Critical phenomena in the 2 matrix model and problems in graphical enumeration

Journée cartes à Jussieu

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Overview

- Overview of the 1- and 2-matrix models
- Universality and critical phenomena: conjectures from physics
- A 'new' critical phenomena: The $(3, 4)$ string equation
- Implications in graph combinatorics

Universality and critical phenomena in the 1-matrix model

The 1-matrix model

Ingredients of the 1-matrix model:

- $\mathcal{H}_N$: space of $N \times N$ hermitian matrices, and
- dX, the Haar measure on $\mathcal{H}_N$
- $V(X)$: (monic) polynomial of even degree

One then defines the probability measure

$$d\mathbb{P}_V(X) = Z_N^{-1} \exp\left[-N \operatorname{tr} V(X)\right] dX$$

where $Z_N = Z_N(V)$ is called the *partition function*.
Let

$$\mu_V = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

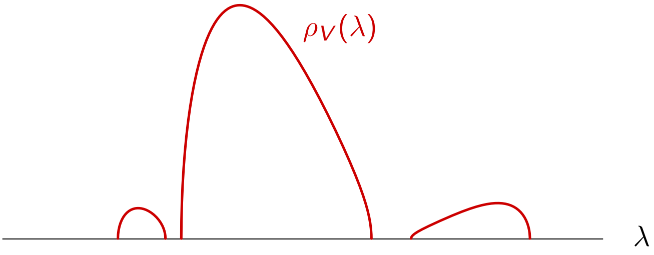
where λ_i are the eigenvalues of a matrix X sampled from $\mathbb{P}_V$.

Universality in the 1-matrix model

As $N \to \infty$, μ converges to an a.c. probability measure on $\mathbb{R}$:

$$d\mu_V(\lambda) \longrightarrow \rho_V(\lambda) d\lambda.$$

For **generic** V, ρ_V looks like:



$\rho_V(\lambda) \sim |\lambda - \lambda_0|^{1/2}$ if λ_0 is an endpoint of supp ρ_V.

Universality in the 1-matrix model: Correlation kernels

Local statistics, behavior of the partition function, etc. are all **universal** in the generic situation (= independent of exact form of V).
For instance, if one considers the correlation kernel

$$K_N(\lambda, \mu) := \mathbb{E}_{N-1}^{V}\left[\det(\lambda - X)\det(\mu - X)\right] e^{\frac{N}{2}(V(\lambda)+V(\mu))},$$

then, as $N \to \infty$, for λ^* an interior point of ρ_V,

$$\frac{1}{cN} K_N\left(\lambda^* + \frac{x}{cN}, \lambda^* + \frac{y}{cN}\right) \longrightarrow \frac{\sin \pi(x-y)}{\pi(x-y)},$$

and for λ^* an endpoint,

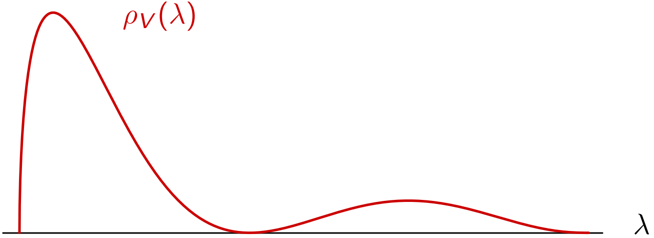
$$\frac{1}{cN^{2/3}} K_N\left(\lambda^* + \frac{x}{cN^{2/3}}, \lambda^* + \frac{y}{cN^{2/3}}\right) \longrightarrow \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

Question: What about the *nongeneric* situation?

Critical phenomena in the 1-matrix model

For some special choices of V, $\rho_V(\lambda)$ can be made to vanish like:

$$\rho_V(\lambda) \sim (\lambda - \lambda_0)^{2k} \qquad \text{or} \qquad \rho_V(\lambda) \sim |\lambda - \lambda_0|^{\frac{4k+1}{2}},$$



We label vanishings type as $(2, 2k)$ and $(2, 4k+1)$.

Critical phenomena in the 1-matrix model

In fact, these 'critical' cases are universal as well: if λ^* is a point of vanishing of type $(2, q)$, then

$$\frac{1}{cN^{\frac{2}{2+q}}} K_N \left(\lambda^* + \frac{x}{cN^{\frac{2}{2+q}}}, \lambda^* + \frac{y}{cN^{\frac{2}{2+q}}} \right) \longrightarrow K^{(2,q)}(x, y; s),$$

where s is a parameter measuring the deviation from criticality, and $K^{(2,q)}(x, y; s)$ is a kernel arising from the Painlevé II $(2, 2k)$ or Painlevé I $(2, 4k+1)$ hierarchies:

$$\underbrace{u''(s) = 6u(s)^2 + s}_{PI}, \qquad \underbrace{u''(s) = u(s)^3 + su(s) + \alpha}_{PII}.$$

These equations are Hamiltonian, and as such carry many 'nice' properties (isomonodromy formulation, Painlevé property,...)

Critical phenomena: Partition function

Suppose ρ has a vanishing of type $(2, 2k+1)$: then $V(\lambda) \to V(\lambda) + \sum_j N^{-\frac{2(k-j)}{2k+1}} s_{2j+1} \delta V_j(\lambda)$ so, near $\xi := \lambda - \lambda^*$,

$$\rho_{V+\delta V}(\xi) \sim \xi^{\frac{2k+1}{2}} + \sum_{j=0}^{k-2} \mathbf{s}_{2j+1}(\xi) \xi^{\frac{2j+1}{2}},$$

$\mathbf{s}_{2j+1}(\xi)$: analytic functions such that, uniformly for $|\xi| \lesssim N^{-\frac{2}{2k+1}}$,

$$N^{\frac{2(k-j)}{2k+1}} \mathbf{s}_{2j+1}(\xi) \xrightarrow[N\to\infty]{} s_{2j+1}.$$

Then, we have the convergence

$$\mathbf{d} \log \check{Z}_N(V + \delta V) \xrightarrow[N\to\infty]{} \mathbf{d} \log(\boldsymbol{\tau}(\vec{s})),$$

where $\boldsymbol{\tau}(\vec{s})$ is a τ-function for the k^{th} member of the PI hierarchy.

Why do we see Painlevé? Some Heuristics

Morally, the reason why such hierarchies appear is the following. For given V, there is a corresponding OPE:

$$\int p_n(\lambda)p_m(\lambda)e^{-NV(\lambda)}d\lambda = h_n\delta_{nm}$$

Three-term recurrence relation $\leftrightarrow$ multiplication by λ is Jacobi-type operator:

$$\lambda \Leftrightarrow P := \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_1 & a_1 & b_1 & 0 & \cdots \\ 0 & b_2 & a_2 & b_2 & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

Polynomial $V \leftrightarrow \frac{d}{d\lambda}$ is a $(\deg V) + 1$-diagonal matrix, Q

Why do we see Painlevé? Some Heuristics

(M. Douglas, '90) If we tune parameters of V to be critical, then we can arrange that

$$P = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_1 & a_1 & b_1 & 0 & \cdots \\ 0 & b_2 & a_2 & b_2 & \cdots \\ & & \ddots & \ddots & \ddots \\ 0 & 0 & & & \end{pmatrix} \longrightarrow \partial_s^2 + u(s),$$

$$Q \to \partial_s^q + v_{q-2}(s)\partial_s^{q-2} + ... + v_0(s)$$

The trivial identity $\left[\frac{d}{d\lambda}, \lambda\right] = 1$ at the level of the scaled operators becomes a *string equation*:

$$[Q, P] = 1.$$

The PI hierarchy is precisely this set of equations!

What does it mean for graphical enumeration?

E. Bender, Z. Gao, L. Richmond (2008), paraphrased: the number $\mathcal{N}_g(j)$ of connected, p-regular, genus g maps on j vertices grows asymptotically as:

$$\mathcal{N}_g(j) = \kappa_g \cdot j!\,(C_p)^j \cdot j^{\frac{1}{2}(5g-7)} \left(1 + \mathcal{O}(j^{-1/2})\right),$$

where κ_g appear in the asymptotic expansion of the PI τ-function:

$$\mathbf{d}\log \boldsymbol{\tau}_{PI}(x) \sim \sum_{g=0}^{\infty} \kappa_g \cdot (-x)^{3/2-\mathbf{5g/2}}, \qquad x \to \infty.$$

So, $\boldsymbol{\tau}_{PI}(x)$ contains combinatorial information.

The 2-matrix model

Introducing the 2-matrix model

Consider the unitary invariant measure

$$d\mathbb{P}_{V_1,V_2}(X,Y) = Z_N^{-1} \exp\left[N \operatorname{tr}\left(\tau XY - V_1(X) - V_2(Y)\right)\right] dXdY,$$

where V_1, V_2- polynomials, and $\tau \in \mathbb{R}$.
Define counting measures

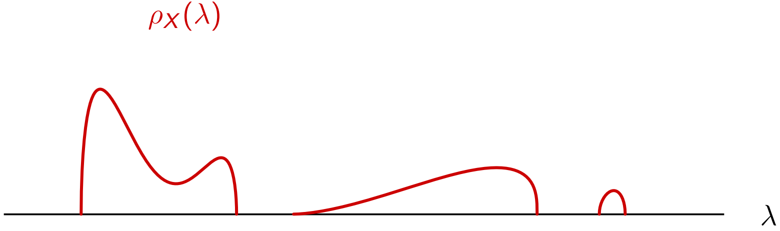
$$\mu_X = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, \qquad \mu_Y = \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i}.$$

It is believed (and in some cases proven, cf. Guionnet et. al., Kuijlaars-Duits et. al.) that $d\mu_X \to \rho_X(\lambda)d\lambda$, $d\mu_Y \to \rho_Y(\lambda)d\lambda$ as $N \to \infty$, with the same **generic** behavior for $\rho_X(\lambda)$, $\rho_Y(\lambda)$ as the 1-matrix model

Conjectures from physics

However, the universality classes of critical phenomena in the 2-matrix model are believed to be much wider: For any* (p, q) coprime, one can find potentials V_1, V_2 such that $\rho_X(\lambda)$ can be made to vanish like

$$\rho_X(\lambda) \sim |\lambda - \lambda^*|^{\frac{p}{q}}$$



if one allows for analytic continuation.

Douglas' Heuristics revisited

Similarly to the 1-matrix model, λ^* is a point of vanishing of type (p, q), quantities like correlation kernels and partition functions should be described by the (p, q) *string equation*:

$$[Q, P] = 1,$$

where $Q = \partial_s^q + \cdots$, and $P = \partial_s^p + \cdots$
Heuristic: orthogonal polynomials $\Rightarrow$ *biorthogonal* polynomials,
Multiplication by λ: tridiagonal matrix $\Rightarrow$ $2q-1$-diagonal matrix

More conjectures from physics

Correlation Kernel: Suppose λ^* is a point of vanishing of type (p, q) of $\rho_X(\lambda)$. One can define X-X, Y-Y, and X-Y correlations. Then, for example,

$$\frac{1}{cN^{\frac{p}{p+q}}} K_N^{XX}\left(\lambda^* + \frac{x}{cN^{\frac{p}{p+q}}}, \lambda^* + \frac{y}{cN^{\frac{p}{p+q}}}\right) \longrightarrow K^{(p,q)}(x, y; s),$$

$K^{(p,q)}$ is a kernel related to a (p, q) *string equation*.
Partition function: Under an appropriate choice of scaling of parameters,

$$\mathbf{d} \log \check{Z}_N(V + \delta V) \longrightarrow \mathbf{d} \log \boldsymbol{\tau}(\vec{s}),$$

where $\boldsymbol{\tau}(\vec{s})$ is a τ-function for the (p, q) *string equation*.

Mathematically, what's actually known?

Known results from mathematics

Early 2010s: (Delvaux, Duits, Geudens, Kuijlaars, Mo, ...)
Riemann-Hilbert formulation of biorthogonal polynomials to study the 2-matrix model
Identify the *Pearcey kernel*, 4×4 Painlevé II kernel as special eigenvalue correlation kernels in the case when

$$V_1(X) = \frac{t}{2}X^2 + \frac{1}{4}X^4, \qquad V_2(Y) = Y^{2k} + \cdots$$

However, no 'higher-order' critical phenomenon aside from the Pearcey process were found

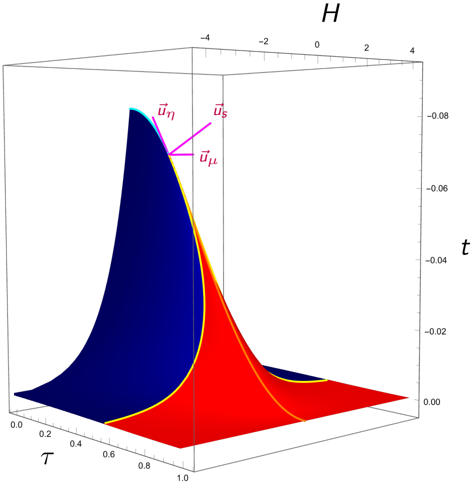
Quartic 2-matrix model: Ising phase transition

V. Kazakov ('86): critical point of type $(3,4)$ if one considers the potentials

$$V_1(X) = \frac{1}{2}X^2 + \frac{te^H}{4}X^4, \qquad V_2(Y) = \frac{1}{2}Y^2 + \frac{te^{-H}}{4}Y^4,$$

with $t_c = -\frac{5}{72}$, $\tau_c = \frac{1}{4}$, $H_c = 0$.
Kazakov, Douglas, Migdal,... ('90): Convergence of $Z_N(V)$ to a τ-function for $(3,4)$ string equation
M. Duits, N.H., & S.-Y. Lee ('24): we can obtain the same phase portrait as Kazakov, and identify the same critical point rigorously using Riemann-Hilbert analysis

Phase Diagram for the quartic model



Our main theorem will be stated in terms of the directions $\vec{u}_s, \vec{u}_\mu, \vec{u}_\eta$.

What is the $(3,4)$ string equation?

The most general equation arising from $[\partial^3 + \cdots, \partial^4 + \cdots] = 1$ $(' = \frac{d}{ds})$:

$$(\star) \begin{cases} 0 = \frac{1}{2}V'' - \frac{3}{2}UV + \frac{5}{2}\eta V + \mu, \\ 0 = \frac{1}{12}U^{(4)} - \frac{3}{4}U''U - \frac{3}{8}(U')^2 + \frac{3}{2}V^2 + \frac{1}{2}U^3 - \frac{5}{12}\eta(3U^2 - U'') + s \end{cases}$$

N.H. ('24), part II: Equation is *Hamiltonian*: there exists polynomial H_1 in $\{p_k, q_k\}_{k=1}^3, s, \mu, \eta$:

$$\frac{\partial p_k}{\partial s} = -\frac{\partial H_1}{\partial q_k}, \quad \frac{\partial q_k}{\partial s} = \frac{\partial H_1}{\partial p_k} \iff (\star)$$

Can further find polynomials H_2, H_5, which Poisson commute with H_1, so that $(\star)$ is Hamiltonian in μ, η as well: then

$$\mathbf{d} \log \tau(\eta, \mu, s) = H_1 ds + H_2 d\mu + H_5 d\eta.$$

Our theorem (soon)

M. Duits, N.H., S.-Y. Lee, ('25) Let $\vec{P} = (\tau, H, t)$, and

$$Z_N(\vec{P}) = \iint \exp N \operatorname{tr}\left[\tau XY - V(X; e^H t) - V(Y; e^{-H} t)\right] dX dY.$$

Put $\vec{P}_c := \left(\frac{1}{4}, 0, -\frac{5}{72}\right)$, and let $\vec{u}_s, \vec{u}_\mu, \vec{u}_\eta$ be the directions from before. Then,

$$\mathbf{d} \log \check{Z}_N\left(\vec{P}_c + \frac{\eta \vec{u}_\eta}{N^{2/7}} + \frac{\mu \vec{u}_\mu}{N^{5/7}} + \frac{s \vec{u}_s}{N^{6/7}}\right) \xrightarrow[N\to\infty]{} H_1 ds + H_2 d\mu + H_5 d\eta$$
$$= \mathbf{d} \log \tau(\eta, \mu, s),$$

i.e. the partition function converges to a τ-function for the $(3, 4)$ string equation.

Properties of $\tau(\eta, \mu, \nu)$

The τ-function itself admits an $\hbar \to 0$ 'topological expansion'

$$\hbar^2 \mathbf{d} \log \tau(\hbar^{-2/7}\eta, \hbar^{-5/7}\mu, \hbar^{-6/7}\nu) \sim \sum_{g=0}^{\infty} \mathbf{d} \log \tau_g(\eta, \mu, \nu) \hbar^{2g}.$$

$\tau_g(\eta, \mu, \nu)$ can be computed iteratively. Contain 'topological data', since

$$\mathbf{d} \log \check{Z}_N(\tau, t, H) \sim \sum_{g=0}^{\infty} \frac{F_g(\tau, t, H)}{N^{2g}},$$

and

$$F_g(\tau, t, H) \xrightarrow{\text{multi-scaling limit}} \tau_g(\eta, \mu, \nu)$$

Implications in graph combinatorics

We would like to say something about the large-j asymptotics of $\mathcal{N}_g(\tau, j)$ near the critical temperature $\tau = \tau_c$.

O. Bernardi, M. Bousquet-Mélou (2011): For $\tau = \tau_c$, as $j \to \infty$,

$$\mathcal{N}_0(\tau_c, j) \sim \tau_0 \cdot K^{-j} \cdot j^{-\frac{1}{3}(10-0\cdot 7)} \left(1 + \mathcal{O}(j^{-1/3})\right).$$

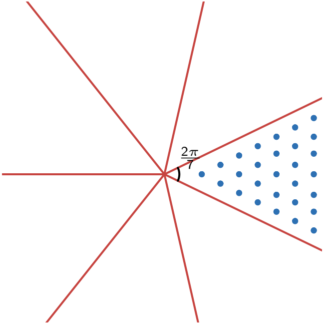
Higher genus formulae seem to be unknown. By analogy to the 1-matrix setting, it seems that the form of this expansion should be determined by the **τ-function for the (3,4) string equation**.

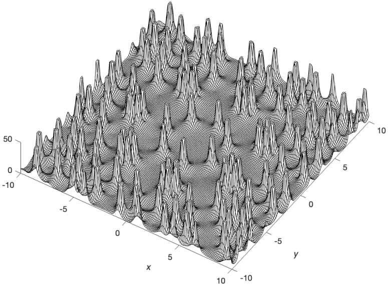
Concluding remarks: What have we studied?

- Critical phenomena in random matrices often involve integrable equations, e.g. critical partition function $\to$ τ-function
- The 2-matrix model is a good source for a much richer class of critical phenomenon, many of which are still unexplored
- We were able to study the first critical phenomenon not appearing in the 1-matrix model, i.e. the $(3, 4)$ cusp: our result has implications for the critical Ising model on random graphs

Thanks!

Question 1: What does our solution to the $(3, 4)$ equation look like? Some speculations





*Second figure from the algorithm of Fornberg/Weideman