=} \exp\{-\partial_s Z(s,\lambda)|_{s=0}\}$$

$$D_{\text{cl}}(\lambda) \overset{formally}{=} \exp \text{“}\int_{-\infty}^{+\infty} (V(q)+\lambda)^{1/2}\,\mathrm{d}q\text{”}$$

$$D_{\text{cl}}(\lambda) \overset{\text{def}}{=} \exp\{-\partial_s Z_{\text{cl}}(s,\lambda)|_{s=0}\}$$

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$$D_{\text{cl}}(\lambda) \overset{formally}{=} \exp\ \text{``}\int_{-\infty}^{+\infty} (V(q)+\lambda)^{1/2}\,\mathrm{d}q\text{''}$$

$$D_{\text{cl}}(\lambda) \overset{\text{def}}{=} \exp\{-\partial_s Z_{\text{cl}}(s,\lambda)|_{s=0}\}$$

identities

$$D_{\text{cl}}^{-}(\lambda) \equiv \Pi_\lambda(0)^{-1/2} \exp \int_0^{+\infty} \Pi_\lambda(q)\,\mathrm{d}q$$

$$\equiv [\psi_\lambda]_{\text{WKB}}(0)$$

$$D_{\text{cl}}^{+}(\lambda) \equiv \Pi_\lambda(0)^{+1/2} \exp \int_0^{+\infty} \Pi_\lambda(q)\,\mathrm{d}q$$

$$\equiv -[\psi'_\lambda]_{\text{WKB}}(0)$$

Semiclassical interpretation of improper action integral

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correspondence

determinant

$$D(\lambda) \;\overset{formally}{=}\; \text{“}\prod_k (\lambda + E_k)\text{”}$$

$$D(\lambda) \overset{\text{def}}{=} \exp\{-\partial_s Z(s,\lambda)|_{s=0}\}$$

basic identities

$$\boxed{D^-(\lambda) \equiv \psi_\lambda(0)}$$

$$\boxed{D^+(\lambda) \equiv -\psi'_\lambda(0)}$$

determinant

$$D_{\text{cl}}(\lambda) \;\overset{formally}{=}\; \exp\ \text{“}\int_{-\infty}^{+\infty} (V(q)+\lambda)^{1/2}\,\mathrm{d}q\text{”}$$

$$D_{\text{cl}}(\lambda) \overset{\text{def}}{=} \exp\{-\partial_s Z_{\text{cl}}(s,\lambda)|_{s=0}\}$$

identities

$$D^-_{\text{cl}}(\lambda) \equiv \Pi_\lambda(0)^{-1/2} \exp \int_0^{+\infty} \Pi_\lambda(q)\,\mathrm{d}q$$

$$\equiv [\psi_\lambda]_{\text{WKB}}(0)$$

$$D^+_{\text{cl}}(\lambda) \equiv \Pi_\lambda(0)^{+1/2} \exp \int_0^{+\infty} \Pi_\lambda(q)\,\mathrm{d}q$$

$$\equiv -[\psi'_\lambda]_{\text{WKB}}(0)$$

The Wronskian identity

- Exact solution $\psi_\lambda(q)$, recessive for $q \to +\infty$ (WKB specification):

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(q')\mathrm{d}q' \,, \quad \Pi_\lambda(q) \stackrel{\mathrm{def}}{=} (V(q)+\lambda)^{1/2} \text{ (classical momentum)}$$

- Adjacent conjugate solution, recessive for $q \to +\,\mathrm{e}^{-\mathrm{i}\varphi/2}\infty$:

$$\Psi_\lambda(q) \stackrel{\mathrm{def}}{=} \psi^{[1]}_{\lambda^{[1]}}(\mathrm{e}^{\mathrm{i}\varphi/2}\, q)$$

- All $q \to +\infty$ expansions ($\psi_\lambda(q)$, $\psi'_\lambda(q)$, $\Psi_\lambda(q)$, $\Psi'_\lambda(q)$) **fully known**, e.g.,

$$\psi_\lambda(q) \sim \mathrm{e}^{\mathcal{C}}\, q^{-N/4\,-\beta_{-1}(0)} \exp\left\{-\sum_{\{\sigma>0\}} \beta_{\sigma-1}(0)\,\frac{q^\sigma}{\sigma}\right\}, \quad \mathcal{C} = \frac{1}{N}\left[(-2\log 2 + \partial_s)\frac{\beta_{-1}(s)}{1-2s}\right]_{s=0}$$

$$\vdots$$

$\Longrightarrow$ **Explicit** Wronskian (evaluated in $q \to +\infty$ limit):

$$\boxed{\psi'_\lambda(q)\Psi_\lambda(q) - \Psi'_\lambda(q)\psi_\lambda(q) \equiv 2\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi/4}\,\mathrm{e}^{\mathrm{i}\varphi\beta_{-1}(0)/2}}$$

Explicit Wronskian:

$$\boxed{\psi'_\lambda(q)\Psi_\lambda(q) - \Psi'_\lambda(q)\psi_\lambda(q) \equiv 2\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi/4}\,\mathrm{e}^{\mathrm{i}\varphi\beta_{-1}(0)/2}}$$

Plus the **basic exact identities**:

$$D^+(\lambda) \equiv -\psi'_\lambda(0) \qquad \Updownarrow \qquad D^-(\lambda) \equiv \psi_\lambda(0)$$

$$\searrow \qquad\qquad \swarrow$$

$$-\,\mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\lambda \mid \mathcal{E}_+)\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) + \mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\,D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(0)/2}$$

Exact quantization condition?

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(0)/2}$$

Degenerate cases:

- $N = 2:$ $$\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + (q^2 + \lambda)\right]\psi(q) = 0$$ (harmonic oscillator)

$$\boxed{\varphi = \pi \qquad \beta_{-1} = \lambda/2}$$

$$\mathrm{e}^{+\mathrm{i}\pi/4}\, D^+(-\lambda)\, D^-(\lambda) - \mathrm{e}^{-\mathrm{i}\pi/4}\, D^+(\lambda)\, D^-(-\lambda) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\pi\lambda/4}$$

Exact quantization condition

$$e^{+i\varphi/4} D(e^{-i\varphi}\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4} D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\,e^{+i\varphi\beta_{-1}(0)/2}$$

Degenerate case:

- $N = 2:$ $\left[-\frac{d^2}{dq^2} + (q^2+\lambda)\right]\psi(q) = 0$ (harmonic oscillator)

$$\boxed{\varphi = \pi \qquad \beta_{-1} = \lambda/2}$$

$$e^{+i\pi/4} D^+(-\lambda)\, D^-(\lambda) - e^{-i\pi/4} D^+(\lambda)\, D^-(-\lambda) \equiv 2\,i\,e^{+i\pi\lambda/4}$$

unknowns **real**, hence identity **splits**:

$$\cos \pi/4\,[D^+(-\lambda)\, D^-(\lambda) - D^+(\lambda)\, D^-(-\lambda)] = -2\sin \pi\lambda/4$$

$$\sin \pi/4\,[D^+(-\lambda)\, D^-(\lambda) + D^+(\lambda)\, D^-(-\lambda)] = +2\cos \pi\lambda/4$$

$$\Longrightarrow \qquad D^+(\lambda) \quad D^-(-\lambda) \qquad = \qquad 2\cos \pi(\lambda-1)/4$$

zeros : $\ldots,\ -9,\ -5,\ -1,\quad +3,\ +7,\ +11, \ldots$

$$\Longrightarrow \qquad D^+(\lambda) = \frac{2^{-\lambda/2}\, 2\sqrt{\pi}}{\Gamma(\frac{1+\lambda}{4})} \qquad D^-(\lambda) = \frac{2^{-\lambda/2}\sqrt{\pi}}{\Gamma(\frac{3+\lambda}{4})}$$

• Same for $\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + (q^N + \Lambda\, q^{\frac{N}{2}-1})\right]\psi(q) = 0$

(zero-energy generalized eigenvalue problem),

likewise exactly solvable (**supersymmetric**) with $\boxed{\nu \stackrel{\text{def}}{=} \frac{1}{N+2}}$:

$$D_N^+(\Lambda) = -\frac{2^{-\Lambda/N}(4\nu)^{\nu(\Lambda+1)+1/2}\,\Gamma(-2\nu)}{\Gamma(\nu(\Lambda-1)+1/2)} \qquad D_N^-(\Lambda) = \frac{2^{-\Lambda/N}(4\nu)^{\nu(\Lambda-1)+1/2}\,\Gamma(2\nu)}{\Gamma(\nu(\Lambda+1)+1/2)}$$

Furthermore, now over the whole real line (for N even),

$$\det\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + (q^N + \Lambda\, q^{\frac{N}{2}-1})\right] = \begin{cases} D_N^+(\Lambda)D_N^-(\Lambda) & \text{if } N \equiv 2 \pmod 4 \\ \frac{1}{\sin \pi\nu} \cos \pi\nu\Lambda & \text{if } N \equiv 0 \pmod 4 \end{cases}$$

Exact quantization condition?

$$e^{+i\varphi/4}\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\, e^{+i\varphi\beta_{-1}(0)/2}$$

Exact quantization condition?

$$e^{+i\varphi/4}\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\,e^{+i\varphi\beta_{-1}(0)/2}$$

$$e^{+i\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(e^{i\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - e^{-i\varphi/4}\, D(e^{i\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,i\,e^{-i\varphi\beta_{-1}(0)/2}$$

Exact quantization condition?

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n}$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(0)/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(0)/2}$$

Exact quantization condition?

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n} \iff D(\lambda \mid \mathcal{E}_+) = 0$

$$e^{+i\varphi/4} D(e^{-i\varphi} \lambda \mid \mathcal{E}_+^{[1]}) D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4} D(\lambda \mid \mathcal{E}_+) D(e^{-i\varphi} \lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\,e^{+i\varphi\beta_{-1}(0)/2}$$

$$e^{+i\varphi/4} D(\lambda \mid \mathcal{E}_+) D(e^{i\varphi} \lambda \mid \mathcal{E}_-^{[-1]}) - e^{-i\varphi/4} D(e^{i\varphi} \lambda \mid \mathcal{E}_+^{[-1]}) D(\lambda \mid \mathcal{E}_-) \equiv 2\,i\,e^{-i\varphi\beta_{-1}(0)/2}$$

Exact quantization condition?

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n} \quad \Longleftrightarrow \quad D(\lambda \mid \mathcal{E}_+) = 0$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(0)/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(0)/2}$$

Exact quantization condition

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n} \iff D(\lambda \mid \mathcal{E}_+) = 0$

$$e^{+i\varphi/4} D(e^{-i\varphi}\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4} D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\,e^{+i\varphi\beta_{-1}(0)/2}$$

$$e^{+i\varphi/4} D(\lambda \mid \mathcal{E}_+)\, D(e^{i\varphi}\lambda \mid \mathcal{E}_-^{[-1]}) - e^{-i\varphi/4} D(e^{i\varphi}\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,i\,e^{-i\varphi\beta_{-1}(0)/2}$$

$$\Longrightarrow \quad \frac{D(e^{-i\varphi}\lambda \mid \mathcal{E}_+^{[+1]})}{D(e^{+i\varphi}\lambda \mid \mathcal{E}_+^{[-1]})} = -\,e^{i\,[-\varphi/2+\varphi\beta_{-1}(0)]} \qquad \left(\varphi \overset{\text{def}}{=} \frac{4\pi}{N+2}\right)$$

$$2\arg D(-\,e^{-i\varphi} E \mid \mathcal{E}_+^{[+1]}) - \varphi\,\beta_{-1}(0) = \pi\left[k + \frac{1}{2} + \frac{N-2}{2(N+2)}\right] \qquad \text{for } k = 2n \geq 0$$

Exact quantization condition

For odd spectrum $\mathcal{E}_-$: $\lambda = -E_{2n+1} \iff D(\lambda \mid \mathcal{E}_-) = 0$

$$e^{+i\varphi/4} D(e^{-i\varphi}\lambda \mid \mathcal{E}_+^{[1]}) D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4} D(\lambda \mid \mathcal{E}_+) D(e^{-i\varphi}\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\,e^{+i\varphi\beta_{-1}(0)/2}$$

$$e^{+i\varphi/4} D(\lambda \mid \mathcal{E}_+) D(e^{i\varphi}\lambda \mid \mathcal{E}_-^{[-1]}) - e^{-i\varphi/4} D(e^{i\varphi}\lambda \mid \mathcal{E}_+^{[-1]}) D(\lambda \mid \mathcal{E}_-) \equiv 2\,i\,e^{-i\varphi\beta_{-1}(0)/2}$$

$$\Longrightarrow \quad \frac{D(e^{-i\varphi}\lambda \mid \mathcal{E}_-^{[+1]})}{D(e^{+i\varphi}\lambda \mid \mathcal{E}_-^{[-1]})} = -\,e^{i\,[+\varphi/2+\varphi\beta_{-1}(0)]} \qquad \left(\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}\right)$$

$$2\arg D(-\,e^{-i\varphi} E \mid \mathcal{E}_-^{[+1]}) - \varphi\,\beta_{-1}(0) = \pi\left[k + \frac{1}{2} - \frac{N-2}{2(N+2)}\right] \qquad \text{for } k = 2n+1 > 0$$

Complete set of exact quantization conditions

(for all conjugate, $\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}$ spectra $\mathcal{E}^{[\ell]}_{\pm}$)

$$\frac{1}{\mathrm{i}}\left[\log D(-\,\mathrm{e}^{-\mathrm{i}\varphi}\,E\mid\mathcal{E}^{[\ell+1]}_{\pm})-\log D(-\,\mathrm{e}^{+\mathrm{i}\varphi}\,E\mid\mathcal{E}^{[\ell-1]}_{\pm})\right]-(-1)^{\ell}\varphi\,\beta_{-1}(0)$$

$$=\pi\left[k+\frac{1}{2}\pm\frac{N-2}{2(N+2)}\right]\qquad \text{for}\quad k=\begin{smallmatrix}0,2,4,\cdots\\ 1,3,5,\cdots\end{smallmatrix}\qquad \ell=0,1,\cdots,L{-}1\quad (\mathrm{mod}\ L)$$

+ structure equations:

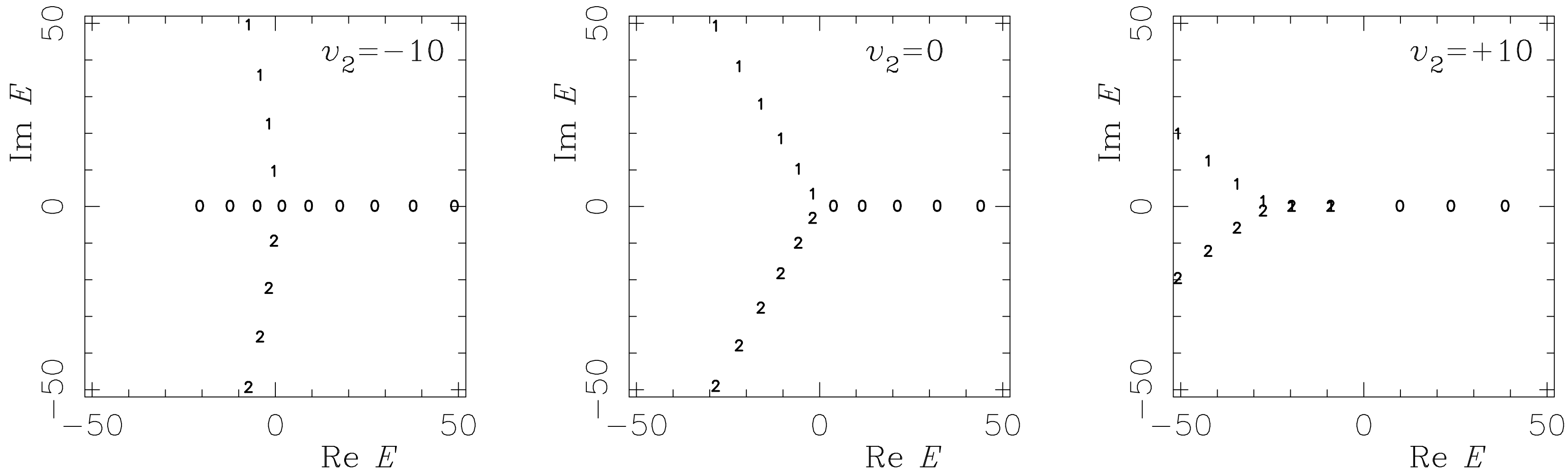
$$\log D(\lambda\mid\mathcal{E}^{[\ell]}_{\pm})\equiv\lim_{K\to+\infty}\left\{\sum_{k<K}\log(E^{[\ell]}_k+\lambda)\ +\ \tfrac{1}{2}\log(E^{[\ell]}_K+\lambda)\right.$$

$$\left(k,K\ \begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right)\qquad\qquad \left.-\sum_{\{\alpha>0\}}\tfrac{1}{2}\,b^{[\ell]}_{\alpha}\left[E^{[\ell]}_K\right]^{\alpha}(\log E^{[\ell]}_K-1/\alpha)\right\}$$

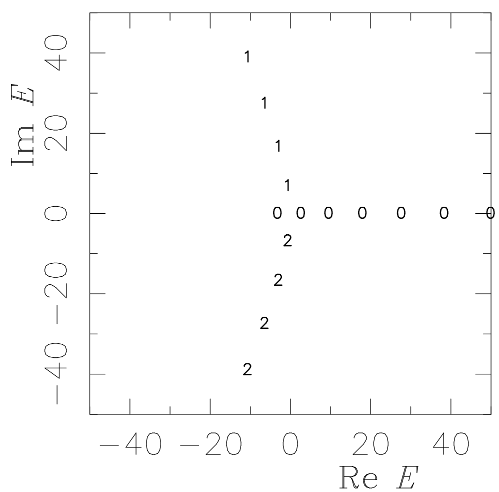
altogether define a formally **complete** set of **fixed-point conditions**

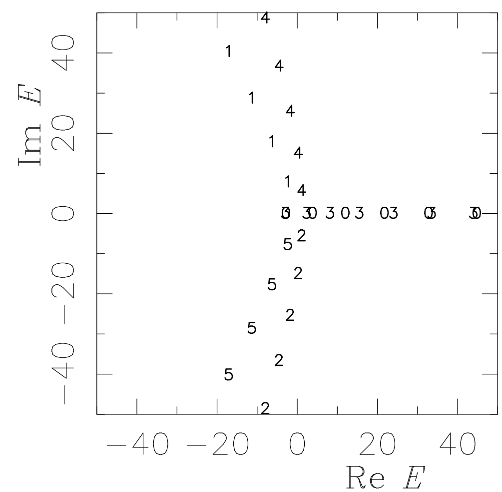
$(\mathcal{M}^{\pm}\{\mathcal{E}^{[\ell]}_{\pm}\}=\{\mathcal{E}^{[\ell]}_{\pm}\}$ for some mappings $\mathcal{M}^{\pm}$).

The three conjugate (odd, and rotated) spectra for $V(q) = q^4 + v_2 q^2$



Same for $V(q) = q^4 - 5\,q^2$ $\quad$ $V(q) \approx q^4 + q^3 - 4.625\,q^2 - 2.4375\,q$





Homogeneous case $V(q) = q^N$ $\boxed{N \neq 2}$

- All conjugate spectra **identical**: $\mathcal{E}_{\pm}^{[\ell]} \equiv \mathcal{E}_{\pm}$
- **Residue polynomial** $\beta_{-1}(s) \equiv 0$ [**except** $N = 2$: $\beta_{-1}(s) \equiv \lambda(-s + \frac{1}{2})$]
- Wronskian identity:

$$e^{+i\varphi/4} D^+(e^{-i\varphi}\lambda)\, D^-(\lambda) - e^{-i\varphi/4} D^+(\lambda)\, D^-(e^{-i\varphi}\lambda) \equiv 2\,i \qquad \boxed{\varphi = \frac{4\pi}{N+2}}$$

- **Exact quantization condition**:

$$2\,\Sigma_+(E_k) = k + \tfrac{1}{2} + \tfrac{1}{2}\kappa \qquad k = 0, 2, 4, \ldots$$

$$2\,\Sigma_-(E_k) = k + \tfrac{1}{2} - \tfrac{1}{2}\kappa \qquad k = 1, 3, 5, \ldots$$

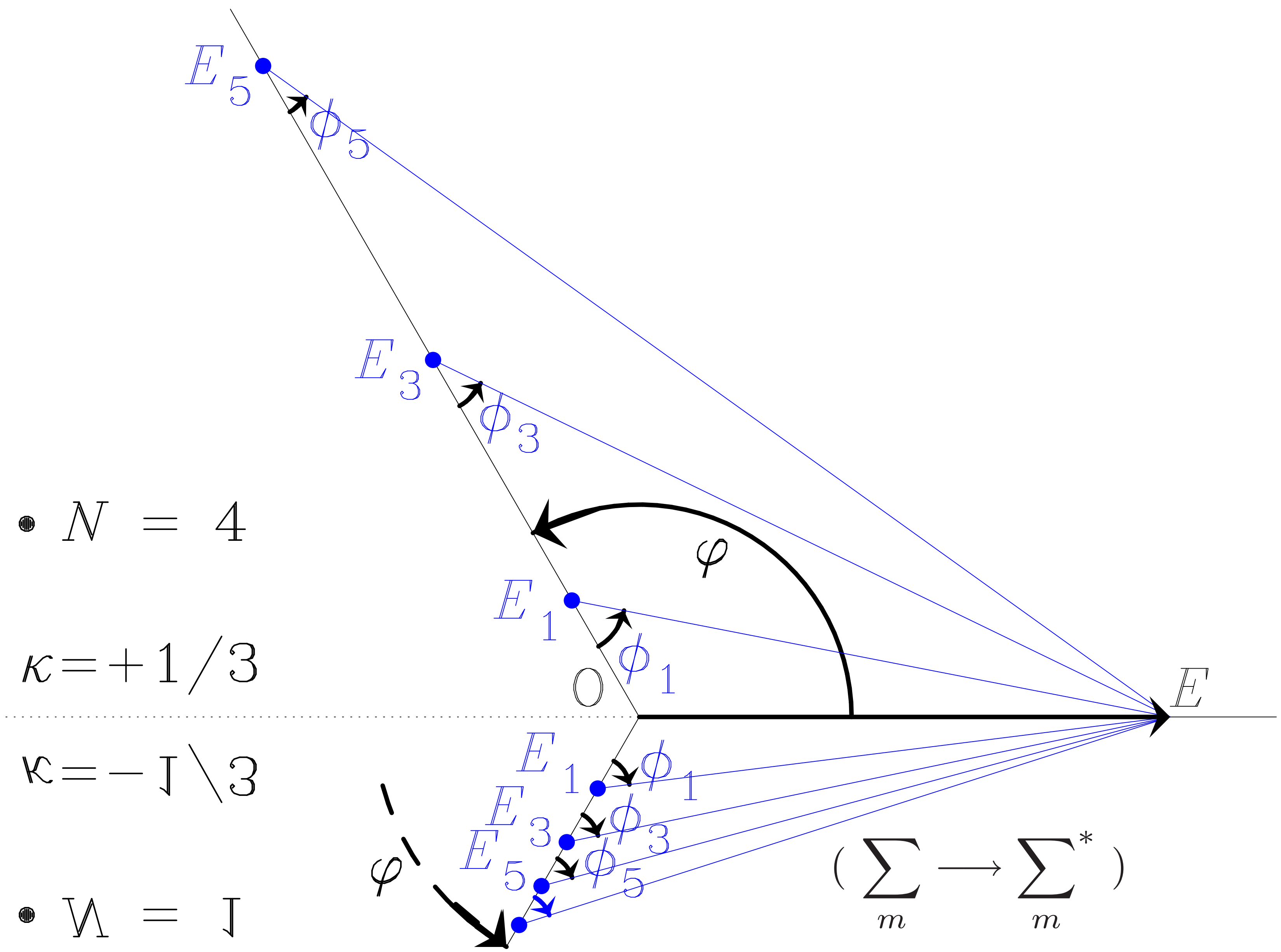
$$\kappa \overset{\text{def}}{=} (N-2)/(N+2)$$

$$\Sigma_{\pm}(E) \overset{\text{def}}{=} \frac{1}{\pi} \sum_{m\ \substack{\text{even}\\ \text{odd}}} \underbrace{\arg(E_m - e^{-i\varphi} E)}_{\phi_m(E)} \qquad (N > 2)$$

\+ boundary condition $\quad b_\mu E_k^\mu \sim k + \frac{1}{2} \quad$ for $k \to +\infty$

$\Longleftrightarrow$ fixed-point equations $\mathcal{M}^{\pm}\{\mathcal{E}_{\pm}\} = \{\mathcal{E}_{\pm}\}$

(mappings $\mathcal{M}^{\pm}$ proved **globally contractive** for $N > 2$, by Avila).



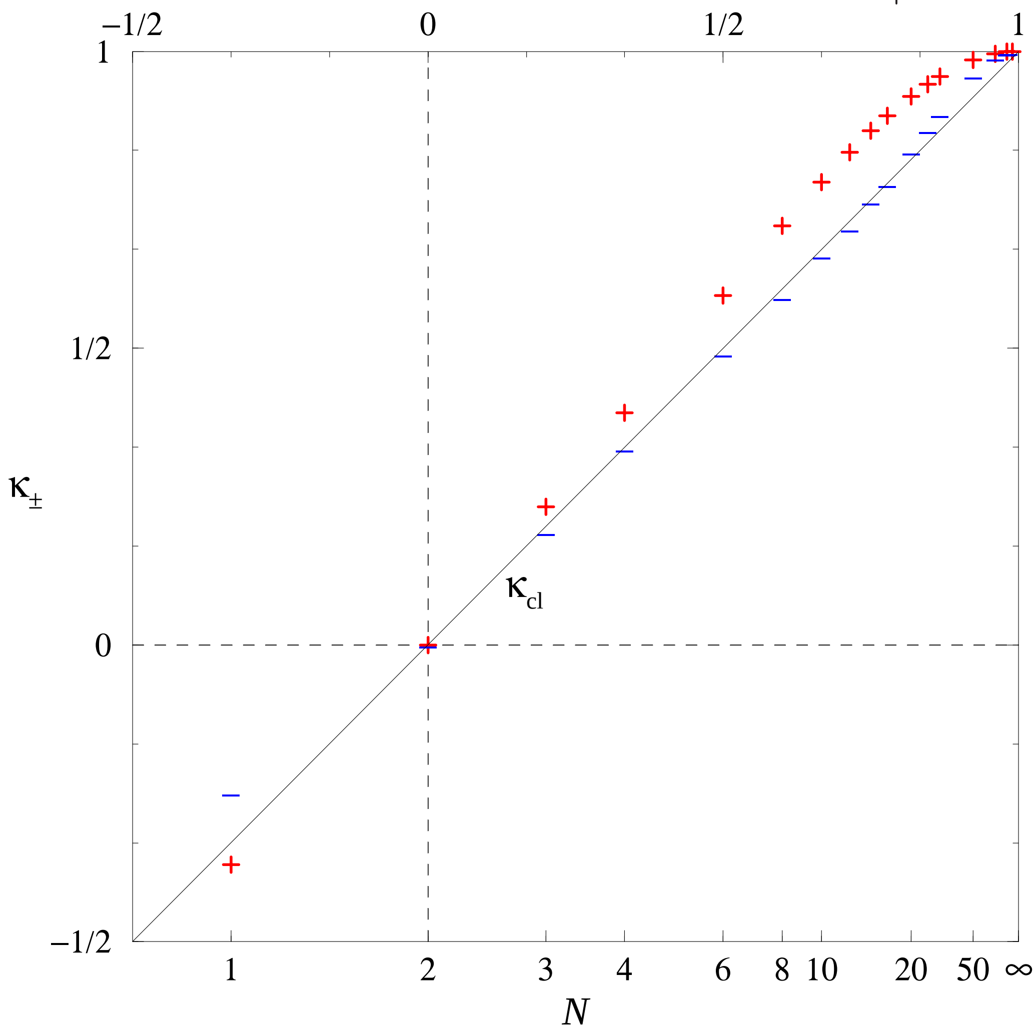
E_5
ϕ_5
E_3
ϕ_3
$\bullet\ N = 4$
φ
E_1
ϕ_1
$\kappa = +1/3$
O
E
E_1
ϕ_1
E_3
ϕ_3
E_5
ϕ_5
φ
$(\sum_m \longrightarrow \sum_m^*)$

Numerical tests

Ex. $\kappa_-(120) \approx 0.9980$

$\kappa_+(400) \approx 0.99975$

Horizontal scale $\kappa_{\text{cl}} = \kappa \overset{\text{def}}{=} \frac{N-2}{N+2}$



Exact wave-function analysis

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + [V(q)+\lambda]\right)\psi(q) = 0$$

and, e.g., $\psi(q)$ recessive for $q \to +\infty$ (λ arbitrary, input).

Restrict to half-line $[Q, +\infty)$ (Q a parameter):

$$V_Q(q) \stackrel{\text{def}}{=} [V(q) - V(Q)] \qquad \text{for } q \in [Q, +\infty)$$

$$D_Q^{\pm}(\lambda) \stackrel{\text{def}}{=} \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V_Q(q) + \lambda\right)^{\pm} \qquad \left[\begin{smallmatrix}\text{Neumann}\\ \text{Dirichlet}\end{smallmatrix} \text{ boundary conditions at } q = Q\right]$$

Translated **basic identities**:

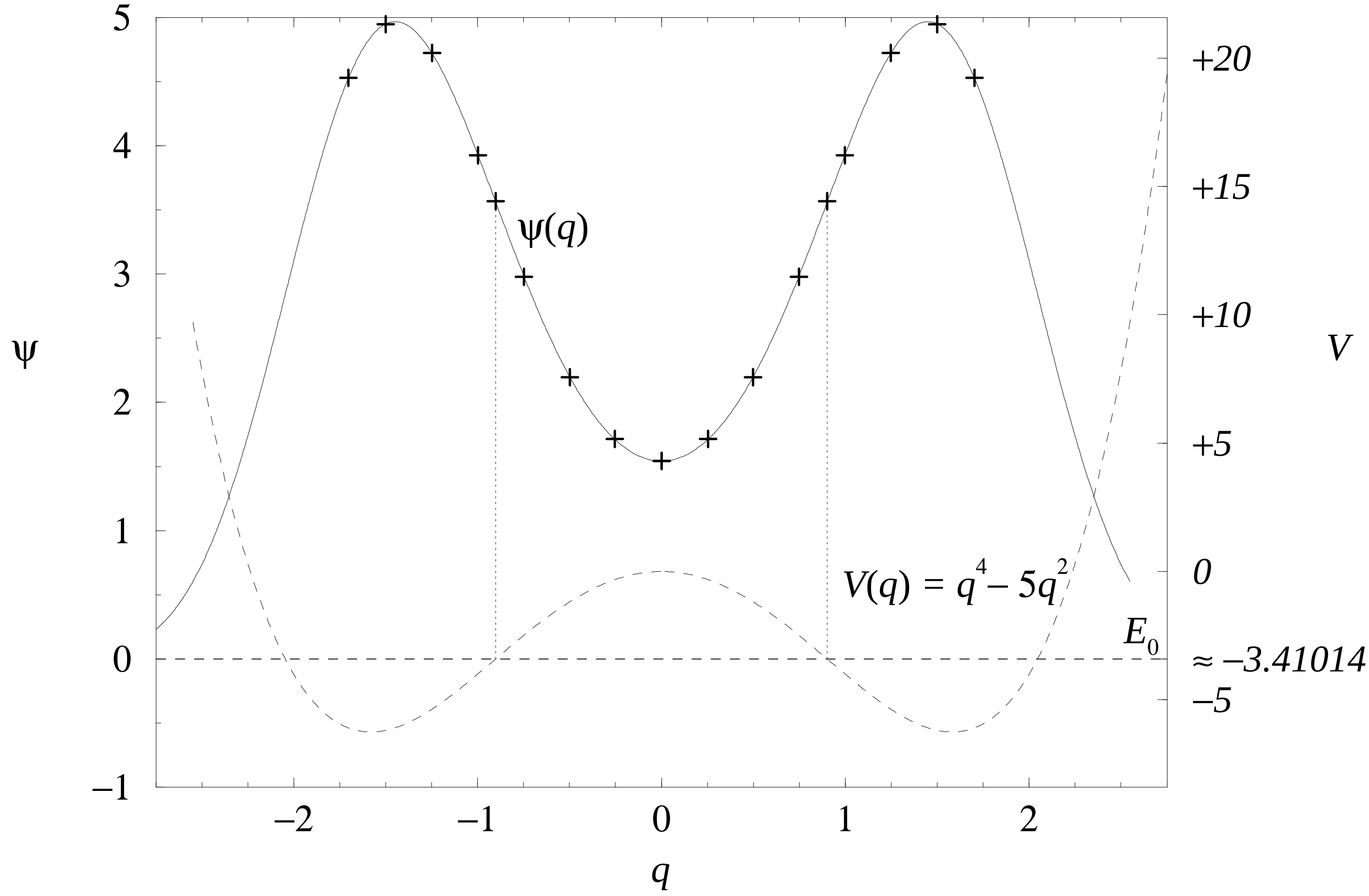
$$\psi_\lambda(Q) \equiv D_Q^-(\lambda + V(Q)), \qquad \psi'_\lambda(Q) \equiv -D_Q^+(\lambda + V(Q))$$

hence $\psi_\lambda(Q)$ follows by solving a parametric fixed-point problem

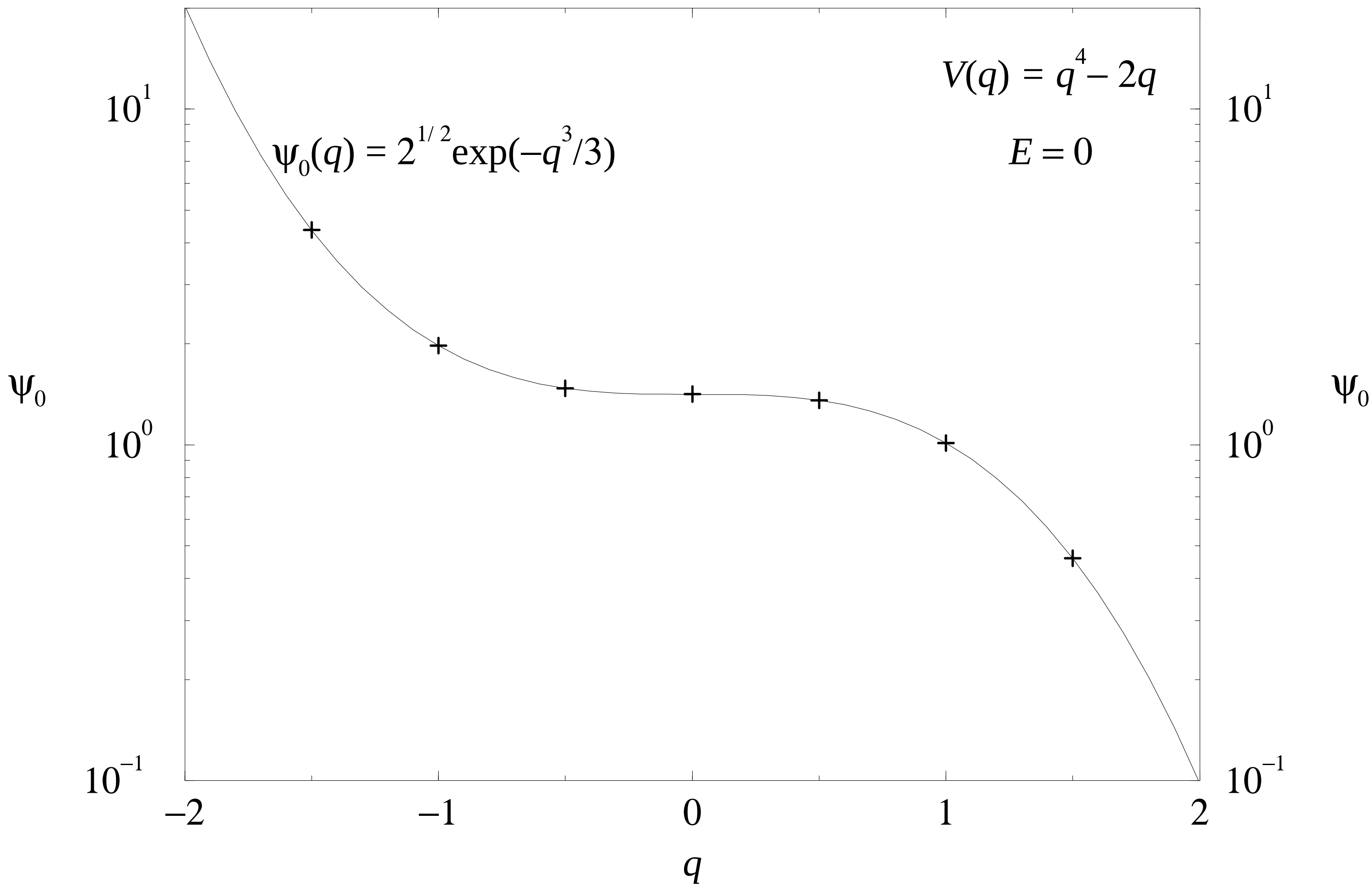
$$\mathcal{M}_Q^-\{\mathcal{E}_{Q,-}^{[\ell]}\} = \{\mathcal{E}_{Q,-}^{[\ell]}\}$$

for the Dirichlet spectrum $\mathcal{E}_{Q,-}$ of the potential V_Q.

Ground-state eigenfunction $\psi(q)$ for the potential $V(q) = q^4 - 5q^2$.



Non-square-integrable solution $\psi_0(q)$ for the potential $V(q) = q^4 - 2q$ at energy $E = 0$.



“ODE/IM correspondence”

(Dorey–Tateo, Suzuki, Bazhanov–Lukyanov–Zamolodchikov,...)

Dictionary between *some* 2D exactly solvable models and *some* 1D Schrödinger eqns.

Ordinary **D**ifferential **E**quations	$\longleftrightarrow$	**I**ntegrable **M**odels
1D Schrödinger equation with **homogeneous potential** q^{2M}		**2D 6-vertex model** with **twist** $\phi = \pi/(2M+2)$
Spectral parameter λ		Spectral parameter ν
Degree of potential $2M$	$\mathrm{e}^{2\pi \mathrm{i}/(2M+2)} = -\,\mathrm{e}^{-2\mathrm{i}\eta}$	Anisotropy η
Stokes multiplier $C(\lambda)$		Transfer matrix $T(\nu)$
$D^{-}(\lambda) = \psi_\lambda(0)$		$Q(\nu)$ operator
Exact quantization conditions		Bethe Ansatz equations

(cf. Dorey–Dunning–Tateo, *The ODE/IM correspondence* `[hep-th/0703066]`)

Extension to $\mathcal{PT}$-symmetric potentials: Dorey–Dunning–Tateo, *Ordinary Differential Equations and Integrable Models* `[hep-th/0010148]`

Open issues

- Inhomogeneous polynomial potentials:
  - contractivity of fixed-point mapping? (Numerically OK near $\vec{v} = \vec{0}$)
  - correspondence with integrable models (generalized Bethe Ansatz).

- More general problems:
  - rational potentials (e.g., centrifugal term)
  - all Heun equations
  - higher-order equations/systems, higher-dimensional Schrödinger equations, ...

- **Consistency with perturbative regime.**

Toward singular quantum perturbation theory

(here $N > M \geq 0$)

$$\hat{H}(v) = -\mathrm{d}^2/\mathrm{d}q^2 + q^N + vq^M \ (\textit{coupled}\ \text{problem}) \approx v^{2/(M+2)}\left[-\mathrm{d}^2/\mathrm{d}q^2 + q^M + gq^N\right]$$

$$\hat{H}_0(v) = -\mathrm{d}^2/\mathrm{d}q^2 + vq^M \qquad (\textit{uncoupled}\ \text{problem}) \approx v^{2/(M+2)}\left[-\mathrm{d}^2/\mathrm{d}q^2 + q^M\right]$$

hence: relate $\det^{\pm}(\hat{H}(v)+\lambda)$ to $\det^{\pm}(\hat{H}_0+\lambda)$ for $v \to +\infty \Leftrightarrow g \to 0^+$?

$g \to 0$: *a most singular limit!* E.g., in exact quantization condition

$$2\arg D(-\,\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda_k \mid \mathcal{E}_+^{[+1]}) - \varphi\,\beta_{-1}(0) = \pi\left[k + \frac{1}{2} + \frac{N-2}{2(N+2)}\right] \qquad \text{for } k = 2n,$$

- the degree jumps ($N \to M$), hence the angle φ as well;
- the anomaly type, hence $\beta_{-1}(0)$ as well, may jump (e.g., $\mathbf{N} \to \mathbf{A}$ for $q^2 + gq^4$).

Main theoretical estimate

$$\det{}^{\pm}(\hat{H}(v)+\lambda) \sim \left[\frac{\det_{\mathrm{cl}}(\hat{H}(v)+\lambda)}{\det_{\mathrm{cl}}(\hat{H}_0(v)+\lambda)}\right]^{1/2} \det{}^{\pm}(\hat{H}_0(v)+\lambda)$$

Practical implication

There only remains to compute *two improper actions*,

$$\left(\tfrac{1}{2}\log\det_{\mathrm{cl}}(\hat{H}(v)+\lambda) = \right) \int_0^{+\infty} \Pi_\lambda(q,v)\,\mathrm{d}q = \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2}\,\mathrm{d}q \quad \text{(coupled)}$$

$$\left(\tfrac{1}{2}\log\det_{\mathrm{cl}}(\hat{H}_0(v)+\lambda) = \right) \int_0^{+\infty} \Pi_{0,\lambda}(q,v)\,\mathrm{d}q = \int_0^{+\infty} (vq^M + \lambda)^{1/2}\,\mathrm{d}q \quad \text{(uncoupled)}$$

Binomial $\Pi(q)^2 = uq^N + vq^M \qquad (N > M \geq 0)$

- **Exact** evaluation of improper action integral:

$$\int_0^{+\infty} (uq^N + vq^M)^{1/2}\, \mathrm{d}q = \frac{\Gamma(\frac{M+2}{2(N-M)})\,\Gamma(-\frac{N+2}{2(N-M)})}{(N-M)\,\Gamma(-1/2)}\, u^{-\frac{M+2}{2(N-M)}}\, v^{\frac{N+2}{2(N-M)}}$$

when the RHS factor is finite, i.e., in **N**ormal case: $\dfrac{N+2}{2(N-M)} \notin \mathbb{N}$.

Binomial $\Pi(q)^2 = uq^N + vq^M \qquad (N > M \geq 0)$

- **Exact** evaluation of improper action integral:

$$\int_0^{+\infty} (uq^N + vq^M)^{1/2}\,\mathrm{d}q = \frac{\Gamma(\frac{M+2}{2(N-M)})\,\Gamma(-\frac{N+2}{2(N-M)})}{(N-M)\,\Gamma(-1/2)}\, u^{-\frac{M+2}{2(N-M)}}\, v^{\frac{N+2}{2(N-M)}}$$

when the RHS factor is finite, i.e., in **N**ormal case: $\dfrac{N+2}{2(N-M)} \notin \mathbb{N}$.

Else, namely in **A**nomalous case: $\dfrac{N+2}{2(N-M)} = j \ (\in \mathbb{N}^*)$,

$$\int_0^{+\infty} (uq^N + vq^M)^{1/2}\,\mathrm{d}q \ = \ \frac{(-1)^{j-1}}{N+2}\,\frac{(2j-2)!}{2^{2j-2}\,[(j-1)!]^2}\, u^{1/2\,-j} v^j \ \times$$

$$\left[-\log v + \sum_{m=1}^{j} \frac{1}{m} + \frac{2M}{N}\left(\log 2 + \tfrac{1}{2}\log u - \sum_{m=1}^{j-1} \frac{1}{2m-1}\right)\right].$$

Trinomial $\Pi(q)^2 = q^N + vq^M + \lambda \qquad (N > M > 0)$

- **Asymptotic** $(v \to +\infty)$ evaluation of improper action integral in general:

$$\int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2}\,\mathrm{d}q \sim \int_0^{+\infty} (q^N + vq^M)^{1/2}\,\mathrm{d}q \quad \left[= C_{N,M}\, v^{\frac{N+2}{2(N-M)}} \qquad [\mathbf{N}]\right]$$

$$+ \int_0^{+\infty} (vq^M + \lambda)^{1/2}\,\mathrm{d}q \quad \left[= C'_M \begin{cases} v^{-\frac{1}{M}}\,\lambda^{\frac{1}{2}+\frac{1}{M}} & M \neq 2 \\ v^{-\frac{1}{2}}\,\lambda(1-\log\lambda) & M = 2 \end{cases}\right]$$

$$+ \ \delta_{M,2}\ C''_N\,\lambda v^{-\frac{1}{2}}(\log v + 2\log 2).$$

- **Exactly computable case:** $N = 4$ (in terms of complete elliptic integrals)

$$\int_0^{+\infty} (q^4 + vq^2 \ + \ \lambda)^{1/2}\,\mathrm{d}q \equiv$$

$$\begin{cases} \frac{1}{3}\,\lambda^{1/4}\big[(2\sqrt{\lambda}+v)K(\tilde{k}) - 2vE(\tilde{k})\big], & \tilde{k} = \dfrac{(2\sqrt{\lambda}-v)^{1/2}}{2\,\lambda^{1/4}} \qquad (v \le 2\sqrt{\lambda}) \\ \frac{1}{3}(v+2\sqrt{\lambda})^{1/2}\big[2\sqrt{\lambda}K(k) - vE(k)\big], & k = \left(\dfrac{v-2\sqrt{\lambda}}{v+2\sqrt{\lambda}}\right)^{1/2} \qquad (v \ge 2\sqrt{\lambda}) \end{cases}$$

$$\sim -\tfrac{1}{3}\,v^{3/2} + 0\,v^{1/2}\log v + 0\,v^{1/2} - \tfrac{1}{4}\,\lambda v^{-1/2}\left[\log(\lambda/v^2) - 4\log 2 - 1\right].$$

Samples of end results ($g \to 0^+$ limit, E fixed)

$$\frac{\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^M + gq^N - E)}{\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^M - E)} \sim g^{-\frac{4}{N(N+2)}\beta_{-1}(0)} \times \quad [\mathbf{A}]$$

$$\exp\, 2\int_0^{+\infty} (q^N + vq^M)^{1/2}\,\mathrm{d}q \times$$

$$\exp\left\{\delta_{M,2}\,\frac{1}{N-2}\,[\log g - N\log 2]\,E\right\}$$

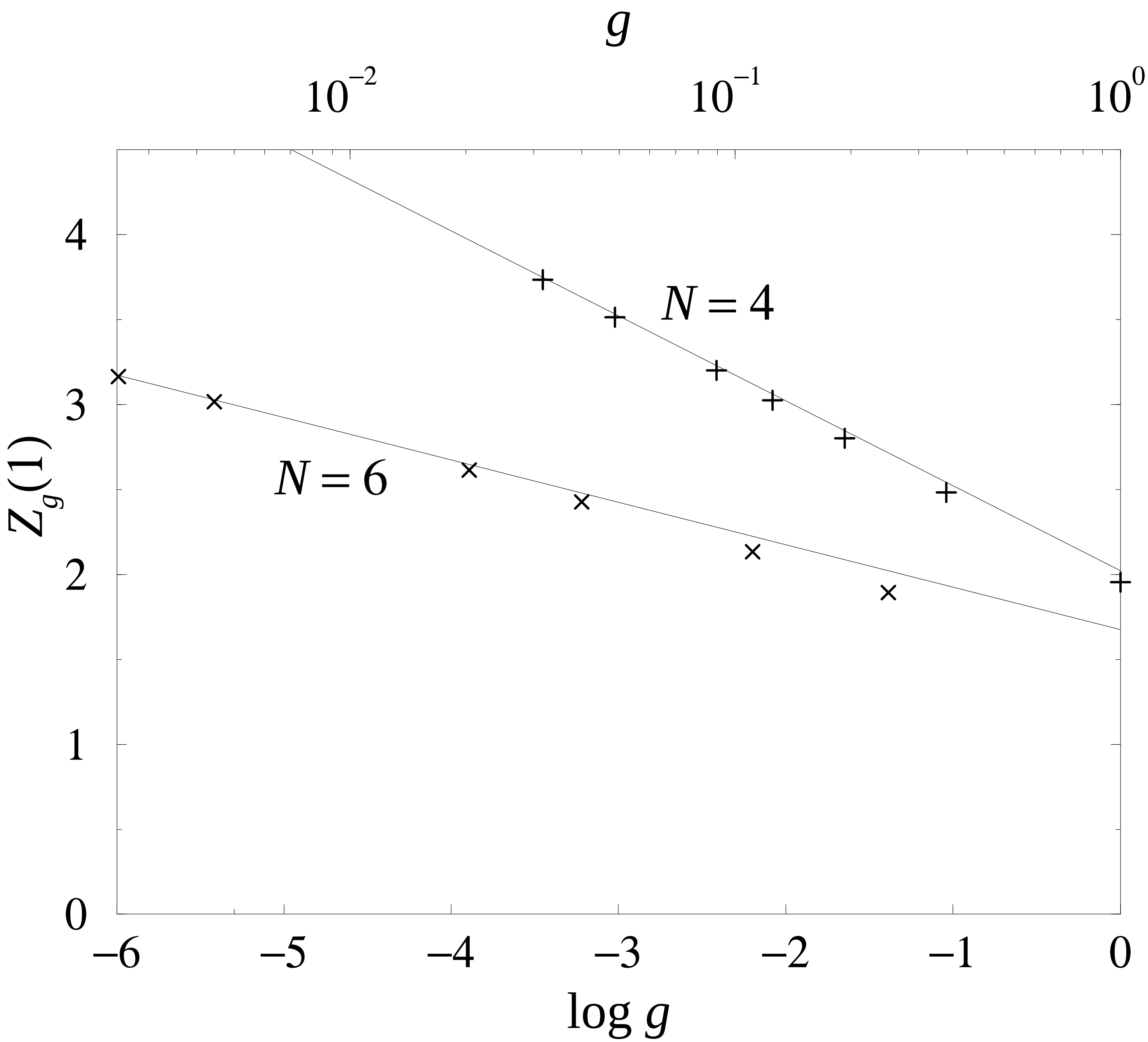
(with $\int_0^{+\infty}(q^N + vq^M)^{1/2}\,\mathrm{d}q \propto g^{-(M+2)/\,2(N-M)}$).

Basic example $N = 4,\ M = 2$:

$$\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^2 + gq^4 - E\right) \sim \exp\left\{-\frac{2}{3g} + \left[\tfrac{1}{2}\log g - 2\log 2\right]E\right\} \underbrace{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^2 - E\right)}_{2^{E/2}\sqrt{2\pi}/\,\Gamma\left(\frac{1}{2}(1-E)\right)}$$

$$\sum_{k=0}^{\infty}\frac{1}{E_k(g) - E} \sim -\left[\tfrac{1}{2}\log g - 2\log 2\right] - \tfrac{1}{2}\left[\log 2 + \psi\left(\tfrac{1}{2}(1-E)\right)\right]$$

$$Z_g(1) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty}\frac{1}{E_k(g)} \sim -\tfrac{1}{2}\log g + \tfrac{1}{2}(\gamma + 5\log 2) \qquad (g \to 0^+).$$



g
10^−2
10^−1
10^0
N = 4
N = 6
Z_g(1)
4
3
2
1
0
−6
−5
−4
−3
−2
−1
0
log g