Non-perturbative Insights into QCD Phase Structure via Functional Renormalization Group

Fabrizio Murgana

SUBATECH

In collaboration with

Dipartimento di Fisica e Astronomia Ettore Majorana, Università degli studi di Catania

Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität Frankfurt am Main

Nantes

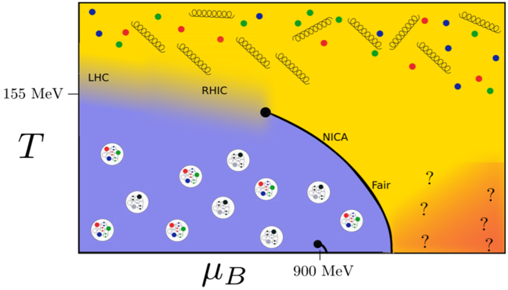
15 October 2025

Motivation

Motivation

- Field theories, and in particular QCD, change their behaviour changing energy scale $\Longrightarrow$ Phase transitions:

- turn from weak to strong coupling;
- change in the relevant degrees of freedom
- Different realization of the fundamental symmetries.



Guenther, J.N. Overview of the QCD phase diagram. Eur. Phys. J. A 57, 136 (2021).

- A non-perturbative approach is needed.
- Possible solution $\Longrightarrow$ Functional Renormalization Group (FRG)

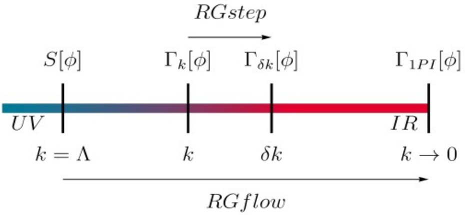
Motivation

- **Why FRG?**
  - Non-perturbative;
  - Fluctuations are taken into account not all at once but from scale to scale;
  - No a priori limitations. However...
- Difficult application to full QCD $\Rightarrow$ Effective field theories and models.
- **Advantages**:
  - Capture the (expected) essential features of the system in a given regime;
  - Insight on the relevant degrees of freedom;
  - Simpler calculations;
- **Disadvantage**:
  - Not the full theory $\Rightarrow$ loss of information.
- In this work we focus on chiral symmetry of QCD:
  - Quark-Meson model.

The Functional Renormalization Group

FRG Flow Equation

- FRG implements Wilson's RG approach ⇒ Fluctuations integrated by momentum shells.
- We consider the *(scale dependent) effective average action* Γ_k



$$\Gamma_{k\to\Lambda} = S_{bare} \quad \Gamma_{k\to 0} = \Gamma$$

- Γ_k can be constructed defining an IR regulated generating functional

$$e^{W_k[J]} \equiv Z_k[J] := \int_\Lambda \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi}$$

- where ΔS_k is a regulator term of the form

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\,\varphi(-p)\,R_k(p)\,\varphi(p)$$

- The effective average action is given by:

$$\Gamma_k[\phi] = \sup_J\left(\int J\phi - W_k[J]\right) - \Delta S_k[\phi]$$

FRG Flow Equation

The **Wetterich flow equation** describes the k-(or t-)evoultion of Γ_k:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr}\left[\partial_t R_k (\Gamma_k^{(2)}[\phi] + R_k)^{-1}\right] \qquad t = -\ln\frac{k}{\Lambda}$$

Key features:

- Exact one-loop structure;
- The purpose of the regulator is twofold:
  - IR Regularization;
  - Implements the idea of integrating over momentum shells $p^2 \sim k^2$;
- The flow equation is a functional integro-differential equation for Γ_k;
- Difficult to solve exactly $\Rightarrow$ we need some ansatz.
- We will use a *derivative expansion*:

$$\Gamma_k[\phi] = \int d^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + \mathcal{O}((\partial^2)^2)\right] .$$

[1] C. Wetterich, Phys. Lett. B 301 (1993) 90-94.
[2] K. G. Wilson, Phys. Rev. B 4, (1971) 3174, Phys. Rev. B 4, (1971) 3184.
[3] J. Berges, N. Tetradis, C. Wetterich, Phys.Rept. 363 (2002) 223-386.

The Quark-Meson model

Quark-Meson model

- The $N_f = 2$ QM model uses as fundamental degrees of freedom mesons coupled to quarks

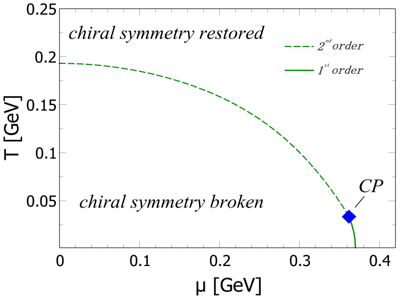
$$\mathcal{L}_{QM}^{E} = \bar{\psi}(\gamma_\mu \partial^\mu + h(\sigma + i\gamma^5 \vec{\tau}\vec{\pi}))\psi + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + U(\sigma^2 + \vec{\pi}^2)$$

- Chiral phase transition: SSB $O(4) \rightarrow O(3)$

$$\langle \bar{\psi}\psi \rangle \simeq \langle \sigma \rangle \begin{cases} > 0 \Leftrightarrow & \text{symmetry breaking} & T < T_c \\ = 0 \Leftrightarrow & \text{symmetry restoration} & T > T_c \end{cases}$$

- 3 massless Goldstone bosons (pions).

- Expected features of the QM model phase diagram:
  - 2nd order phase transition at $\mu = 0$;
  - 1st order phase transition at $T = 0$;
  - critical endpoint.



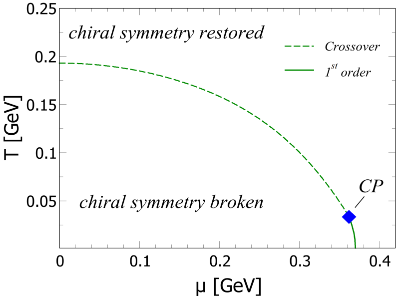
Finite quark mass

- In order to mimic the presence of a finite current quark mass we use a term

$$\mathcal{L}_m = -c\sigma$$

- The O(4) symmetry is (also) explicitly broken by the term $-c\sigma \Rightarrow$
  - spontaneous symmetry-breaking pattern is not exact;
  - $\langle\sigma\rangle \to 0$ never exactly;
  - the O(4) symmetry is never exactly restored;
  - Pions turn into massive pseudo-Goldstone mesons.

- Phase diagram:
  - Crossover at $\mu = 0$;
  - 1st-order phase transition at $T = 0$;
  - critical endpoint.



Quark-Meson model: FRG setup

- Ansatz for effective action: LPA

$$\Gamma_k[\bar{\Psi}, \Psi, \phi] = \int_0^\beta dx_0 \int d^3\mathbf{x} \left\{ \bar{\psi}\big(\gamma_\mu \partial^\mu + h(\sigma + i\gamma^5 \vec{\tau}\vec{\pi}) - \mu\gamma_0\big)\psi + \frac{1}{2}(\partial_\mu \phi)^2 + U_k(\phi^2) \right\}$$

- We can express the flow equation in terms of $u_k(\sigma) = \partial_\sigma U_k(\sigma)$

$$\partial_t u_k(\sigma) + \partial_\sigma f_k(\sigma, u_k(\sigma)) = \partial_\sigma g_k(u_k'(\sigma)) + N_c \partial_\sigma S_k(\sigma)$$

- Advection and diffusion fluxes

$$f_k(\sigma, u_k) = f_k(E_{k,\pi}) \qquad g_k(u_k') = g_k(E_{k,\sigma})$$

where

$$E_{k,\pi} = \sqrt{k^2 + u_k(\sigma)/\sigma} \qquad E_{k,\sigma} = \sqrt{k^2 + u_k'(\sigma)}\,.$$

- Advection and diffusion coefficients:

$$\partial_{u_k} f_k(\sigma, u_k) < 0 \qquad \partial_{u_k'} g_k(u_k') > 0 \qquad \forall \sigma > 0$$

- Source term:

$$S_k(\sigma) = S_k(E_{k,\Psi}) \qquad E_{k,\,\Psi} = \sqrt{k^2 + (h\sigma)^2}$$

[4] E. Grossi and N. Wink (2019), arXiv:1903.09503.
[5] A. Koenigstein, M. J. Steil, N. Wink, E. Grossi, J. Braun, M. Buballa, and Dirk H. Rischke, Phys. Rev. D 106, 065012 (2022)

Thermodynamic geometry

Thermodynamic geometry: The concept

- An equilibrium state for a thermodynamic system can be characterized by the pair $(\beta = 1/T\,,\, \gamma = -\mu/T)$.
- **Key idea**: we consider the (β, γ)-space as a two-dimensional manifold.
- We introduce a distance in this space

$$dl^2 = g_{\beta\beta} d\beta d\beta + 2 g_{\beta\gamma} d\beta d\gamma + g_{\gamma\gamma} d\gamma d\gamma \,,$$

  where the metric tensor is

$$g_{ij} = \frac{\partial^2 \log \mathcal{Z}}{\partial \beta^i \partial \beta^j} = \frac{\partial^2 \phi}{\partial \beta^i \partial \beta^j} \equiv \phi_{,ij} \,,$$

  with $\phi = \beta P$, $P = -\Omega$, $\beta^1 = \beta$ and $\beta^2 = \gamma$.
- One can define the Riemann tensor as

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl},$$

  with the Christoffel symbols

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

- Ricci tensor $R_{ij} = R^k_{ikj}$, and scalar curvature $R = R^i_i$. Within thermodynamic geometry, R is called the **thermodynamic curvature**.

Thermodynamic geometry: The concept

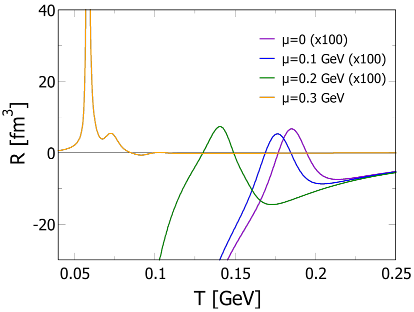
- For our two-dimensional manifold we have

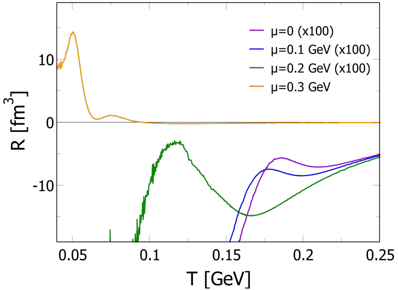
$$R = -\frac{1}{2\,g^2}\begin{vmatrix} \phi_{,\beta\beta} & \phi_{,\beta\gamma} & \phi_{,\gamma\gamma} \\ \phi_{,\beta\beta\beta} & \phi_{,\beta\beta\gamma} & \phi_{,\beta\gamma\gamma} \\ \phi_{,\beta\beta\gamma} & \phi_{,\beta\gamma\gamma} & \phi_{,\gamma\gamma\gamma} \end{vmatrix},$$

- R depends on the second- and third-order moments of the thermodynamic variables $\Rightarrow$information about the fluctuation of the physical quantities.
- Close to a second-order phase transition $|R| \propto \xi^3 \to \infty \Rightarrow$ information on the correlation volume.
- R can convey details about the nature of the interaction:
  - $R > 0$ indicates an attractive interaction;
  - $R < 0$ corresponds to a repulsive one.
- These interactions include also the statistical attraction and repulsion in phase space :
  1. $R < 0$ for an ideal Fermi gas;
  2. $R > 0$ for an ideal Bose Gas;
  3. $R = 0$ for an ideal classical gas.

Thermodynamic geometry: Results

- **Crossover region** ($\mu \ll \mu_c$):
  - R peaked around the pseudo-critical temperature $\Rightarrow$ R sensitive to the chiral crossover;
  - MF positive peaks, FRG negative ones $\Rightarrow$ the sign is sensitive to the approximation;
- **Critical region** ($\mu \sim \mu_c$):
  - R enhanced close to the critical point $\Rightarrow$ R sensitive to the chiral phase transition;
  - For both MF and FRG, R shows a positive peak $\Rightarrow$ Qualitative behavior of R independent of the approximation.





[6]Murgana et al Phys. Rev. D 109, no.9, 096017 (2024)

Testing reconstruction from imaginary chemical potential

Testing reconstruction: Motivation

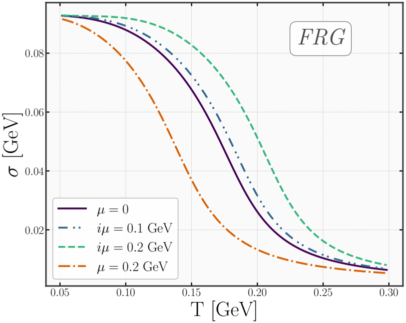
- **Motivation**: Study the QCD phase diagram using both FRG (applied to effective models) and lQCD.
- **Challenge**:
  - lQCD faces the "sign problem" at finite μ, making direct simulations difficult;
  - Reconstruction techniques from imaginary μ can be used.
- **Key Questions**:
  - How reliable is the extrapolation from imaginary μ?
  - How to test it?
- **Idea**: Use the FRG (applied to the QM model):
  - Non-perturbative.
  - No a priori limitations (sign problem)
- In this framework one has access to both:
  - Direct results at both real and imaginary μ.
  - Extrapolated results from imaginary μ.
- Comparison and test is possible and well controlled.

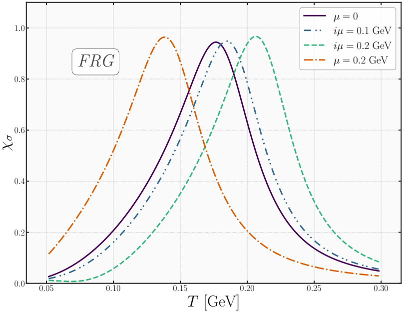
Testing reconstruction: Results

- **Phase Boundary**:
  - Direct calculation: T_c defined by the peak of the chiral susceptibility:

$$\chi_\sigma = -\frac{\partial \langle \sigma \rangle}{\partial T}$$

  - Used for both real and imaginary μ.





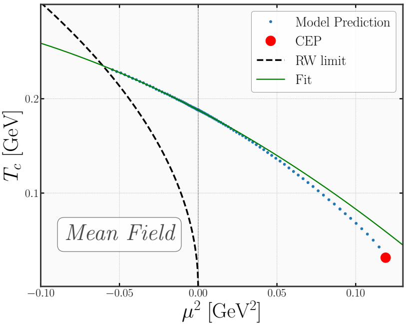
[7] F. Murgana and M. Ruggieri Phys.Rev.D 112 (2025)

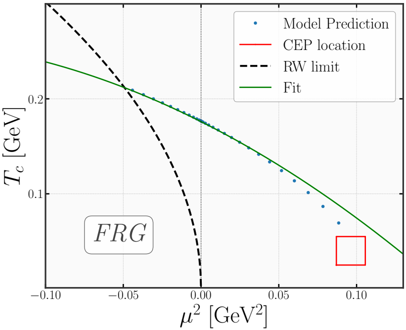
Testing reconstruction: Results

- **Phase Boundary**:
  - Reconstructed from imaginary μ:

$$\frac{T_c(\mu)}{T_c} = 1 - \kappa_2 \left(\frac{\mu}{T_c}\right)^2 - \kappa_4 \left(\frac{\mu}{T_c}\right)^4$$

- **Results**:
  - Excellent agreement at low μ (crossover region).
  - Growing discrepancy near CEP.





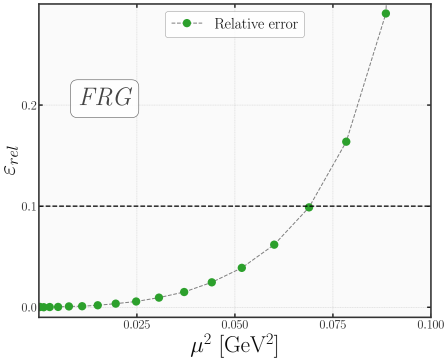
[7] F. Murgana and M. Ruggieri Phys.Rev.D 112 (2025)

Testing reconstruction: Results

- **Convergence Radius**:

$$\varepsilon_{rel} = \frac{|T_c - T_c^{(fit)}|}{T_c};$$

  - μ_{conv} such that $\varepsilon_{rel} \leq 0.1$;
  - Effective radius $\mu_{\text{conv}} \approx 260$ MeV (both MF and FRG);
  - $\varepsilon_{\text{rel}} \approx 1.0$ near CEP (model dependent), limiting extrapolation reliability.



[7] F. Murgana and M. Ruggieri Phys.Rev.D 112 (2025)

Conclusions and Outlook

Conclusions and Outlook

- We described the **FRG** approach to QFT and in particular FRG flow equation;
- We applied an **hydrodynamic approach** to the FRG method in order to study the phase diagram of the **Quark-Meson model**;
- We explored the QM model phase diagram using the **Thermodynamic Geometry technique**:
  - R is peaked around the pseudo-critical temperature$\Rightarrow$ R sensitive to chiral crossover;
  - R exhibits a positive sharper peak around the critical temperature$\Rightarrow$ R sensitive to chiral phase transition;
  - Qualitative behavior of R independent of the approximation close to criticality.
- We tested **reconstruction techniques** from imaginary μ with FRG:
  - within FRG results from both real and imaginary μ are available for comparison;
  - Extrapolation from imaginary μ works well for $\mu < \mu_{\text{conv}} \simeq 260$ MeV;
  - Near CEP, non-analytic effects cause large errors.

Conclusions and Outlook

What's next?

- Possible generalizations to **higher order truncations** (LPA', $O((\partial^2)^2)$, ...):
  - Extension of the hydrodynamic formulation beyond LPA (inhomogeneous phases);
  - Study the Thermodynamic Geometry for the QM model beyond LPA;
  - Study the influence of truncation on the reconstruction procedure of the phase diagram.
- Inclusion of **different chemical potential axes** in the thermodynamic geometry framework (μ_I and exact comparison with lattice at $\mu_B \sim 0$);
- Improve reconstruction techniques for **critical regions**;
- Extend to **Polyakov-loop** models for confinement-deconfinement transition;
- More realistic QCD description with FRG to cross-validate with other methods (DSE, lattice).

Thanks for your attention!