Block-weighted maps and Liouville quantum duality

Emmanuel Guitter

Institut de Physique Théorique
CEA Paris-Saclay

Joint work with Bertrand Duplantier

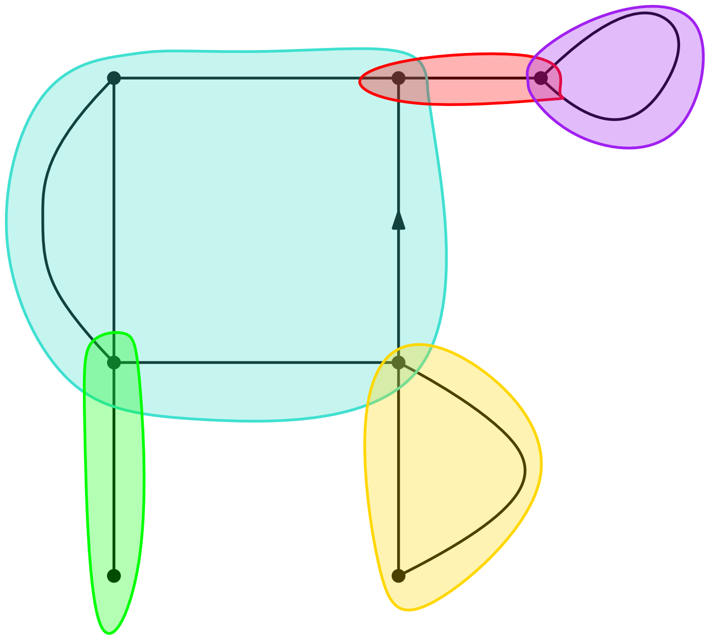
arXiv 2507.12203

Journée Cartes 17 novembre 2025

Block-weighted planar maps

Electronic Journal of Probability

Electron. J. Probab. **29** (2024), article no. 34, 1–61.
ISSN: 1083-6489 https://doi.org/10.1214/24-EJP1089

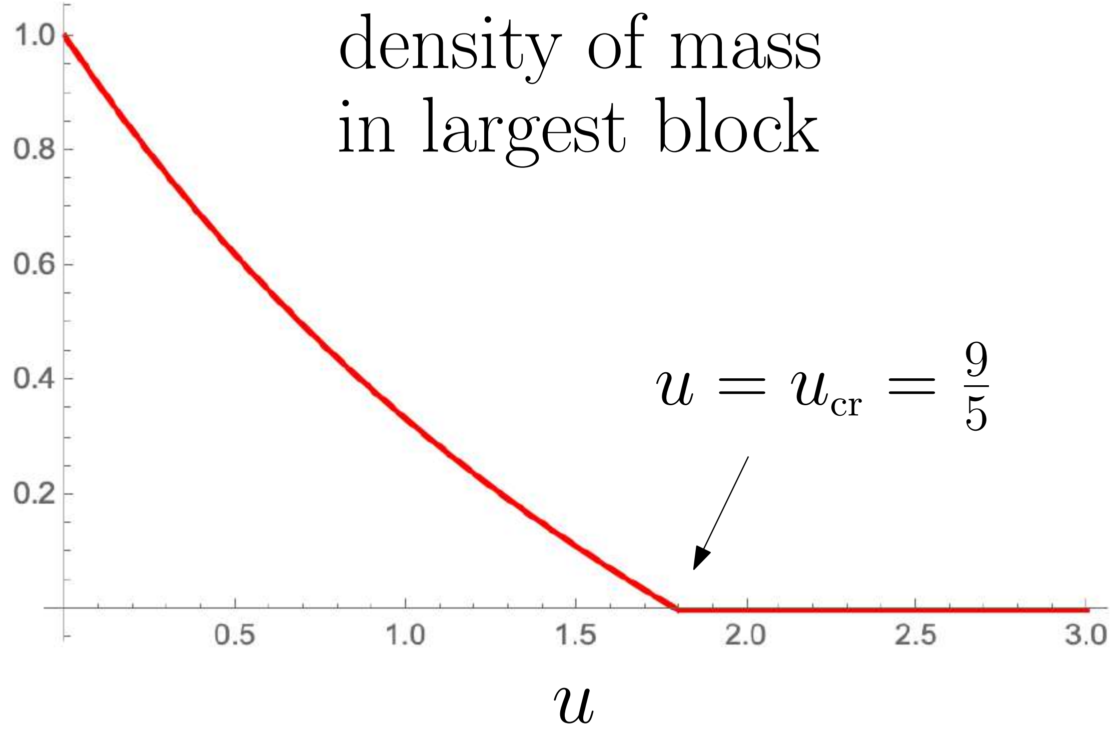


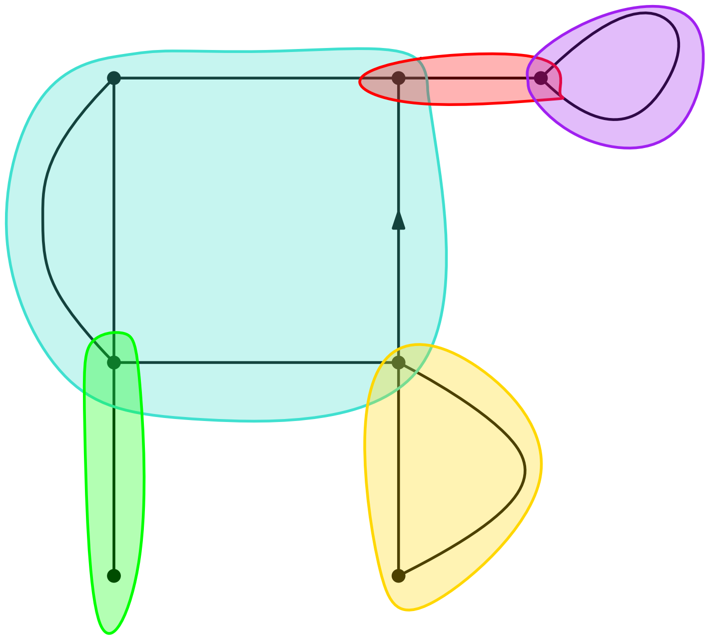
A phase transition in block-weighted random maps

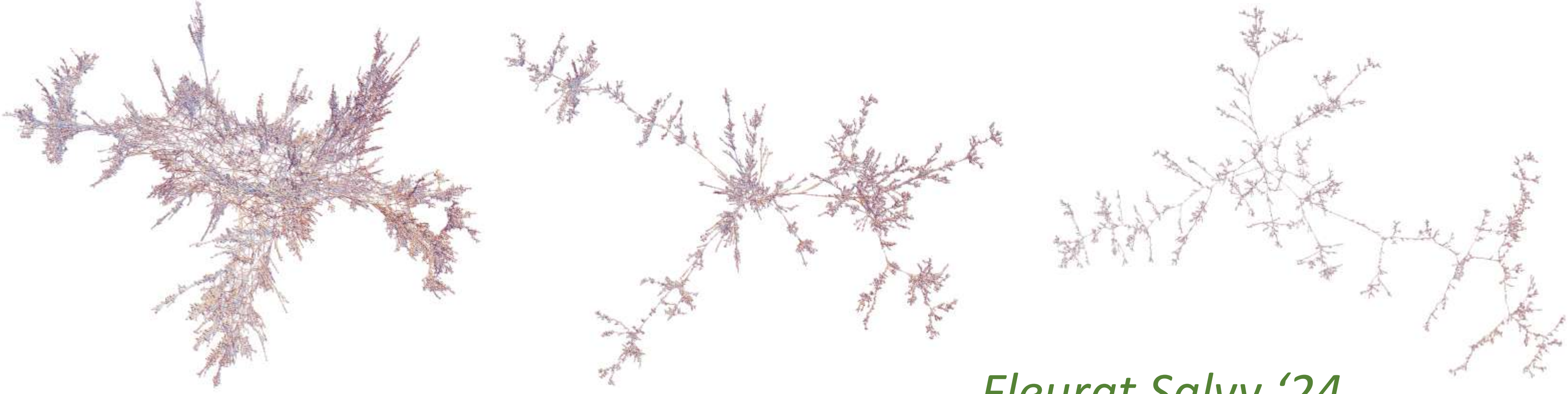
William Fleurat* Zéphyr Salvy†

(Rooted) planar maps with a weight u per 2-connected block

-> **transition** at $u = 9/5$





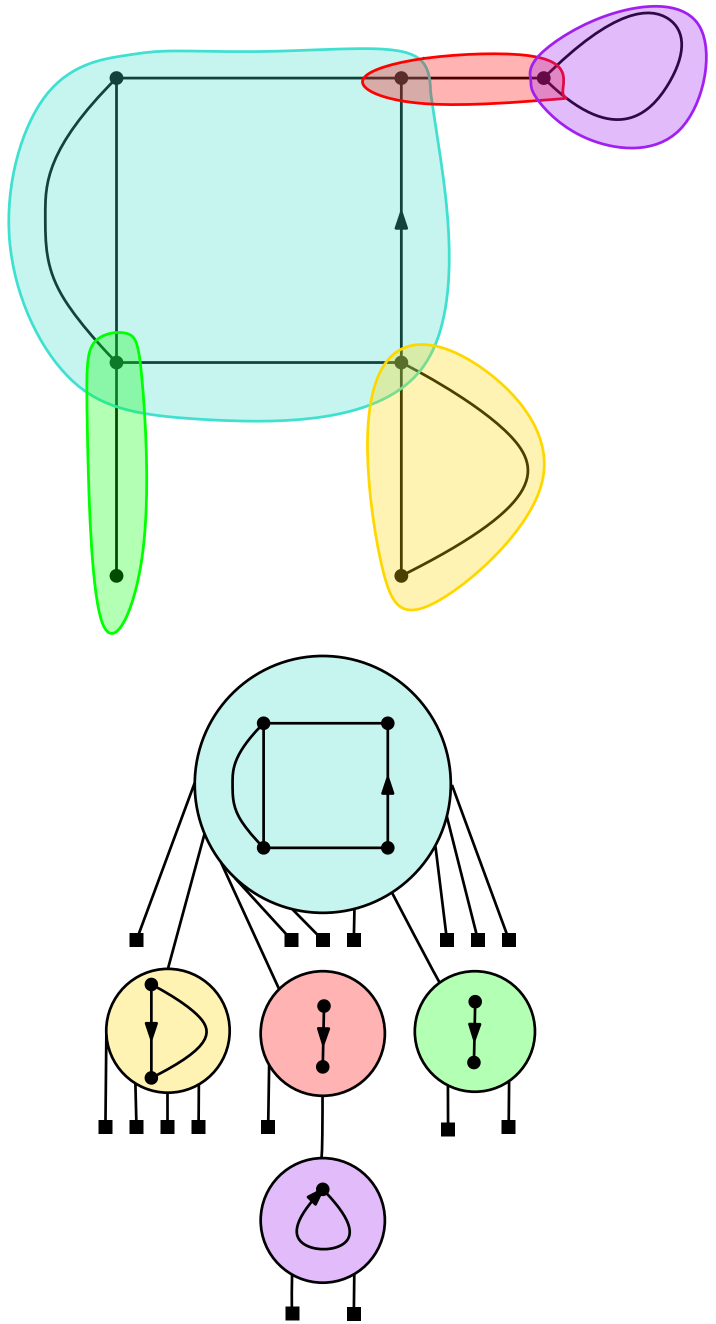


Fleurat Salvy '24

Maps with n edges

$$M_n \propto \begin{cases} \rho(u)^{-n}\ n^{-5/2} & u < 9/5 \\ \rho(u)^{-n}\ n^{-5/3} & u = 9/5 \\ \rho(u)^{-n}\ n^{-3/2} & u > 9/5 \end{cases}$$

Analysis of the **block-tree** = tree whose inner vertices are the blocks and whose edges code for the connexions of these blocks



Phase transition for tree-rooted maps '24

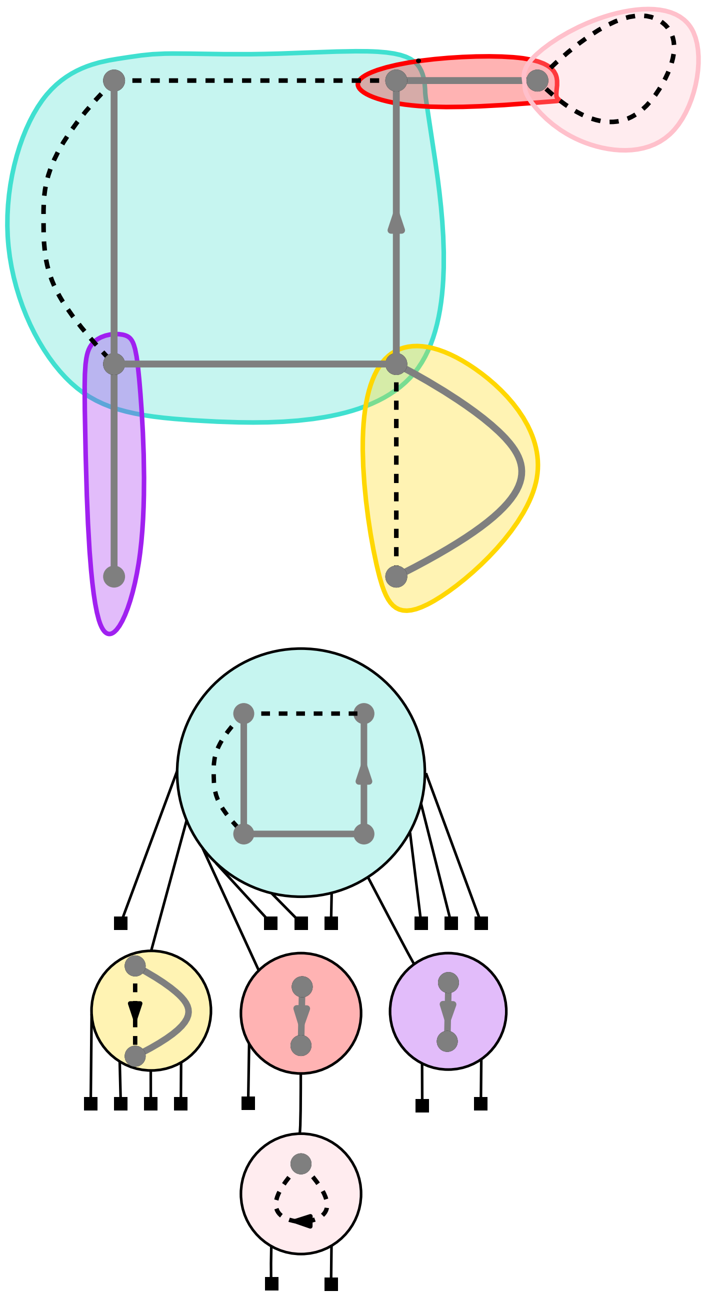
Marie Albenque ✉
IRIF, Université Paris Cité, France

Éric Fusy ✉
Univ Gustave Eiffel, CNRS, LIGM, F-77454 Marne-la-Vallée, France

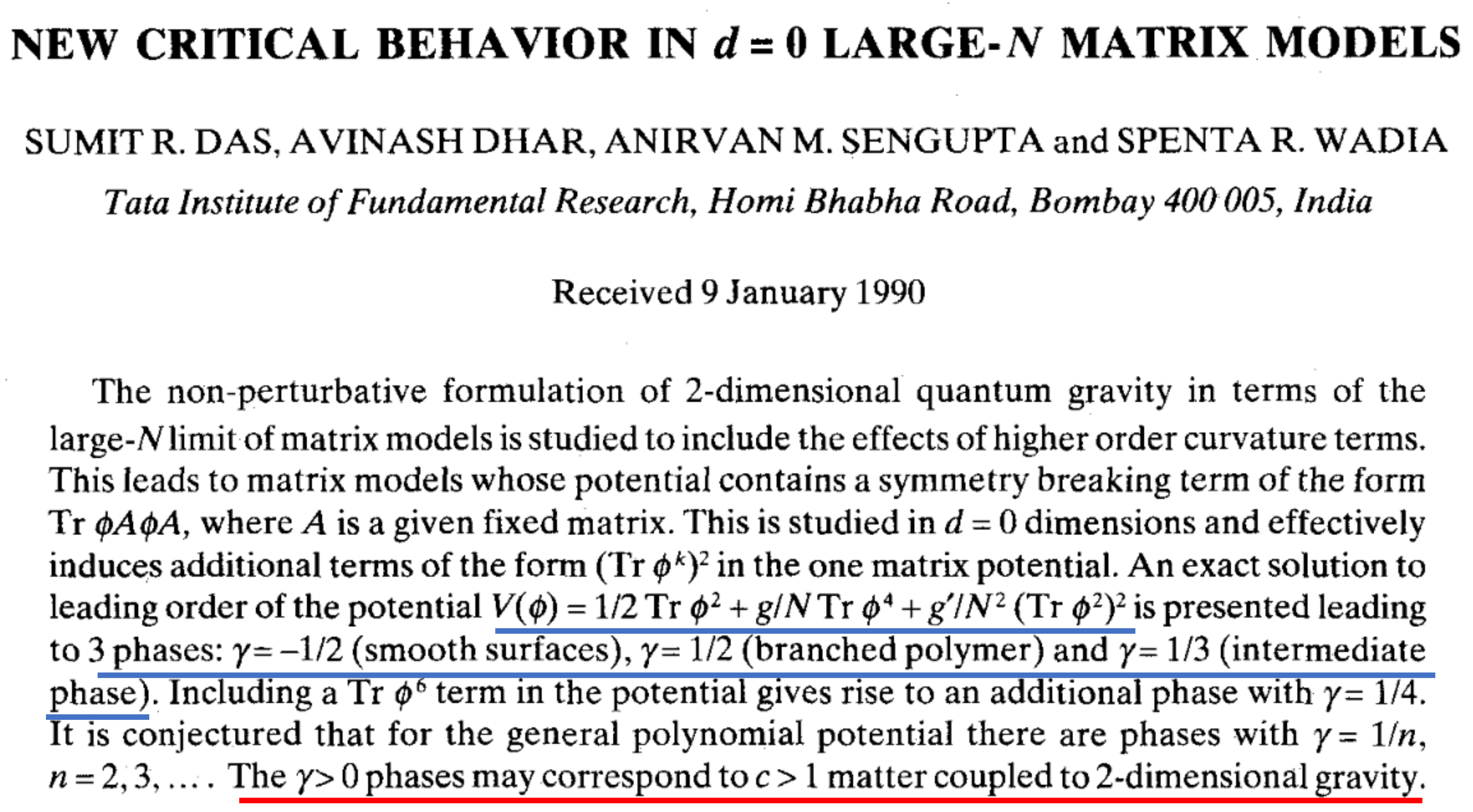
Zéphyr Salvy ✉
Univ Gustave Eiffel, CNRS, LIGM, F-77454 Marne-la-Vallée, France

$$M_n \propto \begin{cases} \rho(u)^{-n}\; n^{-3} & u < u_{\text{cr}} \\ \rho(u)^{-n}\; n^{-3/2} (\log n)^{-1/2} & u = u_{\text{cr}} \\ \rho(u)^{-n}\; n^{-3/2} & u > u_{\text{cr}} \end{cases}$$

$$u_{\text{cr}} = \frac{9\pi\,(4-\pi)}{420\,\pi - 81\,\pi^2 - 512}$$



Back to physics literature



NEW CRITICAL BEHAVIOR IN $d = 0$ LARGE-N MATRIX MODELS

SUMIT R. DAS, AVINASH DHAR, ANIRVAN M. SENGUPTA and SPENTA R. WADIA

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

Received 9 January 1990

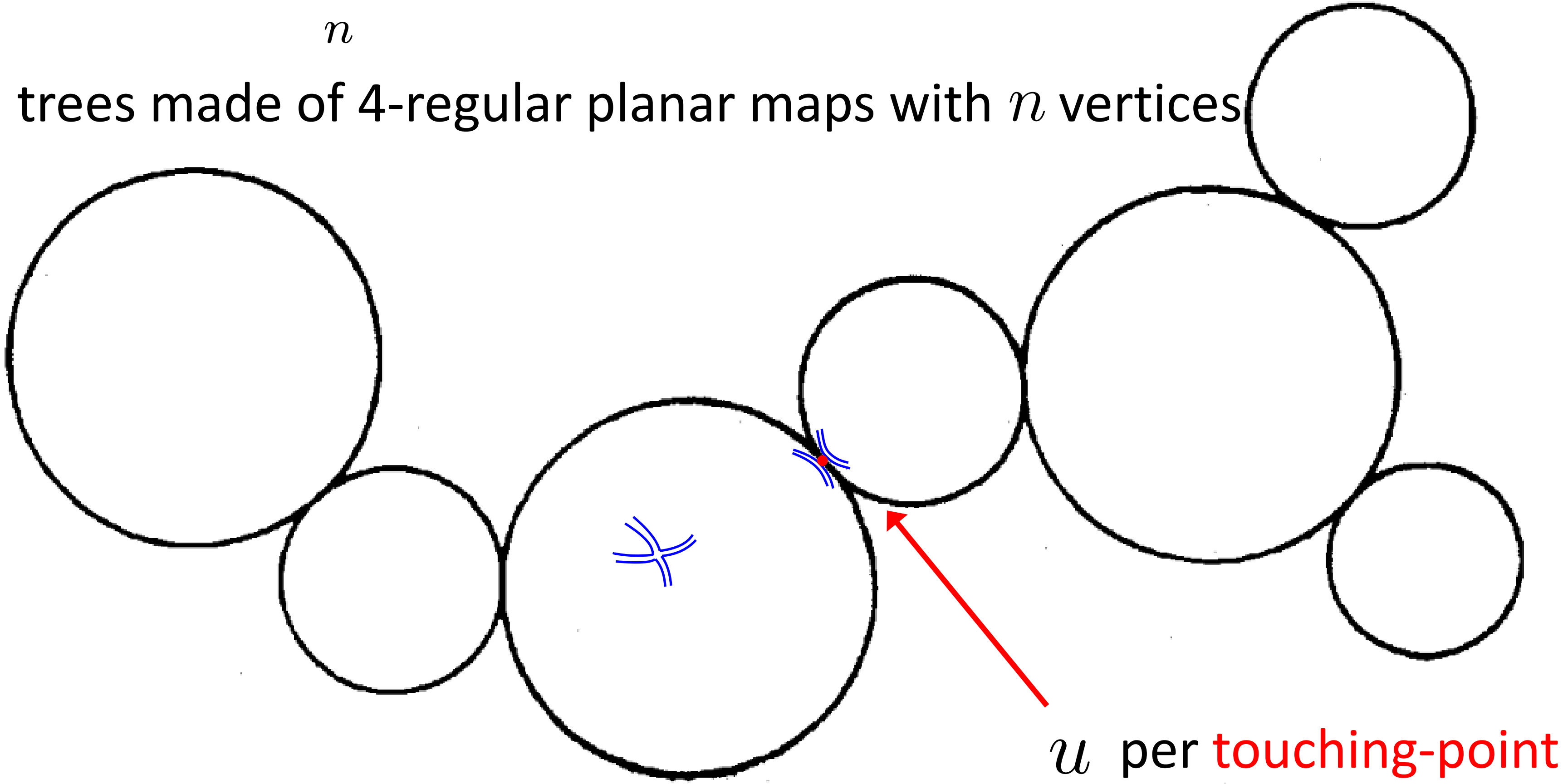
The non-perturbative formulation of 2-dimensional quantum gravity in terms of the large-N limit of matrix models is studied to include the effects of higher order curvature terms. This leads to matrix models whose potential contains a symmetry breaking term of the form $\mathrm{Tr}\,\phi A \phi A$, where A is a given fixed matrix. This is studied in $d = 0$ dimensions and effectively induces additional terms of the form $(\mathrm{Tr}\,\phi^k)^2$ in the one matrix potential. An exact solution to leading order of the potential $V(\phi) = 1/2\,\mathrm{Tr}\,\phi^2 + g/N\,\mathrm{Tr}\,\phi^4 + g'/N^2\,(\mathrm{Tr}\,\phi^2)^2$ is presented leading to 3 phases: $\gamma = -1/2$ (smooth surfaces), $\gamma = 1/2$ (branched polymer) and $\gamma = 1/3$ (intermediate phase). Including a $\mathrm{Tr}\,\phi^6$ term in the potential gives rise to an additional phase with $\gamma = 1/4$. It is conjectured that for the general polynomial potential there are phases with $\gamma = 1/n$, $n = 2, 3, \ldots$. The $\gamma > 0$ phases may correspond to $c > 1$ matter coupled to 2-dimensional gravity.

Matrix integral with potential $V(\Phi) = \frac{1}{2}\,\mathrm{Tr}\,\Phi^2 - \frac{g}{4N}\,\mathrm{Tr}\,\Phi^4 - \frac{u}{4N^2}(\mathrm{Tr}\,\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} \exp(-N^2 Z(g,u))$

$$Z(g,u) = \sum_n g^n Z_n(u)/(4n)$$

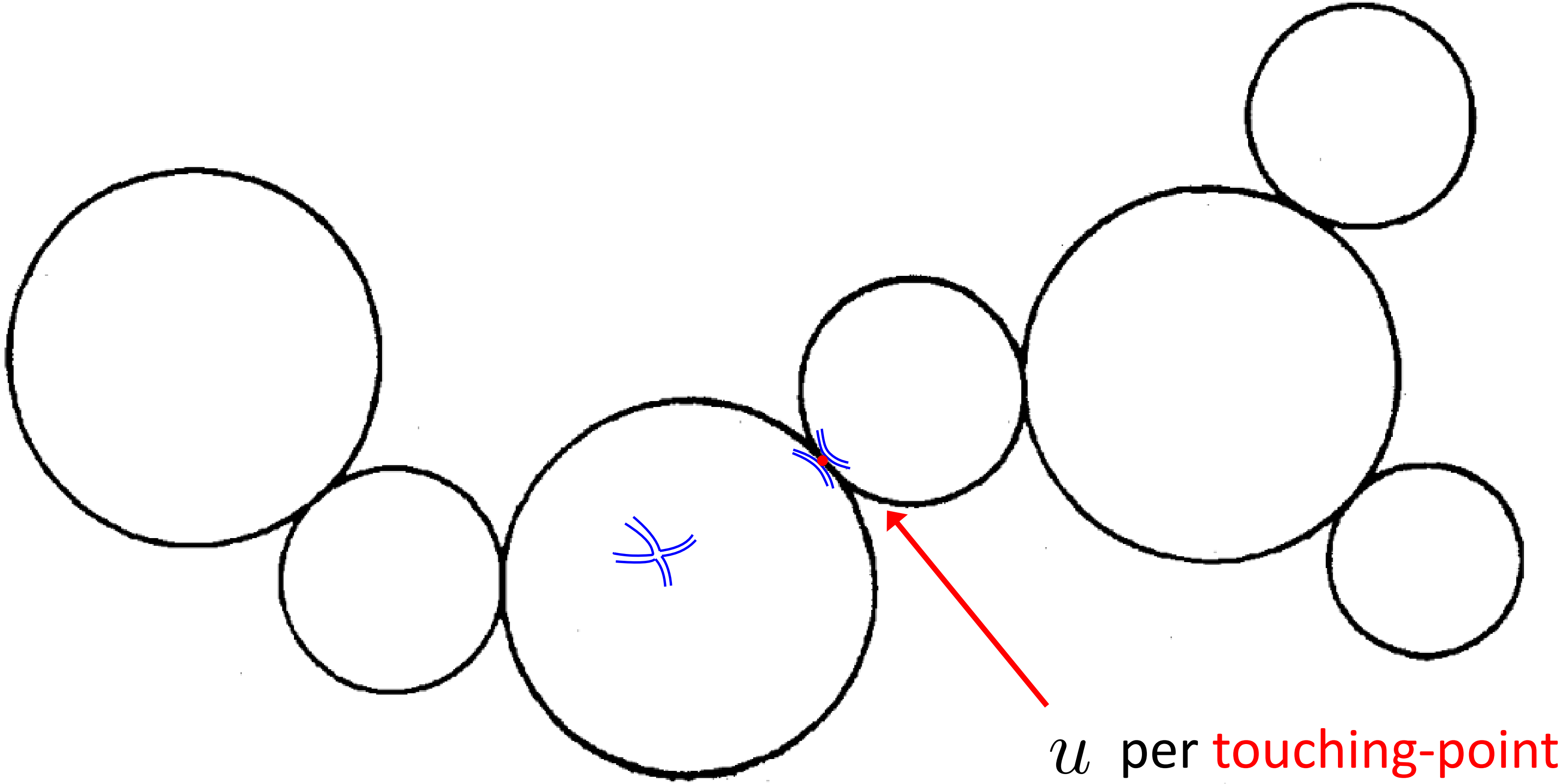
$Z_n(u)$ g.f. for (rooted) trees made of 4-regular planar maps with n vertices



$$Z_n(u) \propto \frac{g_c^{-n}}{n^{2-\gamma_S}}$$

$$2-\gamma_S = \begin{cases} 2-\left(-\frac{1}{2}\right) = 5/2 & u < 9/64 \\ 2-\left(\frac{1}{3}\right) = 5/3 & u = 9/64 \\ 2-\left(\frac{1}{2}\right) = 3/2 & u > 9/64 \end{cases}$$

Baby-universes



… more generally

Loops in the curvature matrix model

G.P. Korchemsky [1,2,3]

Dipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, I-43100 Parma, Italy

Received 8 July 1992

For $g<g_0=16/(\alpha_0 c_0)^2$, the model is in the phase of smooth (Liouville) surfaces with the string susceptibility exponent $\gamma_{\rm str}=-1/m$ ($m=2, 3, \ldots$). The critical values of the parameters are

$$\alpha_{\rm cr}=\alpha_0, \quad c_{\rm cr}=c_0, \quad \bar g(\alpha_{\rm cr})=0\,, \tag{1.3}$$

where the parameter $c<0$ defines the boundary of the cut of the one loop correlator, defined below in (1.6) and (1.7). Near the critical point they scale as

$$\chi \sim c-c_0 \sim (\alpha-\alpha_0)^{1/m}, \quad \bar g(\alpha) \sim \alpha-\alpha_0\,. \tag{1.4}$$

For $g=g_0$, the model turns into the intermediate phase with the critical exponent $\gamma_{\rm str}=1/(m+1)$. The critical values of the parameters are the same (1.3) as in the previous phase, but their scaling is different:

$$\chi \sim \frac{1}{c-c_0} \sim (\alpha-\alpha_0)^{-1/(m+1)}, \quad \bar g(\alpha) \sim (\alpha-\alpha_0)^{m/(m+1)}\,. \tag{1.5}$$

maps equipped with
a minimal $(2, 2m-1)$
conformal model

$$\gamma_{\rm S} = -\frac{1}{m} \to \gamma_{\rm S}' = \frac{1}{m+1}$$

… even more generally

Touching random surfaces and Liouville gravity

Igor R. Klebanov
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 25 July 1994)

Large N matrix models modified by terms of the form $g(\mathrm{Tr}\Phi^n)^2$ generate random surfaces which touch at isolated points. Matrix model results indicate that, as g is increased to a special value g_t, the string susceptibility exponent suddenly jumps from its conventional value γ to $\gamma/(\gamma-1)$. We study this effect in Liouville gravity and attribute it to a change of the interaction term from $Oe^{\alpha_+\phi}$ for $g < g_t$ to $Oe^{\alpha_-\phi}$ for $g = g_t$ (α_+ and α_- are the two roots of the conformal invariance condition for the Liouville dressing of a matter operator O). Thus, the new critical behavior is explained by the unconventional branch of Liouville dressing in the action.

PACS number(s): 11.25.Pm, 11.25.Sq

$$\gamma_{\mathrm{S}} \to \gamma'_{\mathrm{S}} = \frac{\gamma_{\mathrm{S}}}{\gamma_{\mathrm{S}} - 1}$$

i.e.,

$$(1-\gamma_{\mathrm{S}})(1-\gamma'_{\mathrm{S}}) = 1$$

Liouville quantum gravity

Kahane '85, Duplantier Sheffield '08

GFF h, averaged over a circle of radius ε around z : $h_\varepsilon(z) = \mathcal{B}_t$

Brownian motion in $t = -\log \varepsilon$

$\gamma-$LQG: random measure for $0 \leq \gamma < 2$

$$\mu_\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$$

describes the continuum limit of maps in the universality class of central charge $c = 1 - 6\left(\frac{\gamma}{2} - \frac{2}{\gamma}\right)^2$

$$\gamma_{\mathrm{s}} = 1 - \frac{4}{\gamma^2} < 0$$

Liouville quantum gravity

Kahane '85, Duplantier Sheffield '08

GFF h , averaged over a circle of radius ε around z : $h_\varepsilon(z) = \mathcal{B}_t$

Brownian motion in $t = -\log \varepsilon$

$\gamma-$LQG: random measure for $0 \le \gamma < 2$

$\mu_\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$ describes the continuum limit of maps in the universality class of central charge $c = 1 - 6\left(\frac{\gamma}{2} - \frac{2}{\gamma}\right)^2$

$\gamma_{\mathrm{S}} = 1 - \frac{4}{\gamma^2} < 0$ 2nd solution $\gamma' = 4/\gamma > 2$ $\gamma'_{\mathrm{S}} = 1 - \frac{4}{\gamma'^2} > 0$

Liouville quantum duality

Kahane '85, Duplantier Sheffield '08

$\gamma' -$ LQG for $\gamma' > 2$

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma'^2/2} e^{\gamma' h_\varepsilon(z)} dz = 0$$

Liouville quantum duality $\gamma\gamma' = 4$

Duplantier Sheffield '08
Barral Jin Rhodes Vargas '12

$\gamma' -$ LQG for $\gamma' > 2$

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma'^2/2} e^{\gamma' h_\varepsilon(z)} dz = 0$$

Additional random set of *atoms* with localized quantum area η

$$\mu_{\gamma'}(dz) := \int_0^\infty \eta \mathcal{N}_{\gamma'}(dz, d\eta)$$

$\mathcal{N}_{\gamma'}$ Poisson random measure of intensity $\mu_\gamma(dz) d\eta / \eta^{1+1/\alpha}$

$$\alpha = 4/\gamma^2 > 1$$

Liouville quantum duality $\gamma\gamma' = 4$

Duplantier Sheffield '08
Barral Jin Rhodes Vargas '12

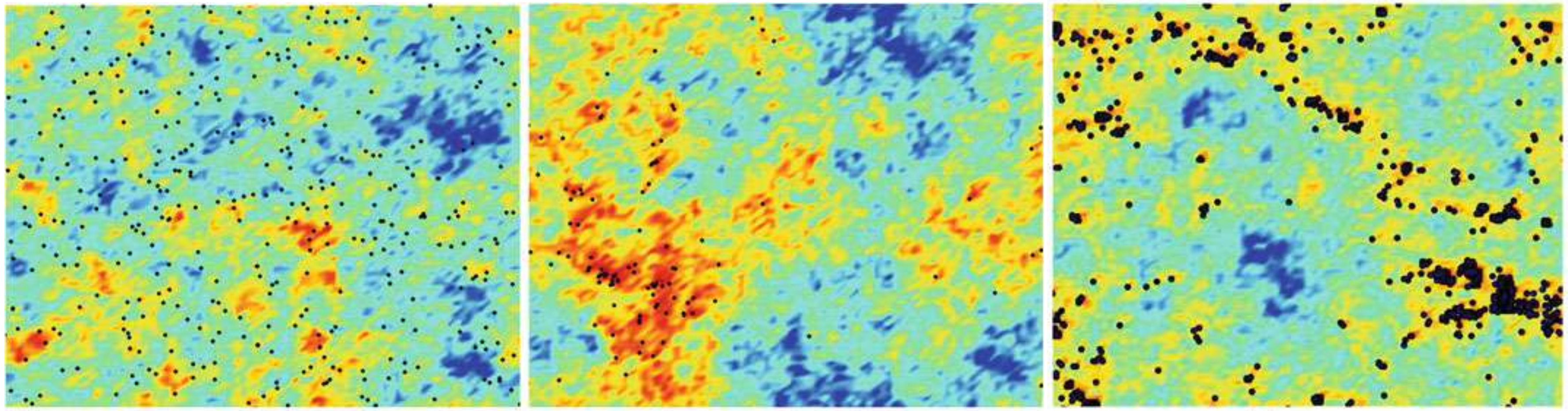
$\gamma' -$ LQG for $\gamma' > 2$

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma'^2/2} e^{\gamma' h_\varepsilon(z)} dz = 0$$

Additional random set of *atoms* with localized quantum area η

$$\mu_{\gamma'}(dz) := \int_0^\infty \eta \mathcal{N}_{\gamma'}(dz, d\eta)$$

$\mathcal{N}_{\gamma'}$ Poisson random measure of intensity $\mu_\gamma(dz) d\eta / \eta^{1+1/\alpha}$



$$\alpha = 4/\gamma^2 > 1$$

Liouville quantum duality $\gamma\gamma' = 4$

Duplantier Sheffield '08
Barral Jin Rhodes Vargas '12

$\gamma' -$ LQG for $\gamma' > 2$

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma'^2/2} e^{\gamma' h_\varepsilon(z)} dz = 0$$

Additional random set of *atoms* with localized quantum area η

$$\mu_{\gamma'}(dz) := \int_0^\infty \eta \, \mathcal{N}_{\gamma'}(dz, d\eta)$$

$\mathcal{N}_{\gamma'}$ Poisson random measure of intensity $\mu_\gamma(dz) d\eta / \eta^{1+1/\alpha}$

$$\alpha = 4/\gamma^2 > 1$$

$$\mathbb{E} \exp[-\lambda \, \mu_{\gamma'}(A)] = \mathbb{E} \exp \left[\Gamma(-\alpha') \lambda^{\alpha'} \mu_\gamma(A) \right]$$

$$\alpha\alpha' = 1$$

KPZ relation.

Fractal set of Hausdorff dimension $D_{\mathrm{H}} = 2 - 2x$

Euclidean vs quantum scaling dimensions (x, Δ_γ)

Duplantier Sheffield '08
Barral et al. '12

$$x = \frac{\gamma^2}{4}\Delta_\gamma^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta_\gamma \qquad \Delta_\gamma \geq 0$$

Duality $\gamma\gamma' = 4$

Dual scaling dimension $\Delta_{\gamma'}$

$$\gamma'(\Delta_{\gamma'} - 1) = \gamma(\Delta_\gamma - 1)$$

KPZ relation. Fractal set of Hausdorff dimension $D_{\mathrm{H}} = 2 - 2x$

Euclidean vs quantum scaling dimensions (x, Δ_γ)

Duplantier Sheffield '08
Barral et al. '12

$$x = \frac{\gamma^2}{4}\Delta_\gamma^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta_\gamma \qquad \Delta_\gamma \geq 0$$

Duality $\gamma\gamma' = 4$

Dual scaling dimension $\Delta_{\gamma'}$

$$\gamma'(\Delta_{\gamma'} - 1) = \gamma(\Delta_\gamma - 1)$$

In terms of $\gamma_{\mathrm{S}} = 1 - \frac{4}{\gamma^2}$

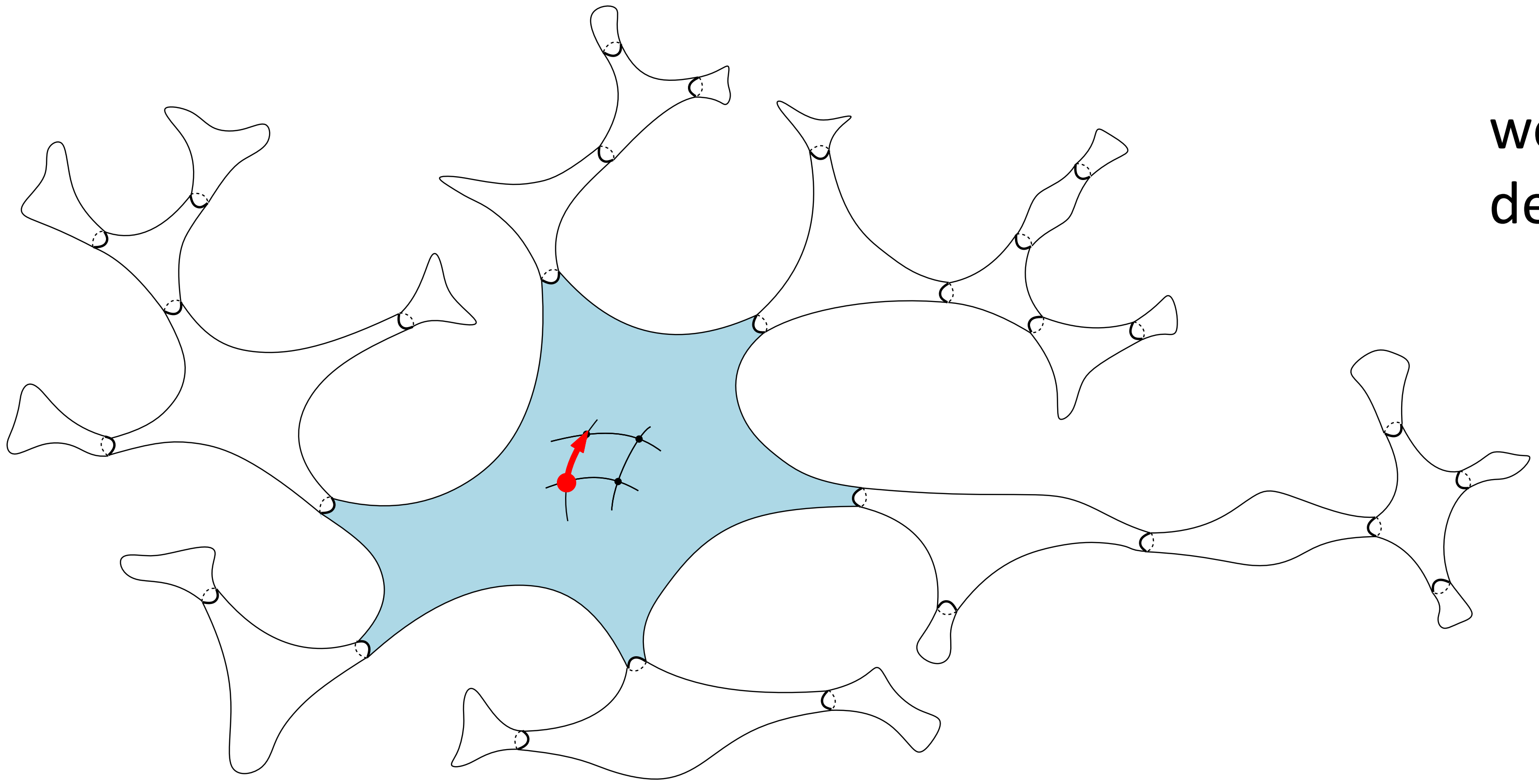
$$\Delta_{\gamma'} = \frac{\Delta_\gamma - \gamma_{\mathrm{S}}}{1 - \gamma_{\mathrm{S}}}$$

$$(1 - \gamma_{\mathrm{S}})(1 - \gamma'_{\mathrm{S}}) = 1$$

$$\Delta_\gamma = \frac{\Delta_{\gamma'} - \gamma'_{\mathrm{S}}}{1 - \gamma'_{\mathrm{S}}}$$

Lessons from analytic combinatorics

Flajolet Sedgewick '09



weight ϕ_i per block with i descending sub-blocks

$$\phi(\tau) = \sum_{i\geq 0} \phi_i \tau^i$$

y_n trees with n blocks

$$y(z) = \sum_{n\geq 1} y_n z^n$$

Substitution relation

$$y(z) = z\,\phi(y(z))$$

$$y(z) = z\,\phi(y(z))$$

Tree critical point where the mapping $z \mapsto y$ is no longer invertible

$$1 = z_c\,\phi'(y(z_c))$$

$\big(z_c, y_c = y(z_c)\big)$ determined by $\phi(y_c) = y_c\,\phi'(y_c) \quad , \quad z_c = \dfrac{y_c}{\phi(y_c)}$ (CRIT)

Singularity of $\phi(\tau)$

$$\phi_i \propto \tau_\phi^{-i}/i^{1+\alpha} \qquad 1 < \alpha < 2$$

$$\phi(\tau) = \phi(\tau_\phi) - (\tau_\phi - \tau)\phi'(\tau_\phi) + K(\tau_\phi - \tau)^\alpha + \cdots$$

When z increases, $y = y(z)$ increases and becomes singular when it reaches y_c or τ_ϕ

- **Supercritical case:** (CRIT) has a solution $y_c < \tau_\phi$

$$z_c - z = \frac{y_c}{\phi(y_c)} - \frac{y}{\phi(y)} =: H(y)$$

$$H(y_c) = 0\ , \qquad H'(y_c) = 0 \qquad \text{(CRIT)}$$

$$z_c - z = \frac{1}{2} H''(y_c)(y_c - y)^2 + \cdots$$

$$y = y_c - \left(\frac{2}{H''(y_c)} (z_c - z) \right)^{1/2} + \cdots$$

$$y_n \propto \frac{z_c^{-n}}{n^{3/2}}$$

- Critical case: (CRIT) has a solution $y_c = \tau_\phi$

$$\phi(y) = \frac{y}{z_c} + K(y_c - y)^\alpha + \cdots$$

$$z_c - z = \frac{z_c^2}{y_c} K(y_c - y)^\alpha + \cdots$$

$$y = y_c - \left(\frac{y_c}{z_c^2 K} (z_c - z) \right)^{1/\alpha} + \cdots$$

$$y_n \propto \frac{z_c^{-n}}{n^{1+1/\alpha}}$$

- Subcritical case: (CRIT) has no solution $y_c \leq \tau_\phi$

$z_\phi = \tau_\phi / \phi(\tau_\phi)$ such that $y(z_\phi) = \tau_\phi$

$$z_\phi - z = K_\phi(\tau_\phi - y) + \frac{K\tau_\phi}{\phi(\tau_\phi)^2}(\tau_\phi - y)^\alpha + \cdots \qquad K_\phi > 0$$

$$y = \tau_\phi - \frac{1}{K_\phi}(z_\phi - z) + \frac{K\tau_\phi}{\phi(\tau_\phi)^2}\frac{1}{K_\phi^{1+\alpha}}(z_\phi - z)^\alpha + \cdots$$

$$y_n \propto \frac{z_\phi^{-n}}{n^{1+\alpha}}$$

Asymptotics for y_n :

	subcritical	critical	supercritical
$1 < \alpha < 2$	$\dfrac{z_\phi^{-n}}{n^{1+\alpha}}$	$\dfrac{z_c^{-n}}{n^{1+1/\alpha}}$	$\dfrac{z_c^{-n}}{n^{3/2}}$
$\alpha = 2 \ (\times \log)$	$\dfrac{z_\phi^{-n}}{n^3}$	$\dfrac{z_c^{-n}}{n^{3/2}(\log n)^{1/2}}$	$\dfrac{z_c^{-n}}{n^{3/2}}$

$$\phi(\tau) = \phi(\tau_\phi) - (\tau_\phi - \tau)\phi'(\tau_\phi) - K(\tau_\phi - \tau)^2 \log(\tau_\phi - \tau) + \cdots$$

Correlators maps with **2** markings

w_n enumerates maps with n **regular** blocks

$$w(z) = \sum_{n\geq 1} w_n z^n$$

$$w(z) = \psi(y(z))$$

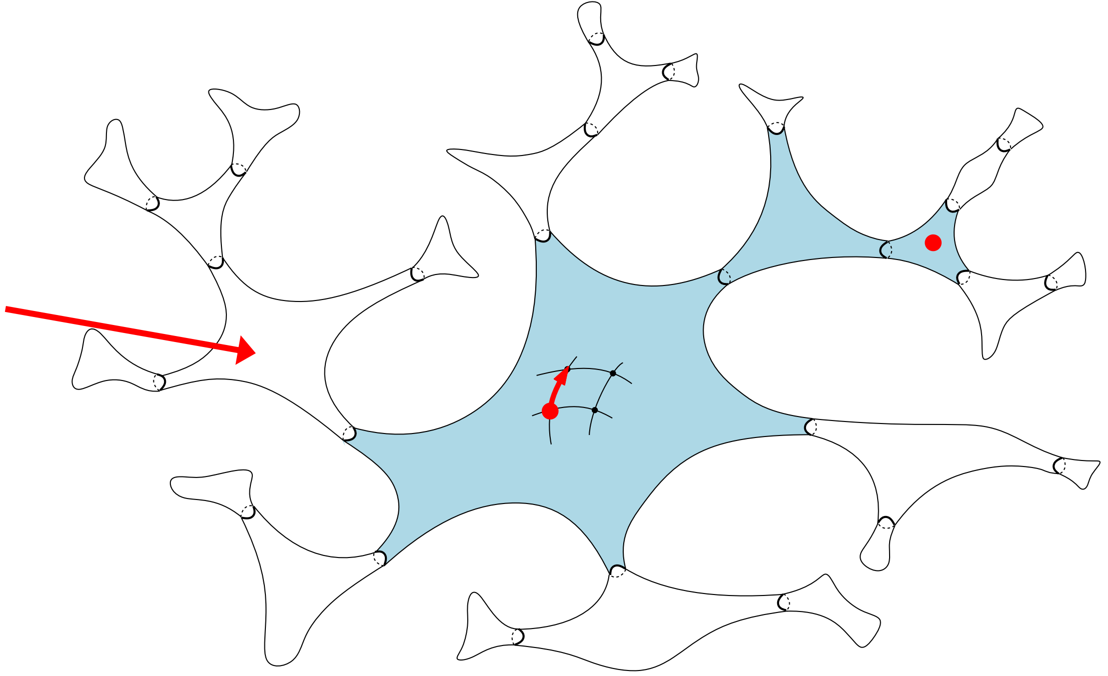
$$\psi(\tau) = \sum_{i\geq 0} \psi_i \tau^i$$

- 2nd marking anywhere

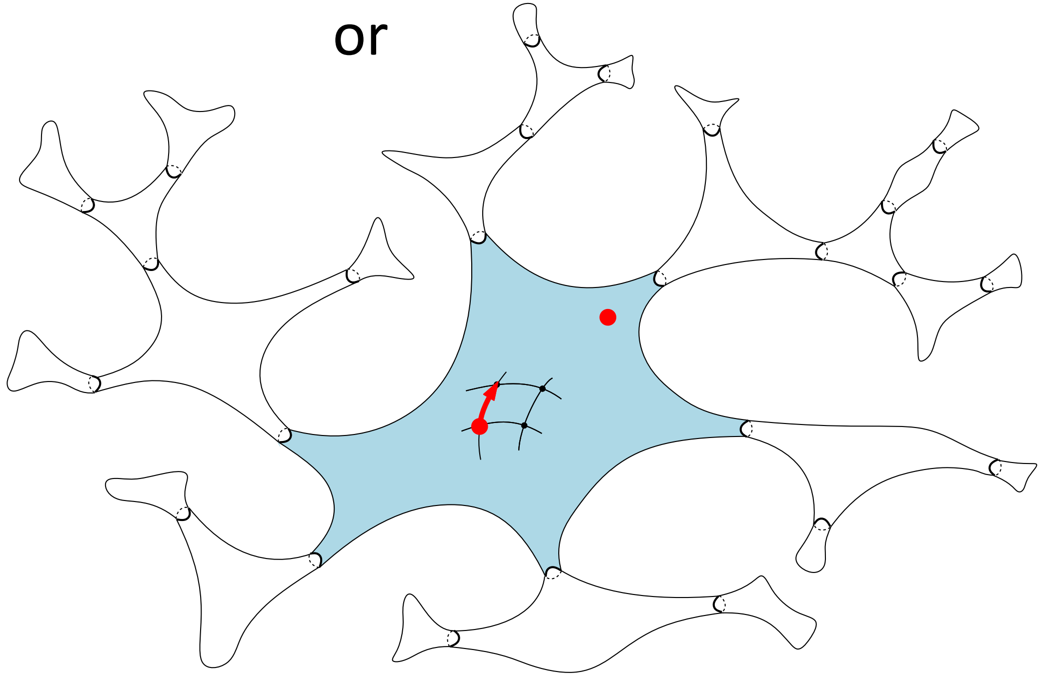
ψ_i sequence of blocks connecting the 2 markings

or • 2nd marking in the root block

ψ_i single block with 2 markings



or



Assume $\psi_i \propto \tau_\phi^{-i}/i^{1+\beta}$ $0 < \beta < 1$ with same τ_ϕ

$$\psi(\tau) = \psi(\tau_\phi) - K_\psi(\tau_\phi - \tau)^\beta + \ \cdots$$

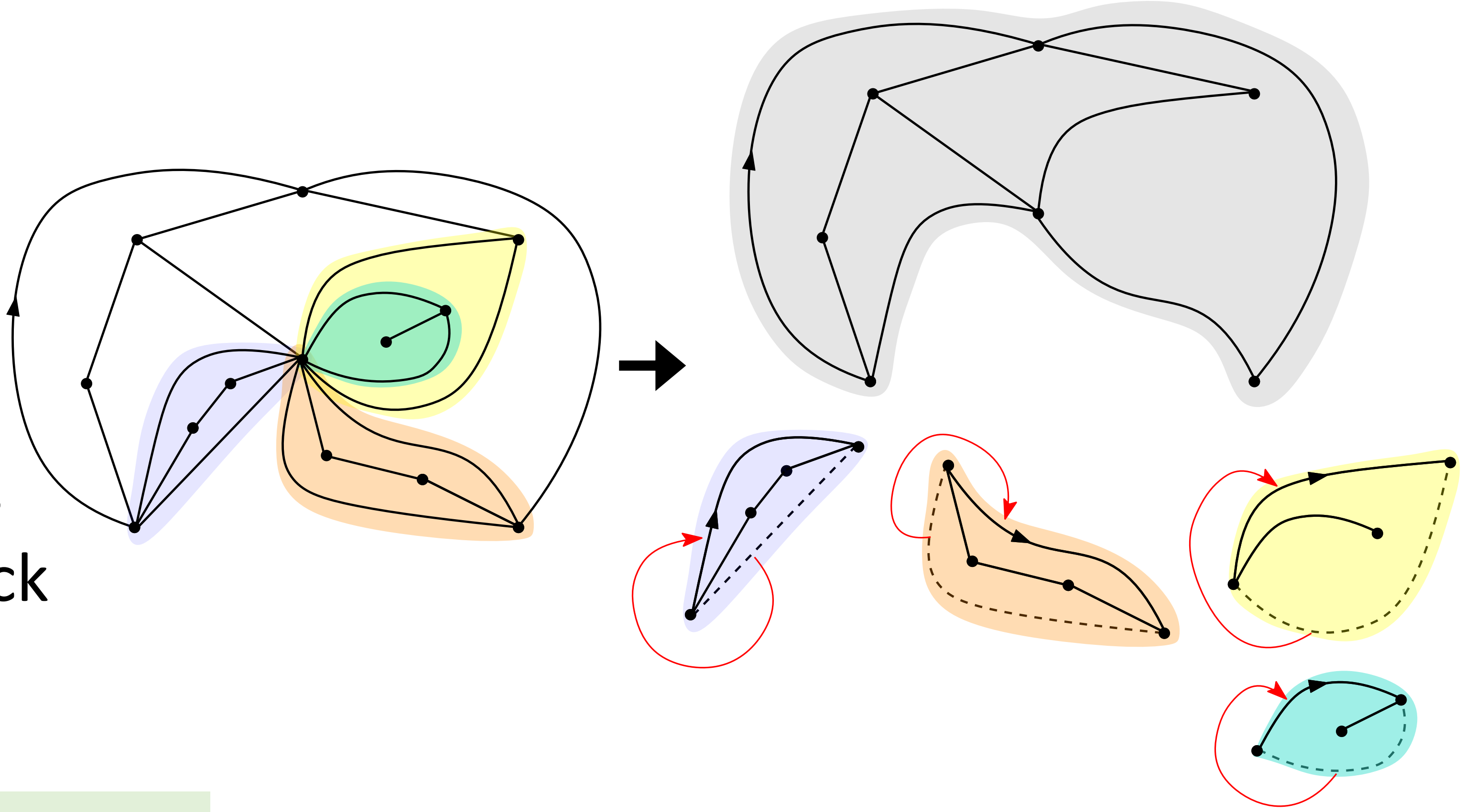
Asymptotics for w_n :

	subcritical	critical	supercritical
$1 < \alpha < 2$ $0 < \beta < 1$	$\frac{z_\phi^{-n}}{n^{1+\beta}}$	$\frac{z_c^{-n}}{n^{1+\beta/\alpha}}$	$\frac{z_c^{-n}}{n^{3/2}}$
$\alpha = 2\ (\times \log)$ $0 < \beta < 1$	$\frac{z_\phi^{-n}}{n^{1+\beta}}$	$\frac{z_c^{-n}}{n^{1+\beta/2}(\log n)^{\beta/2}}$	$\frac{z_c^{-n}}{n^{3/2}}$
$\alpha = 2\ (\times \log)$ $\beta = 1\ (\times \log)$	$\frac{z_\phi^{-n}}{n^2}$	$\frac{z_c^{-n}(\log n)^{1/2}}{n^{3/2}}$	$\frac{z_c^{-n}}{n^{3/2}}$

Block-weighted maps

$$M_u(g) = \sum_{n \geq 0} g^n \, m_n^{(u)}$$

Quadrangulations with n faces and a weight u per **simple** block



$$M_u(g) = 1 + u \left[B\left(g \, M_u(g)^2 \right) - 1 \right]$$

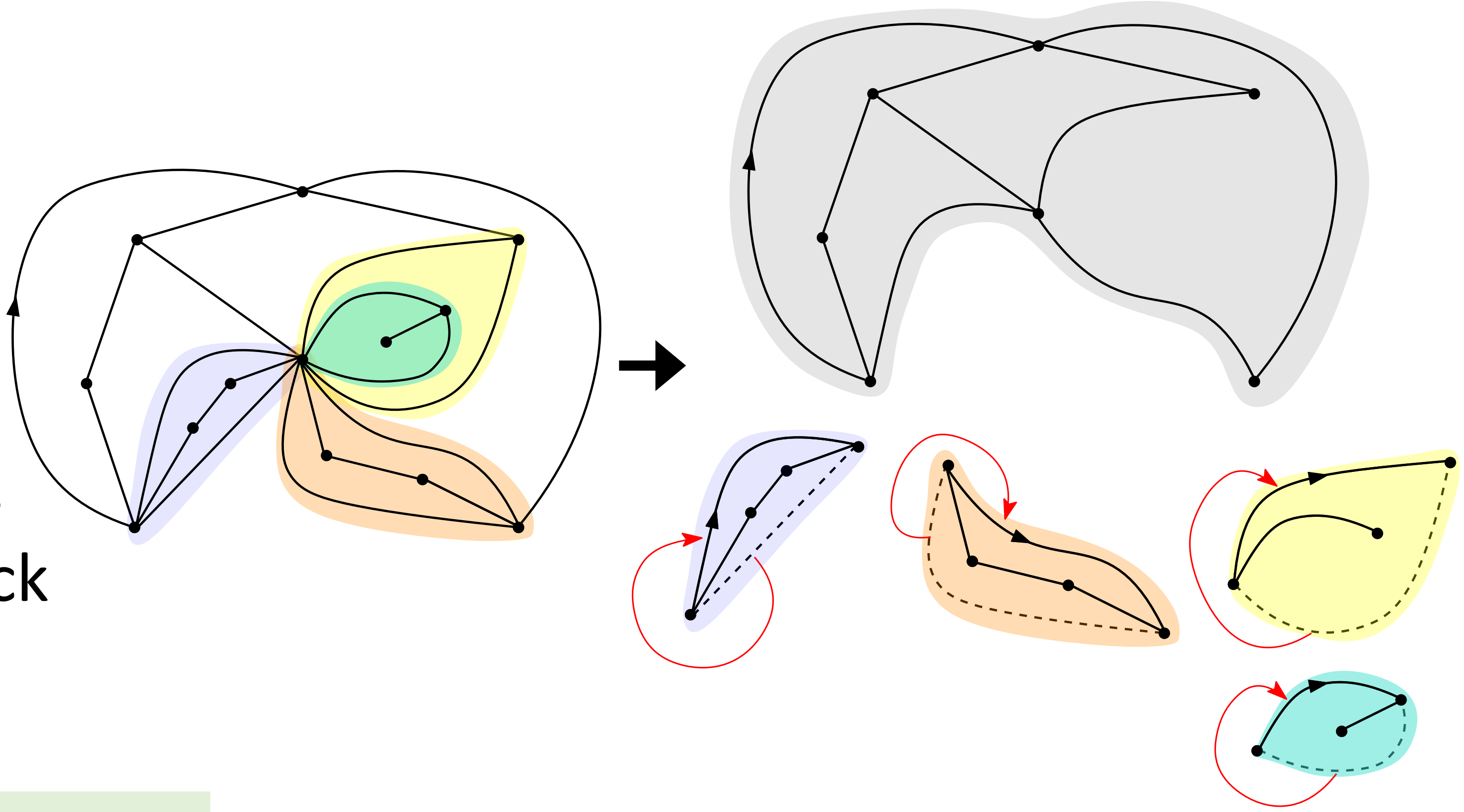
$$B(t) = \sum_{j \geq 0} b_j \, t^j$$

simple quadrangulations with j faces

Block-weighted maps

$$M_u(g) = \sum_{n\geq 0} g^n\, m_n^{(u)}$$

Quadrangulations with n faces and a weight u per **simple** block



$$M_u(g) = 1 + u\left[B\left(g\, M_u(g)^2\right) - 1\right]$$

$$B(t) = \sum_{j\geq 0} b_j\, t^j$$

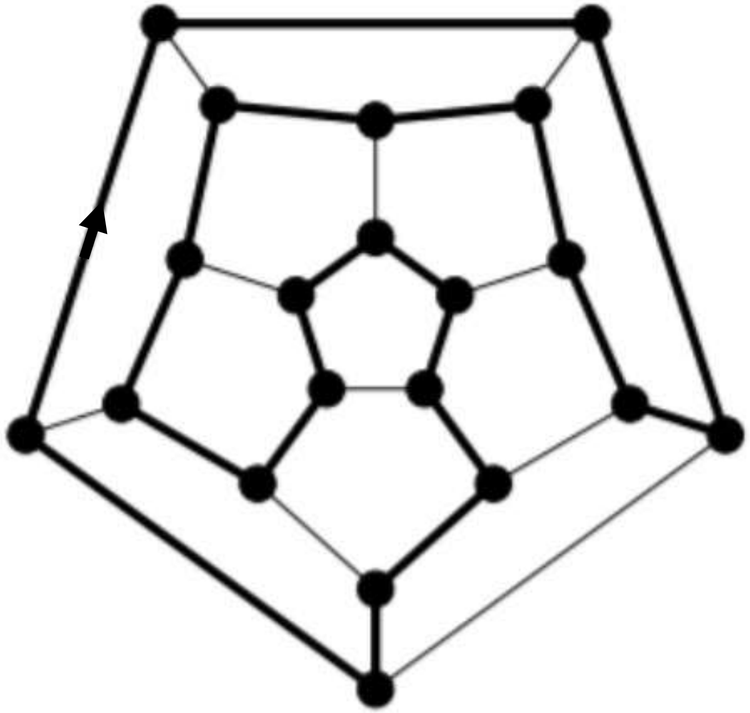
simple quadrangulations with j faces

$$y(z) = z\,\phi(y(z))$$

$$z = \sqrt{g}\ , \quad y(z) = zM_u(z^2)$$

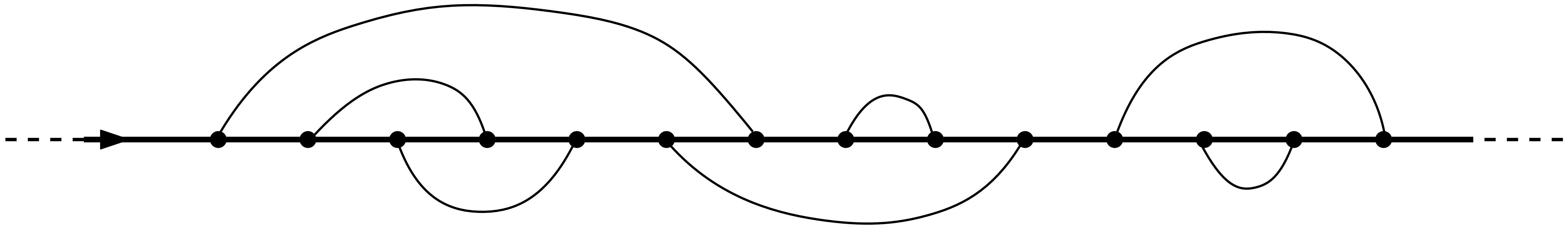
$$\phi(\tau) = 1 + u\left[B(\tau^2) - 1\right]$$

Other examples: maps equipped with a **Hamiltonian cycle**

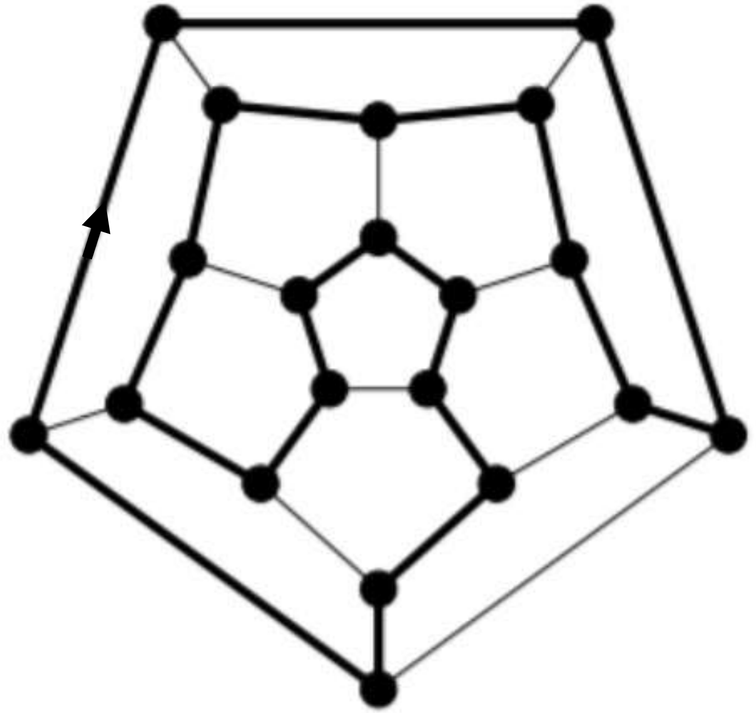


$$c = -2$$

Cubic maps: system of arches



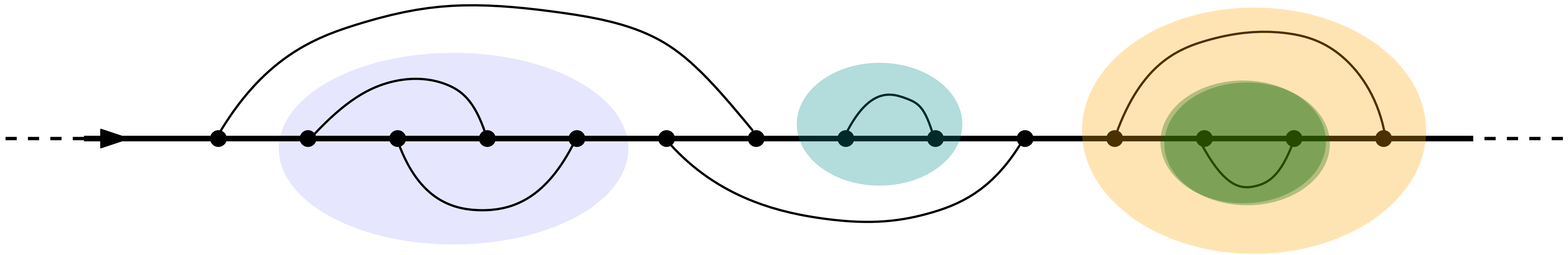
Other examples: maps equipped with a **Hamiltonian cycle**



$$M_u(g) = 1 + u\left[B\left(g\, M_u(g)^2\right) - 1\right]$$

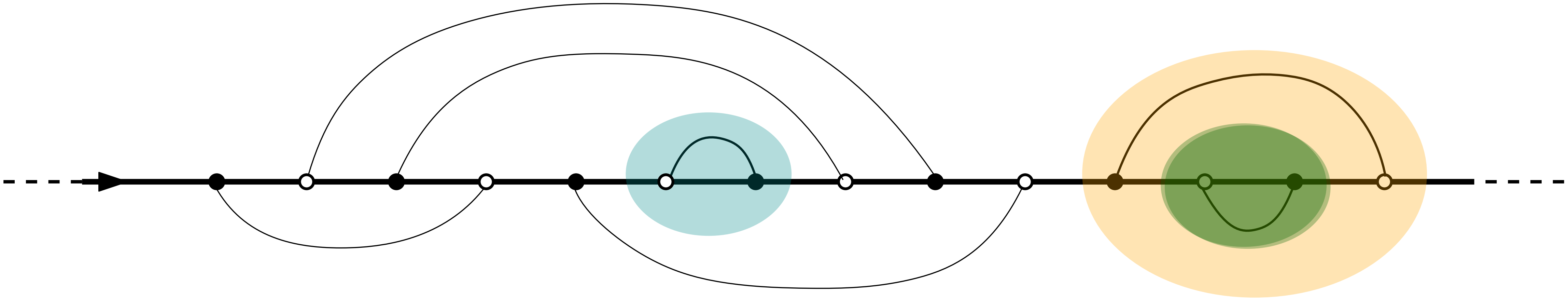
Cubic maps: system of arches

decomposed into **irreducible** blocks



Bicubic maps : system of arches decomposed into irreducible blocks

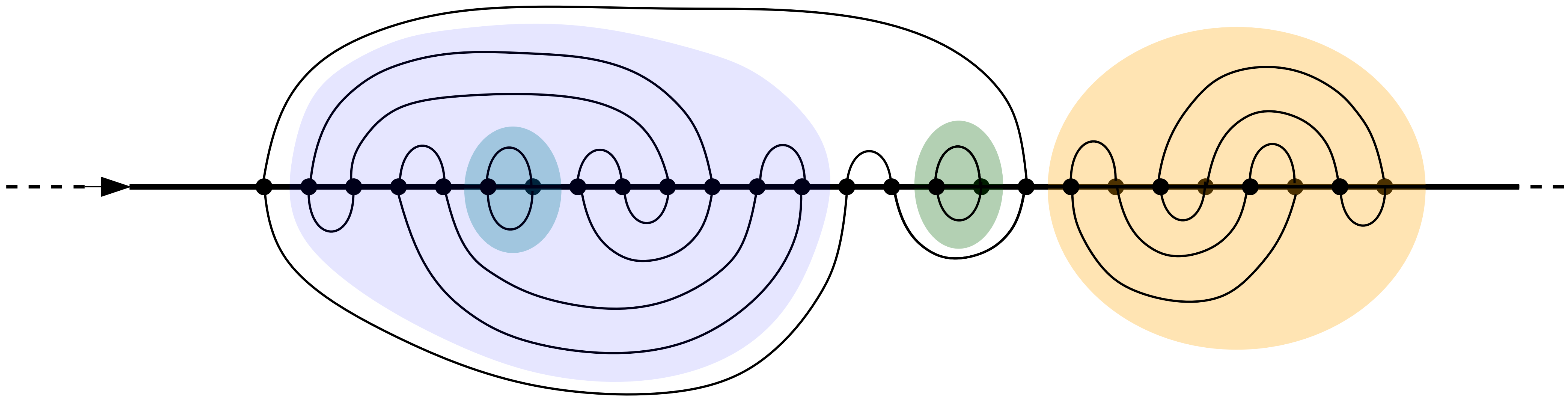
$$c = -1$$



different universality class!

Guitter Kristjansen Nielsen '99
Di Francesco et al. '23
Duplantier Golinelli Guitter '23

Quartic maps: meandric systems decomposed into irreducible blocks



weights u per irreducible block and q per loop

universality class depending on q

In general, we know the case $u = 1$

$$m_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{2-\gamma_{\mathrm{S}}}} \qquad 2 < 2 - \gamma_{\mathrm{S}} < 3$$

From $M_1(g) = B\left(g\, M_1(g)^2\right)$, we deduce

$$B(t) = B(t_{\mathrm{cr}}) - (t_{\mathrm{cr}} - t)B'(t_{\mathrm{cr}}) + K_B(t_{\mathrm{cr}} - t)^{\alpha} + \cdots$$

$$t_{\mathrm{cr}} = g_1\, M_1(g_1)^2 \qquad \alpha = 1 - \gamma_{\mathrm{S}} \qquad 1 < \alpha < 2$$

$$u_{\mathrm{cr}} = \frac{M_1(g_1) + 2g_1 M_1'(g_1)}{M_1(g_1)(1 - M_1(g_1)) + 2g_1 M_1'(g_1)}$$

$$u_{\mathrm{cr}} > 1$$

$$m_n^{(u)} \propto \frac{g_{\mathrm{cr}}(u)^{-n}}{n^{1+\alpha}} \quad \text{for} \quad u < u_{\mathrm{cr}}$$

$$m_n^{(u)} \propto \frac{g_c(u)^{-n}}{n^{1+1/\alpha}} \quad \text{for} \quad u = u_{\mathrm{cr}}$$

$$m_n^{(u)} \propto \frac{g_c(u)^{-n}}{n^{3/2}} \quad \text{for} \quad u > u_{\mathrm{cr}}$$

$$2 - \gamma_{\mathrm{S}} = 1 + \alpha$$

$$2 - \gamma_{\mathrm{S}}' = 1 + 1/\alpha$$

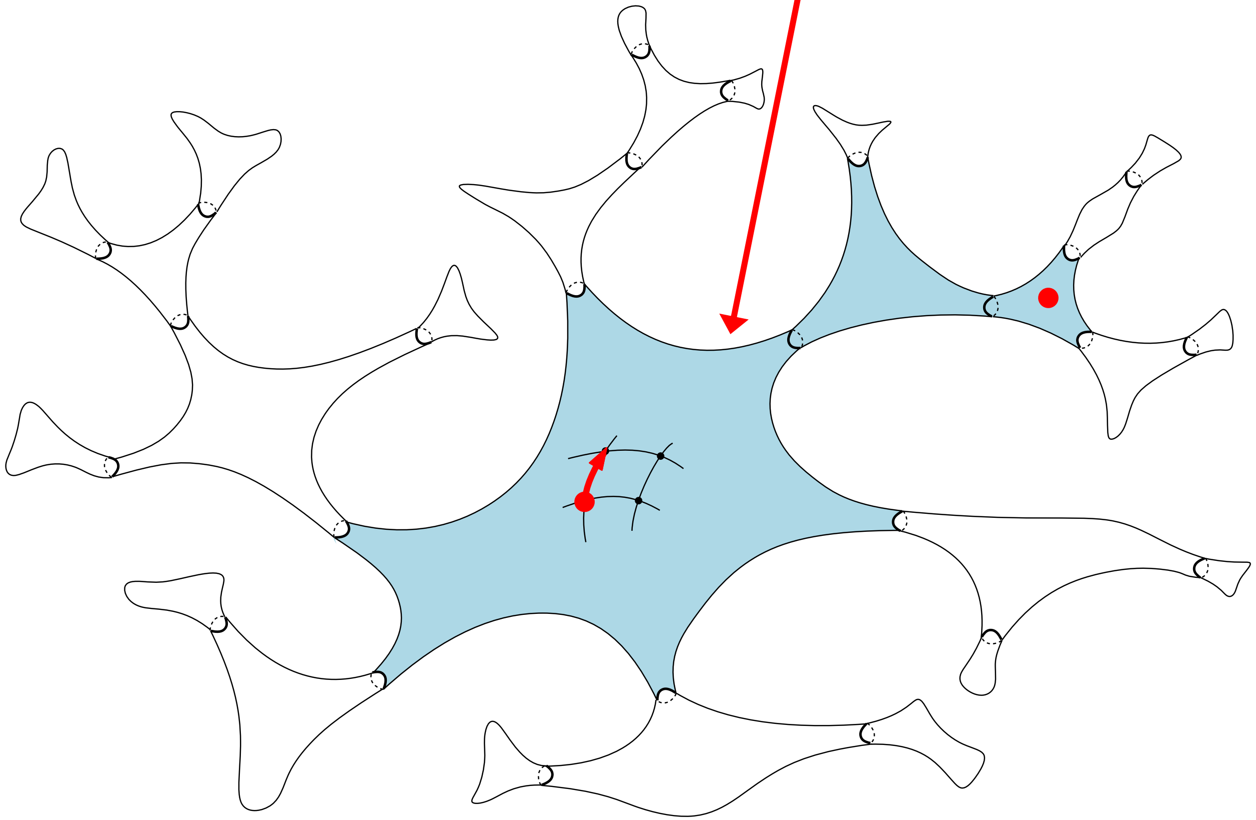
$$(1 - \gamma_{\mathrm{S}})(1 - \gamma_{\mathrm{S}}') = 1$$

Correlators

$$\widetilde{s}_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{1+2\Delta-\gamma_S}}$$

sequence of blocks

$$\widetilde{S}_1(g) = \widetilde{C}\left(g\, M_1(g)^2\right)$$



Correlators

sequence of blocks ↓

$$\widetilde{s}_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{1+2\Delta-\gamma_{\rm S}}}$$

$$\widetilde{S}_1(g) = \widetilde{C}\left(g\, M_1(g)^2\right)$$

$$\widetilde{C}(t) = \widetilde{C}(t_{\rm cr}) - K_{\widetilde{C}}(t_{\rm cr} - t)^{\beta} + \cdots$$

$$\beta = 2\Delta - \gamma_{\rm S}$$

Correlators

sequence of blocks

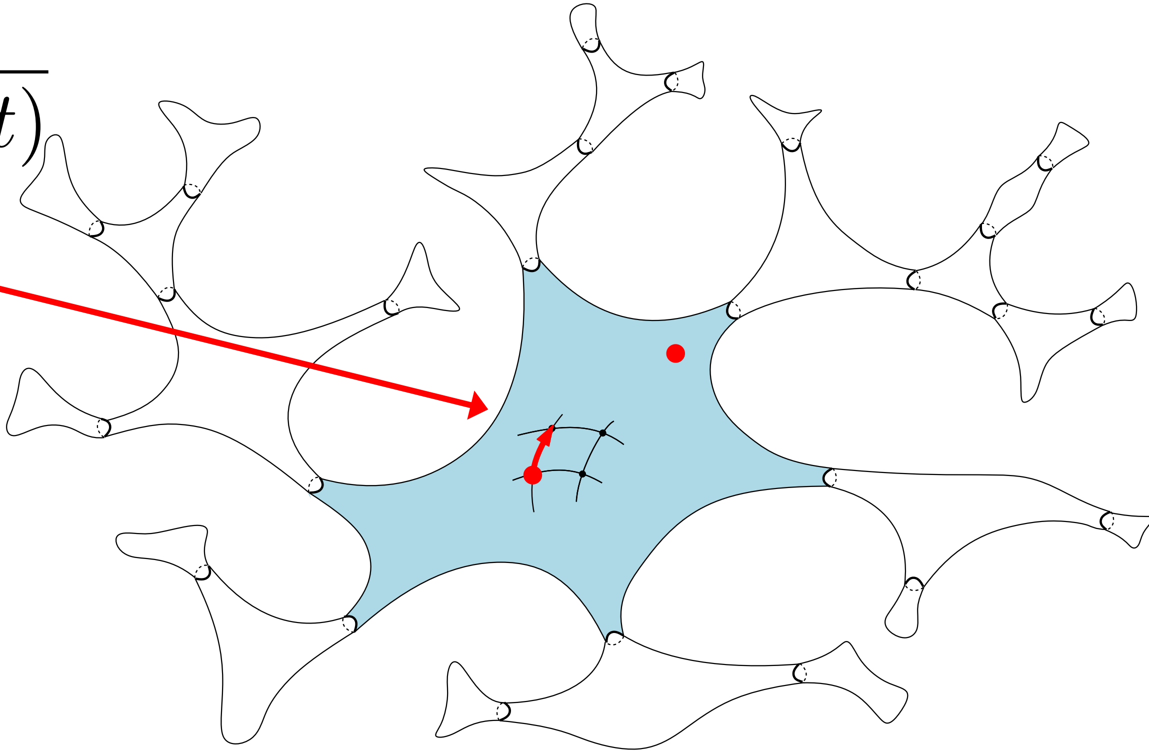
$$\widetilde{s}_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{1+2\Delta-\gamma_S}} \qquad \widetilde{S}_1(g) = \widetilde{C}\left(g\, M_1(g)^2\right)$$

$$\widetilde{C}(t) = \widetilde{C}(t_{cr}) - K_{\widetilde{C}}(t_{cr}-t)^{\beta} + \cdots$$

$$\beta = 2\Delta - \gamma_S$$

Single block → C with $\widetilde{C}(t) = \dfrac{1}{1-C(t)}$

has same singularity



Correlators

$$\widetilde{s}_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{1+2\Delta-\gamma_S}}$$

sequence of blocks

$$\widetilde{S}_1(g) = \widetilde{C}\left(g\, M_1(g)^2\right)$$

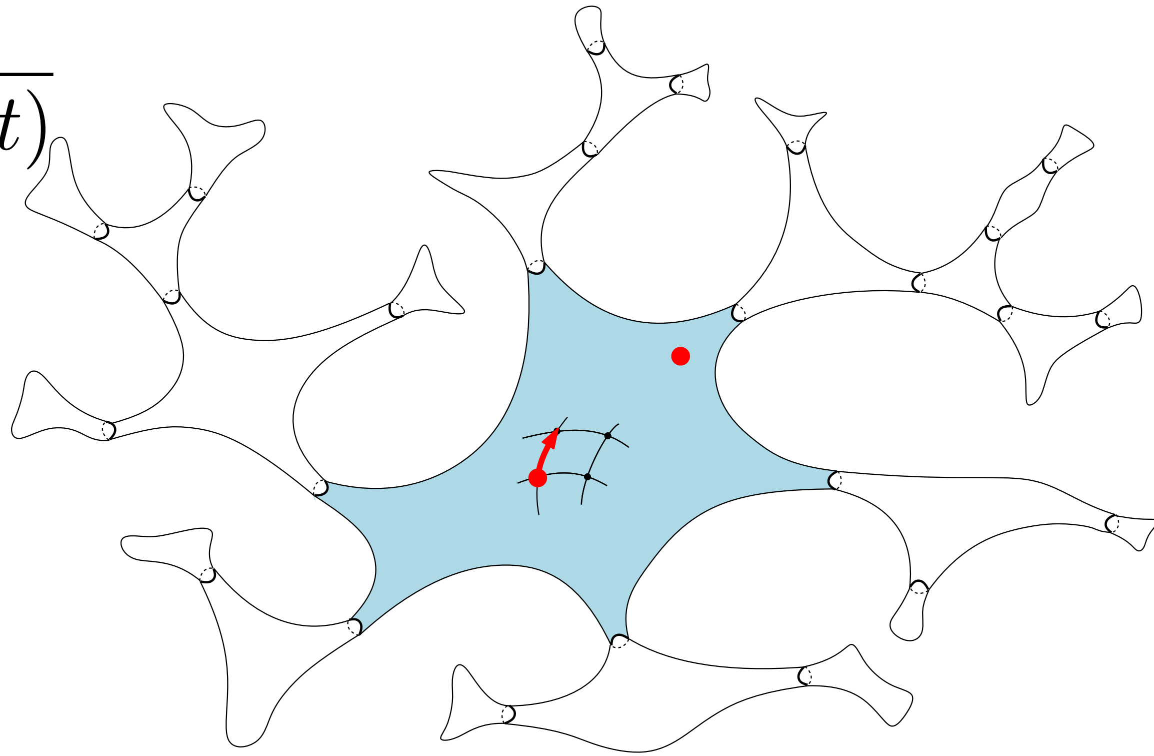
$$\widetilde{C}(t) = \widetilde{C}(t_{\rm cr}) - K_{\widetilde{C}}(t_{\rm cr} - t)^{\beta} + \cdots$$

$$\beta = 2\Delta - \gamma_S$$

Single block → C with $\widetilde{C}(t) = \dfrac{1}{1 - C(t)}$

has same singularity

$$S_u(g) = C\left(g\, M_u(g)^2\right)$$



Correlators

$$\widetilde{s}_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{1+2\Delta-\gamma_S}}$$

sequence of blocks

$$\widetilde{S}_1(g) = \widetilde{C}\left(g\, M_1(g)^2\right)$$

$$\widetilde{C}(t) = \widetilde{C}(t_{cr}) - K_{\widetilde{C}}(t_{cr}-t)^{\beta} + \cdots$$

$$\beta = 2\Delta - \gamma_S$$

Single block → C with $\widetilde{C}(t) = \dfrac{1}{1-C(t)}$

has same singularity

$$2\Delta' - \gamma_S' = \frac{\beta}{\alpha} = \frac{2\Delta-\gamma_S}{1-\gamma_S}$$

$$S_u(g) = C\left(g\, M_u(g)^2\right)$$

$$s_n^{(u_{cr})} \propto \frac{g_c(u_{cr})^{-n}}{n^{1+2\Delta'-\gamma_S'}}$$

$$\Delta' = \frac{\Delta-\gamma_S}{1-\gamma_S}$$

Experiments

$$M_1(g) \to B(t) \to M_u(g)$$

$$m_n^{(1)} \to b_j \to m_n^{(u)} \propto \frac{g_c(u)^{-n}}{n^{2-\gamma_S(u)}}$$

Estimation of exponent

$$t_n \propto \frac{g_*^{-n}}{n^\delta} \qquad \delta_n := n^2 \left(\frac{t_{n+2}\, t_n}{t_{n+1}^2} - 1 \right) \underset{n\to\infty}{\to} \delta$$

Accelerated convergence $(\Delta f)_N := f_{N+1} - f_N$

$$\hat{\delta}_n^{(p)} := n^p \delta_n \qquad \tilde{\delta}_n^{(p)} := \frac{1}{p!} \left(\Delta^p \hat{\delta}^{(p)} \right)_n \underset{n\to\infty}{\to} \delta$$

Quadrangulations with weight u per simple block

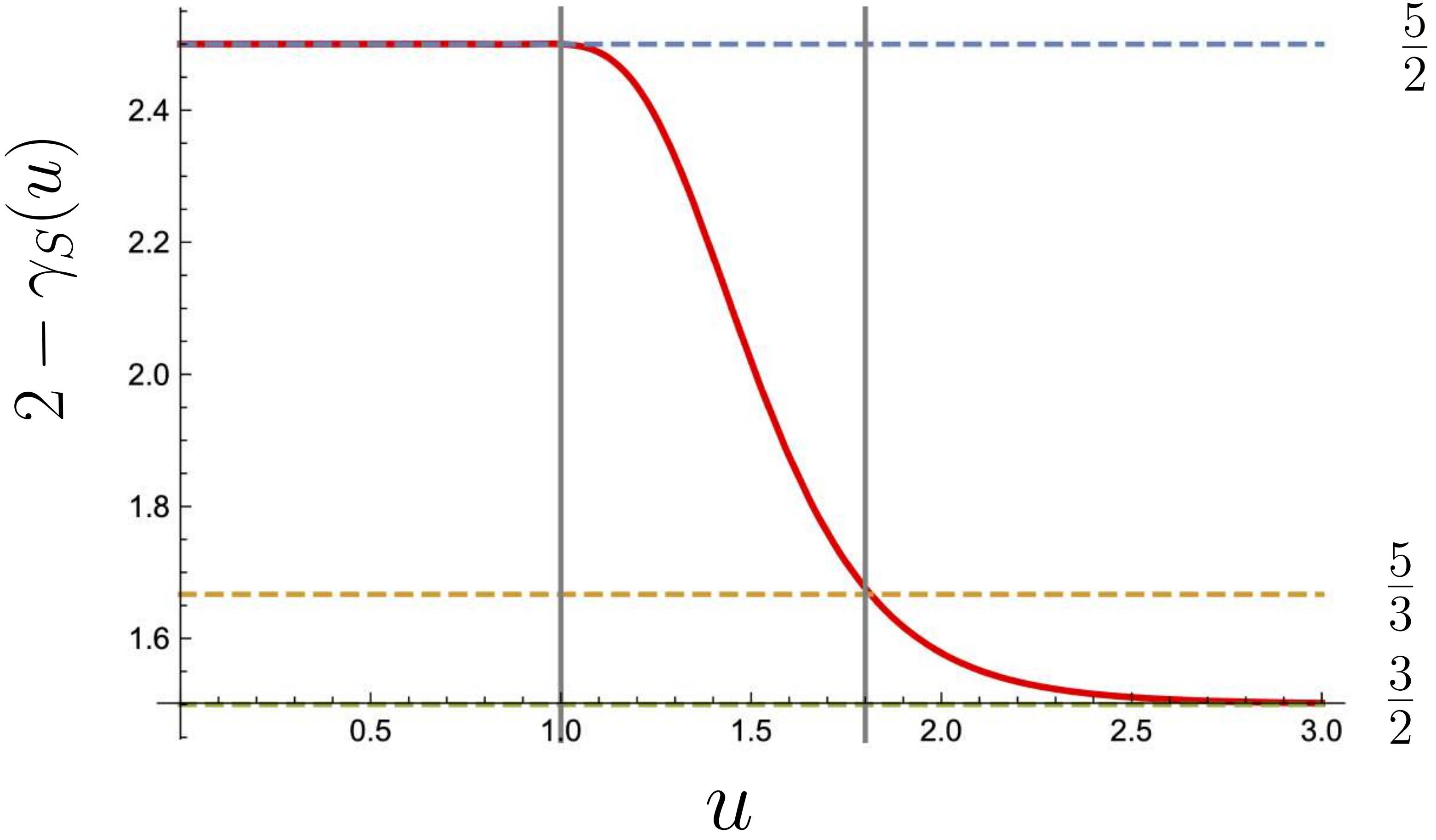
$$M_1(g) = \sum_{n\geq 0} \underbrace{2\frac{3^n}{n+2}\mathrm{Cat}(n)}_{m_n^{(1)}} g^n = \frac{18g - 1 + (1-12g)^{3/2}}{54g^2}$$

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$$

$c = 0$

$\gamma = \sqrt{\frac{8}{3}}$

$u = 1$ $\qquad$ $u = u_{\mathrm{cr}} = \frac{9}{5}$

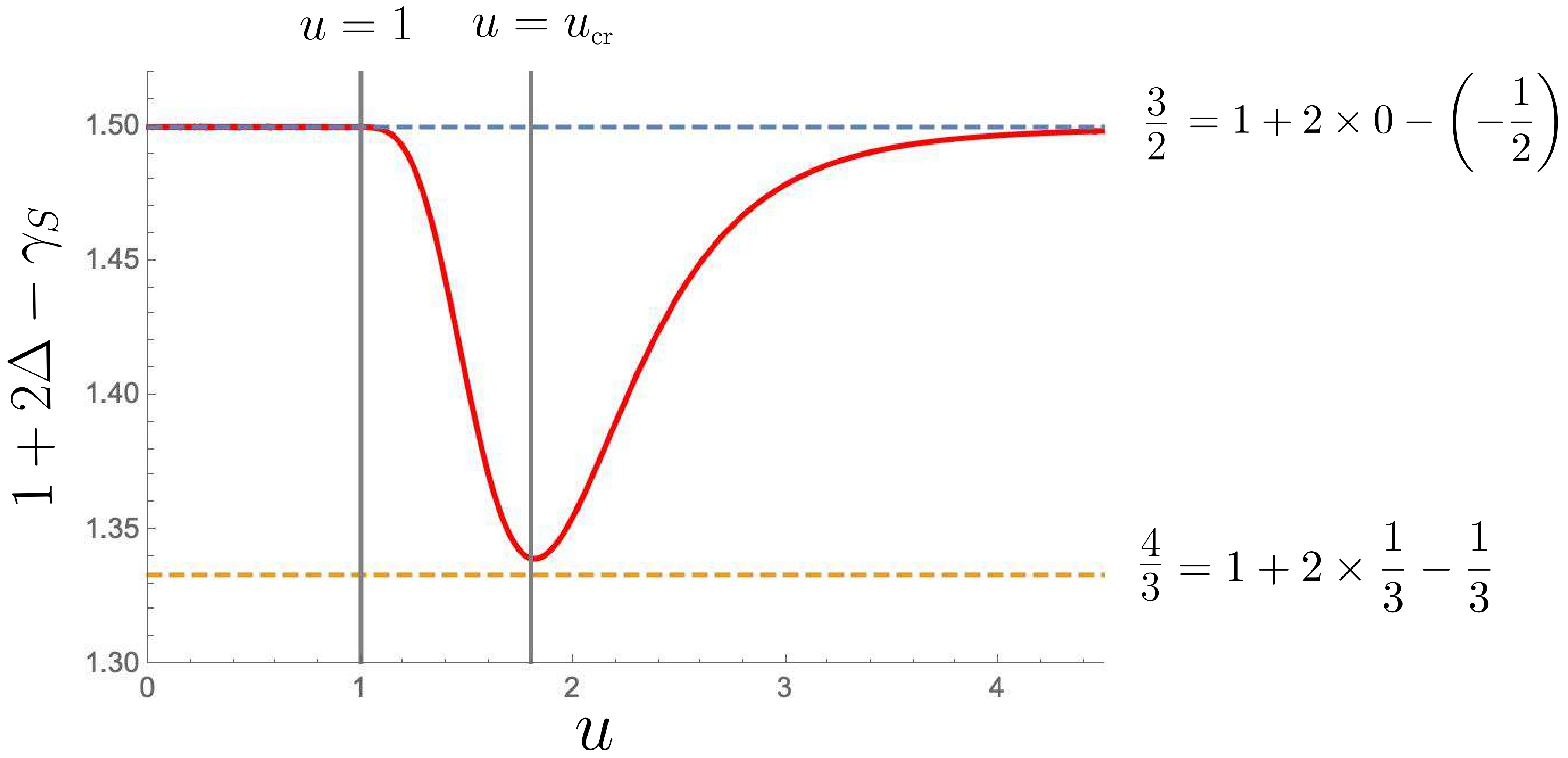


$\gamma_{\mathrm{S}} = -\frac{1}{2}$

$\gamma'_{\mathrm{S}} = \frac{1}{3}$

Rooted quadrangulations with a second marked edge

$$\widetilde{s}_n^{(1)} = (2n-1)m_n^{(1)} \propto \frac{g_1^{-n}}{n^{1+2\Delta-\gamma_{\rm S}}} \quad \text{with } \Delta = 0 \text{ hence } \Delta' = \frac{\gamma_{\rm S}}{\gamma_{\rm S}-1} = \gamma'_{\rm S}$$

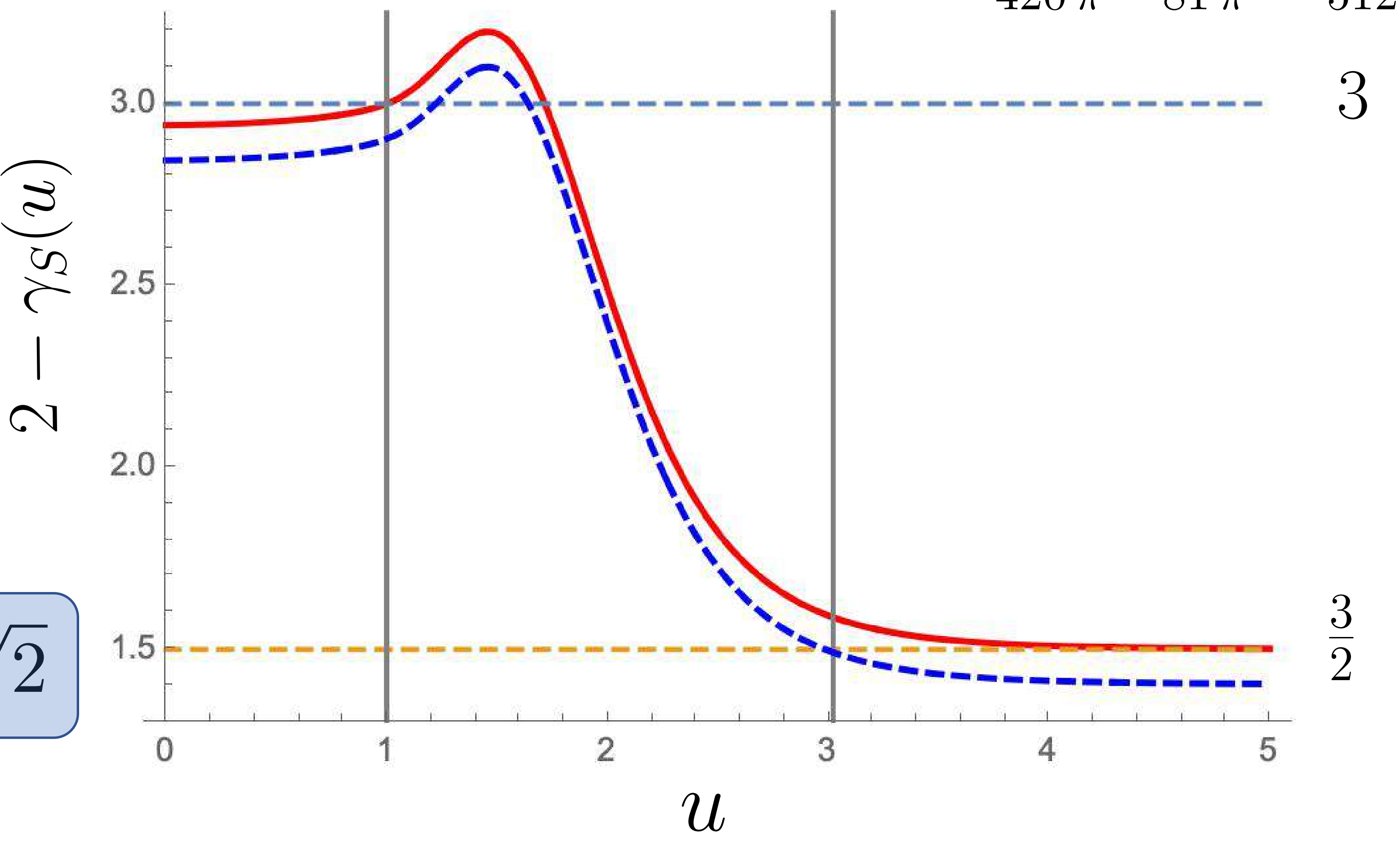


Cubic maps + Hamiltonian cycle with weight u per irreducible block

$$m_n^{(1)} = \sum_{k=0}^{n} \binom{2n}{2k} \mathrm{Cat}(k)\mathrm{Cat}(n-k) = \mathrm{Cat}(n)\mathrm{Cat}(n+1)$$

Albenque Fusy Salvy '24

$u = 1$ $\quad$ $u = u_{\mathrm{cr}} = \dfrac{9\pi\,(4-\pi)}{420\,\pi - 81\,\pi^2 - 512}$



$\gamma = \sqrt{2}$

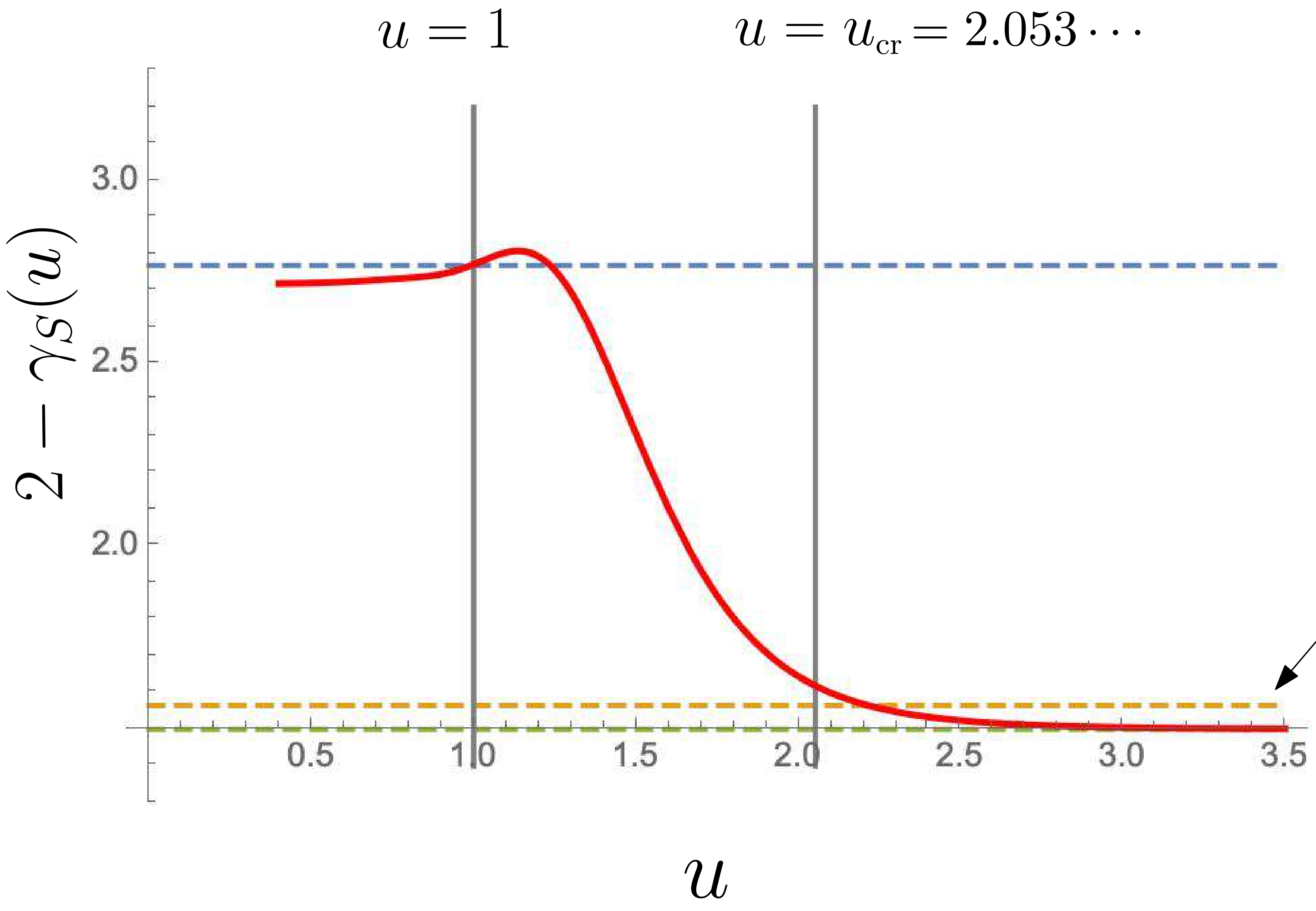
$\gamma_{\mathrm{S}} = -1$

$$\frac{g_c(u_{\mathrm{cr}})^{-n}}{n^{2-\gamma'_{\mathrm{S}}}(\log n)^{1/2}}$$

$\gamma'_{\mathrm{S}} = \dfrac{1}{2}$

Bicubic maps + Hamiltonian cycle with weight u per irreducible block

Guitter Kristjansen Nielsen '99



$\frac{13+\sqrt{13}}{6} = 2 - \gamma_{\mathrm{S}}$

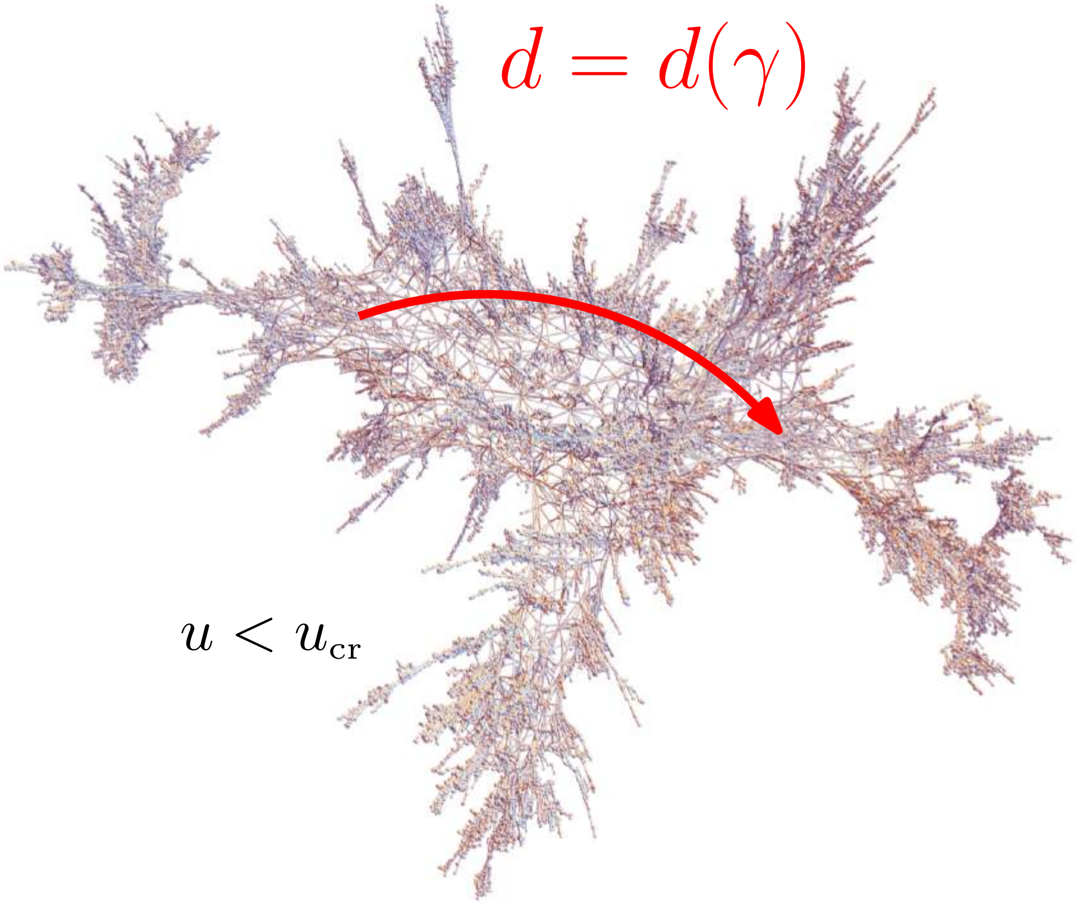
$$\gamma = \frac{\sqrt{13}-1}{\sqrt{3}}$$

$\frac{13-\sqrt{13}}{6} = 2 - \gamma'_{\mathrm{S}}$

$\frac{3}{2}$

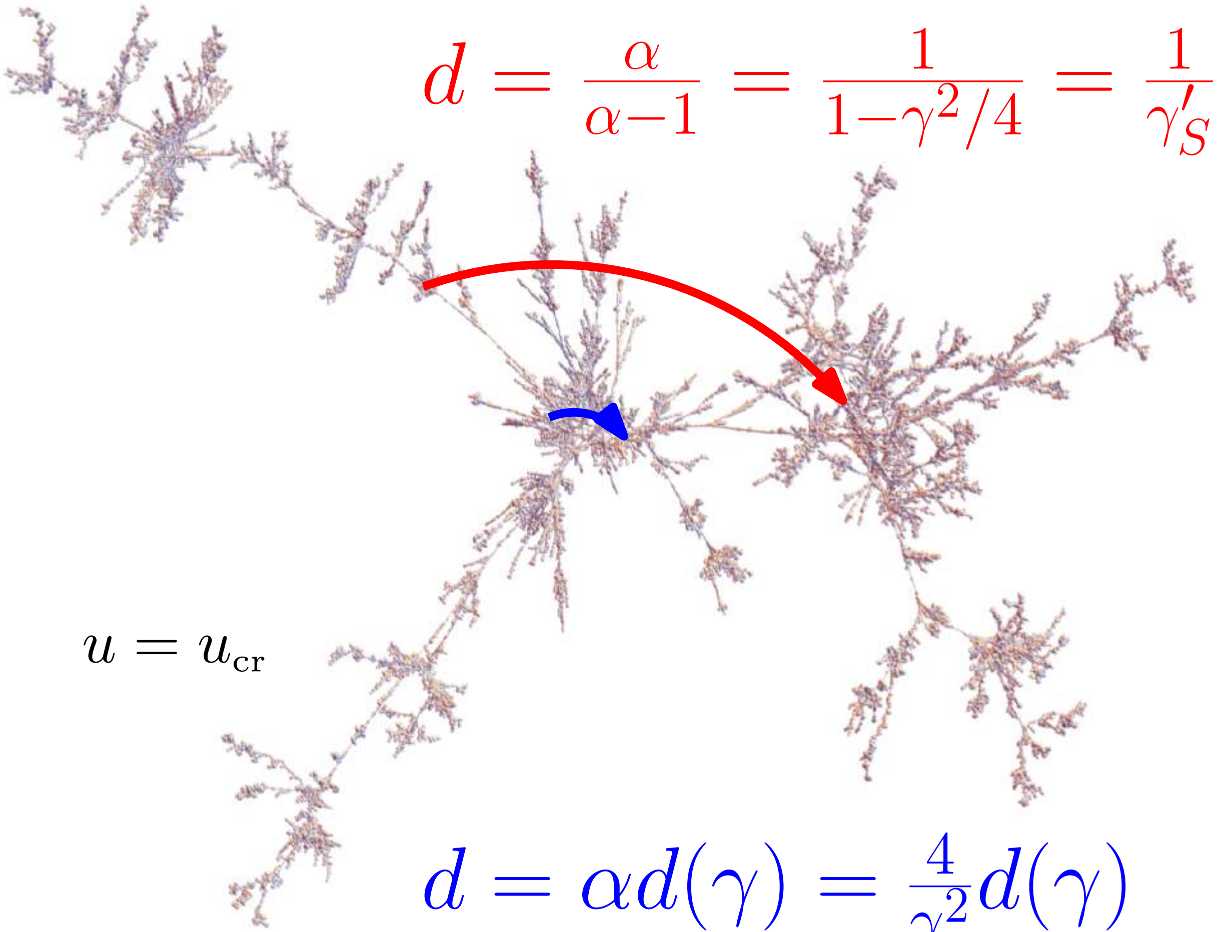
Distances

$$\text{distance} \sim n^{1/d}$$



$$d = d(\gamma)$$

$u < u_{\text{cr}}$



$$d = \frac{\alpha}{\alpha-1} = \frac{1}{1-\gamma^2/4} = \frac{1}{\gamma'_S}$$

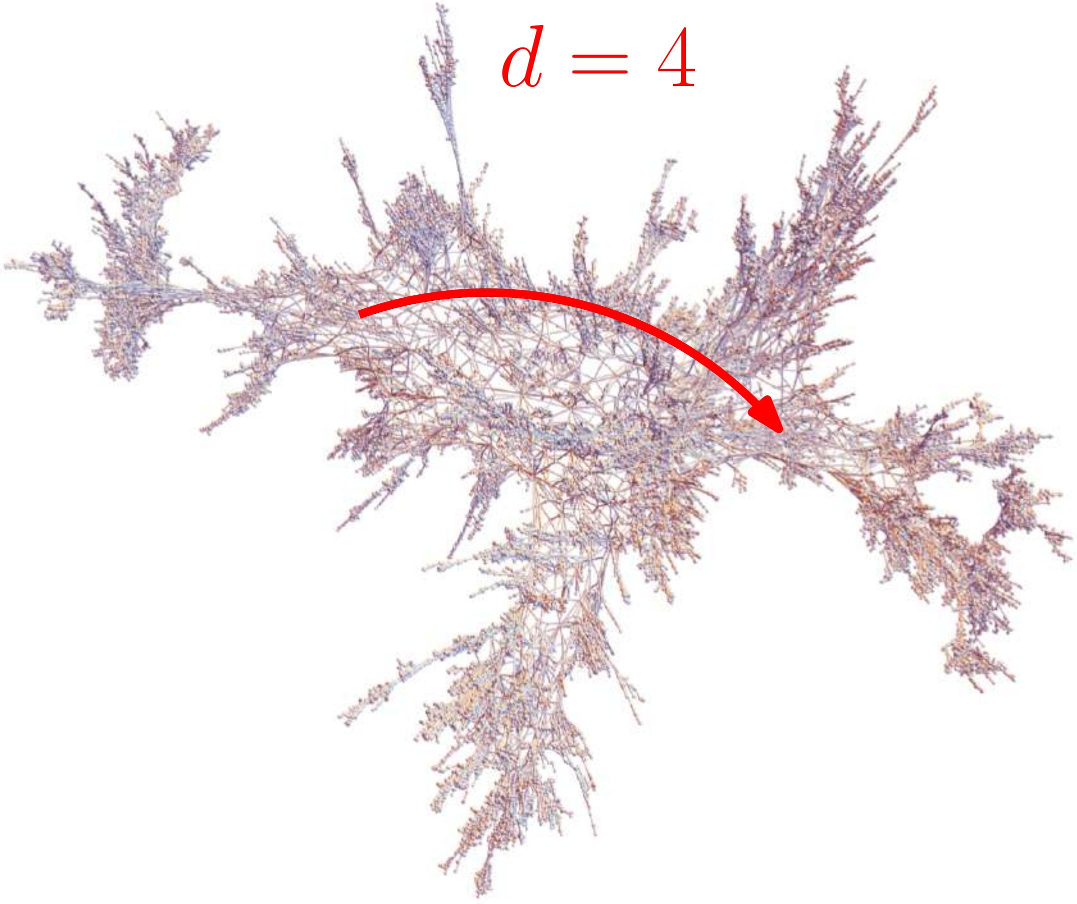
$u = u_{\text{cr}}$

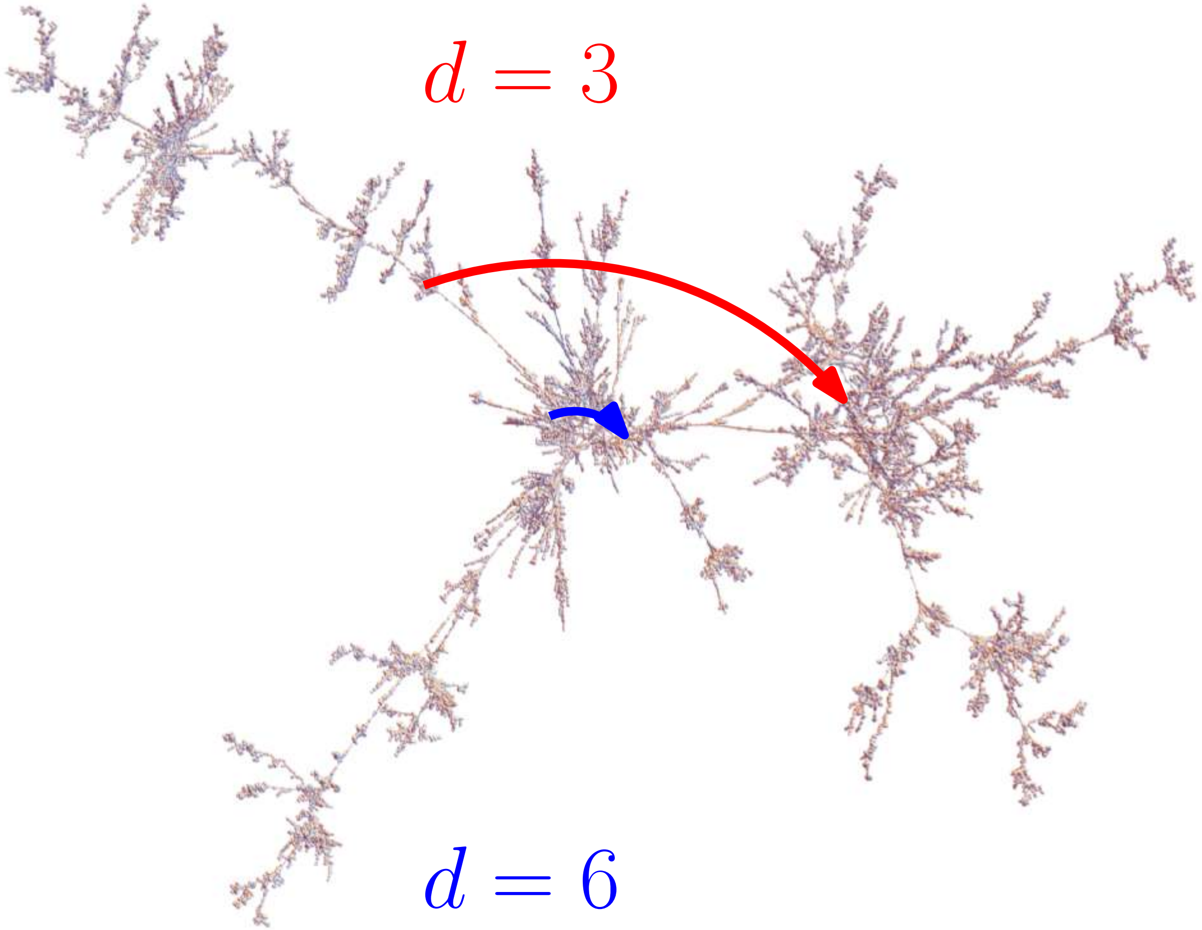
$$d = \alpha d(\gamma) = \frac{4}{\gamma^2} d(\gamma)$$

$$\gamma = \sqrt{\frac{8}{3}}$$

$$\text{distance} \sim n^{1/d}$$

Fleurat Salvy '24





$$\gamma = \sqrt{\frac{8}{3}}$$

Distance profile in the same block at $u = u_{\mathrm{cr}}$

Duplantier Guitter '25
Bouttier Guitter Manet (soon)

$$r = \mathrm{distance}/n^{1/6}$$

$$\rho(r) = \frac{3^2 2^{7/3}}{\Gamma\left(\frac{4}{3}\right)} \int_0^\infty x^{7/2} dx \, \frac{e^{-x^3/r^6}}{r^9} \mathrm{sh}(x) \frac{\mathrm{c}(x)\left(\mathrm{c}(x) + \mathrm{ch}(x)\right) - 2}{\left(\mathrm{c}(x) - \mathrm{ch}(x)\right)^3}$$

$$\mathrm{c}(x) = \cos\left(\frac{\sqrt{3x}}{2^{2/3}}\right) \ , \quad \mathrm{ch}(x) = \cosh\left(\frac{3\sqrt{x}}{2^{2/3}}\right)$$

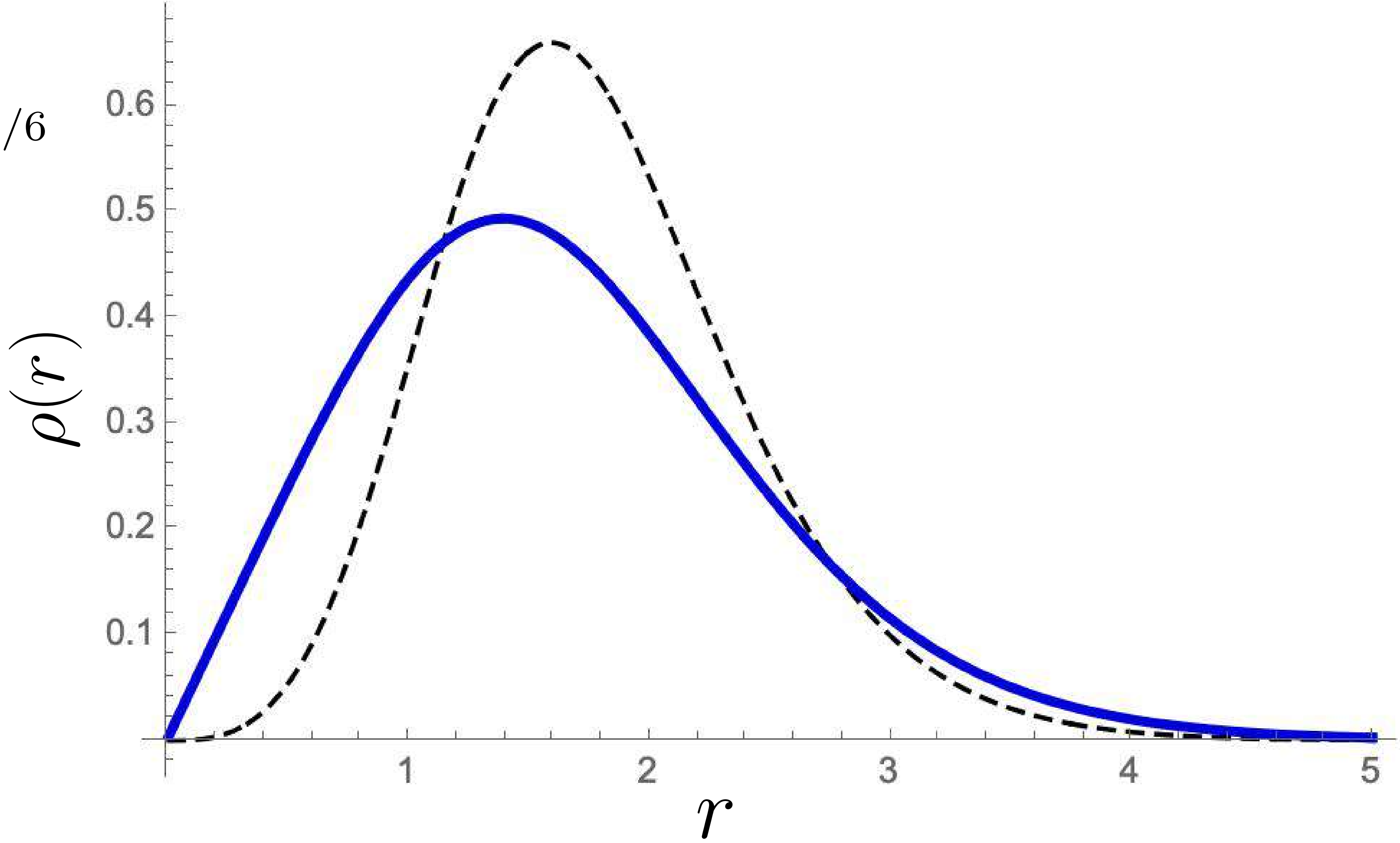
$$\mathrm{s}(x) = \sin\left(\frac{\sqrt{3x}}{2^{2/3}}\right) \ , \quad \mathrm{sh}(x) = \sinh\left(\frac{3\sqrt{x}}{2^{2/3}}\right)$$

$$\gamma = \sqrt{\frac{8}{3}}$$

Distance profile in the same block at $u = u_{\mathrm{cr}}$

Duplantier Guitter '25
Bouttier Guitter Manet (soon)

$$r = \mathrm{distance}/n^{1/6}$$



THANK YOU

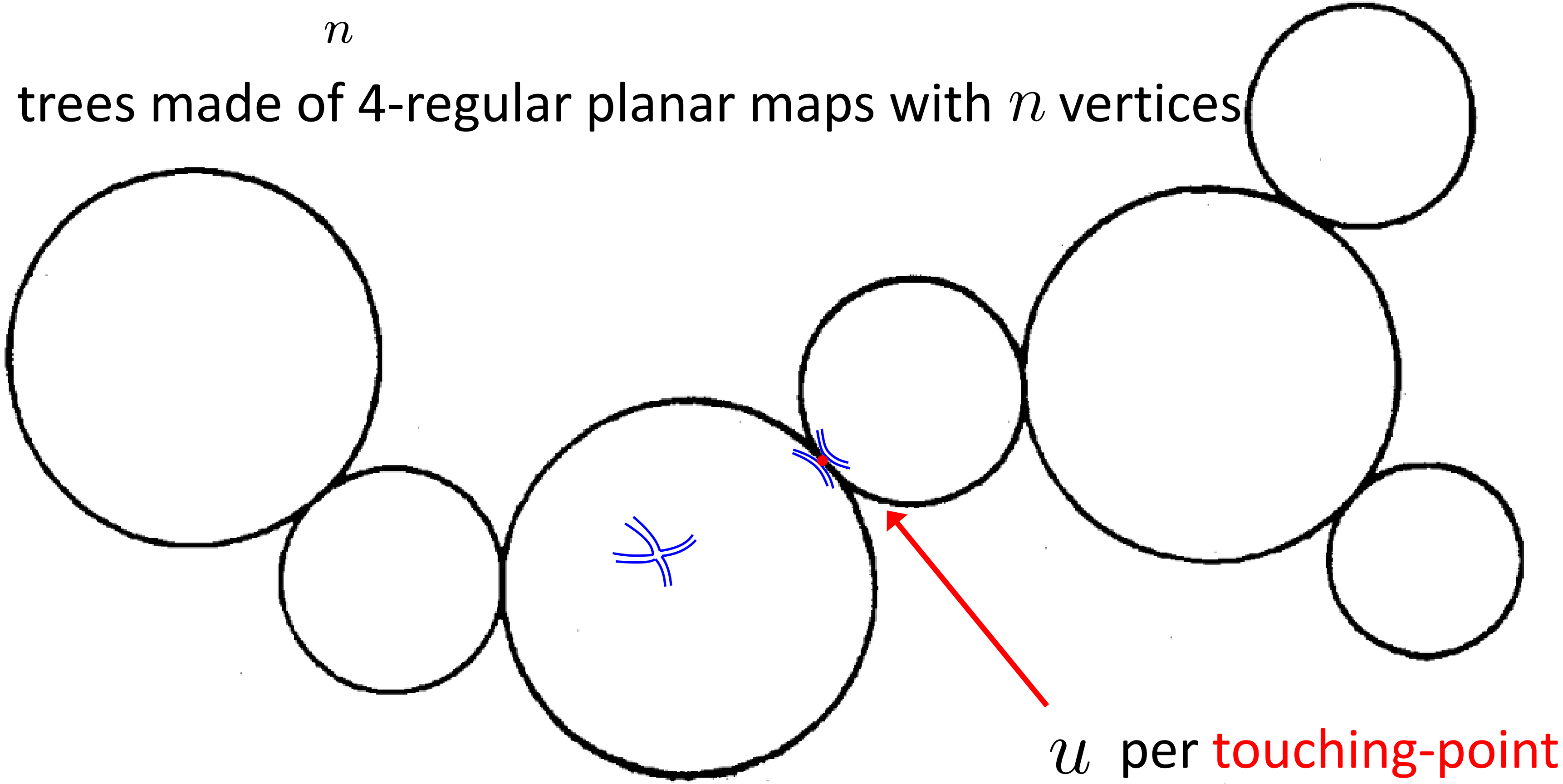
Matrix integral with potential $V(\Phi) = \frac{1}{2}\,\mathrm{Tr}\,\Phi^2 - \frac{g}{4N}\,\mathrm{Tr}\,\Phi^4 - \frac{u}{4N^2}(\mathrm{Tr}\,\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} \exp(-N^2 Z(g,u))$

$$Z(g,u) = \sum_n g^n Z_n(u)/(4n)$$

$Z_n(u)$ g.f. for (rooted) trees made of 4-regular planar maps with n vertices

Das et al. '90



Matrix integral with potential $V(\Phi) = \frac{1}{2}\,\mathrm{Tr}\,\Phi^2 - \frac{g}{4N}\,\mathrm{Tr}\,\Phi^4 - \frac{u}{4N^2}(\mathrm{Tr}\,\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} \exp(-N^2 Z(g,u))$

Substitution relation for $Z^\bullet(g,u) = \sum_n g^n Z_n(u) = 1 + 4g\frac{d}{dg}Z(g,u)$

$$Z^\bullet(g,u) = \frac{1}{1-u\,Z^\bullet(g,u)} M\left(\frac{g}{(1-u\,Z^\bullet(g,u))^2}\right)$$ with $M(g) = \frac{18g - 1 + (1-12g)^{3/2}}{54g^2}$

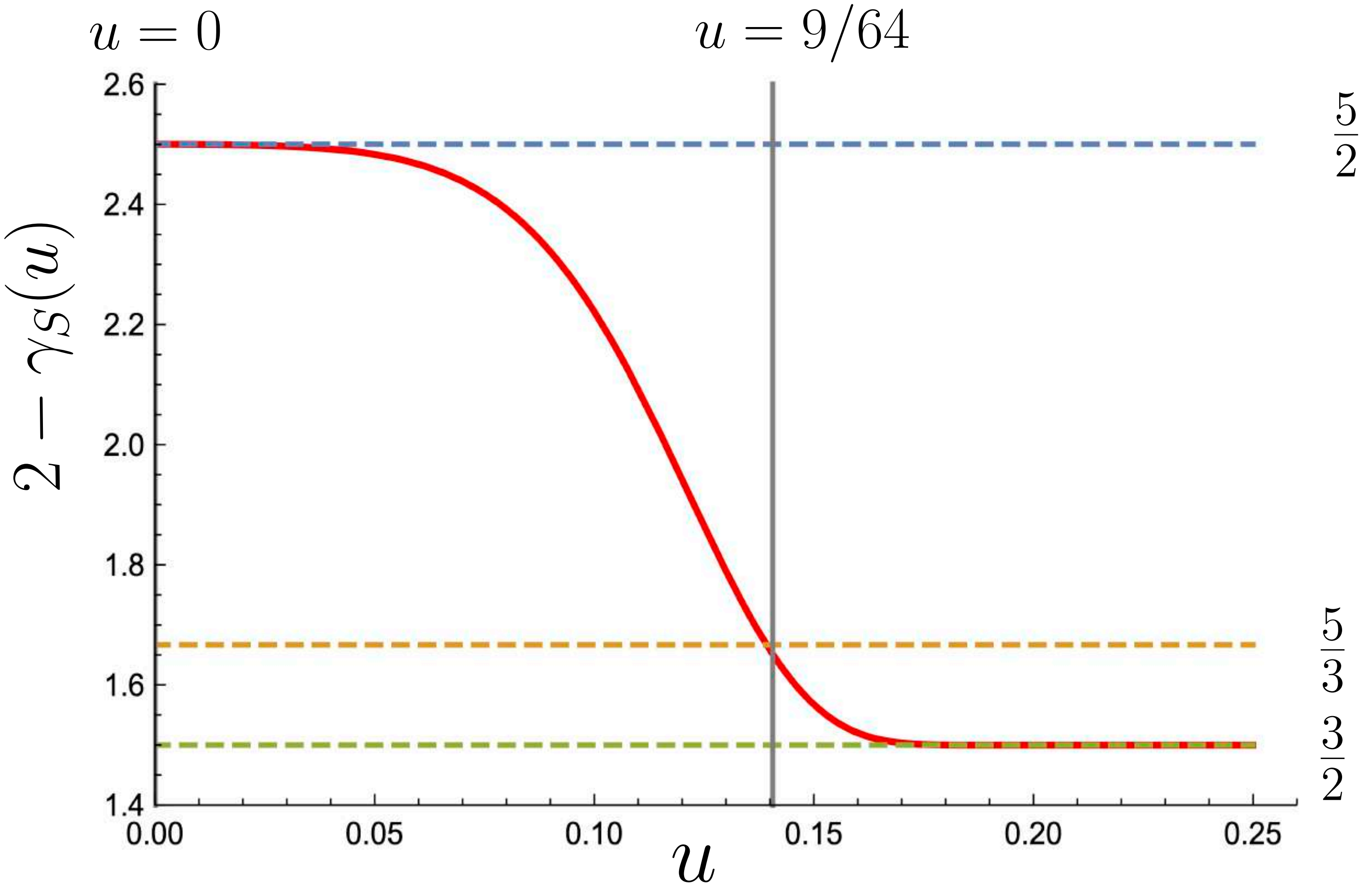
↑ g.f. of rooted planar 4-regular maps

Set $u = \sqrt{g}\,v$, $z = \sqrt{g}$, $y(z) = \frac{z}{1-u\,Z^\bullet(z^2,u)}$

then $y(z) = z\,\phi(y(z))$ with $\phi(\tau) = \frac{1}{1 - v\,\tau M(\tau^2)}$

Value of $u_{\rm cr}$? (CRIT) $\phi(y_c) = y_c\,\phi'(y_c)$, $z_c = \frac{y_c}{\phi(y_c)}$ with $y_c = \tau_\phi$, $\tau_\phi^2 = \frac{1}{12}$

fixes $v = v_{\rm cr} = \frac{3\sqrt{3}}{8}$ and $z_c = \frac{\sqrt{3}}{8}$ so that $u_{\rm cr} = z_c\,v_{\rm cr} = \frac{9}{64}$



$u = 0$
$u = 9/64$
2.6
2.4
2.2
2.0
1.8
1.6
1.4
$2 - \gamma_S(u)$
$\frac{5}{2}$
$\frac{5}{3}$
$\frac{3}{2}$
0.00
0.05
0.10
0.15
0.20
0.25
u