Lecture 3 Notes (VSOP2026): Thermal Freeze-Out

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The basic idea of thermal freeze-out is that perhaps the well-measured abundance of dark matter is controlled by its interactions with Standard Model particles, which could have been much more common in the early universe (with its very high density) compared with today. This is not the only way to explain the dark matter abundance, but it has the advantage that we obtain a prediction for the interactions between dark matter and the Standard Model, which can be tested in various experiments. We will focus on the simple case where dark matter undergoes a number-changing process known as *annihilation*, where two dark matter particles can collide to produce a pair of Standard Model particles (similar to matter-antimatter annihilation in the Standard Model). The rate of this process is controlled by the annihilation cross section σ.

I. THE BOLTZMANN EQUATION FOR ANNIHILATING DARK MATTER

Consider the evolution of the dark-matter number density n in the early Universe. In the absence of annihilation or production processes, the comoving number density is conserved:

$$\frac{d}{dt}\left(na^3\right) = 0. \tag{1}$$

Equivalently,

$$a^3 \frac{dn}{dt} + 3a^2 n \frac{da}{dt} = 0, \tag{2}$$

so

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a} n = 0, \tag{3}$$

where $H \equiv \dot{a}/a$ is the Hubble expansion rate.

With annihilation, the number density is instead depleted by annihilations and replenished by the inverse production processes. Schematically, if $\langle \sigma v \rangle$ is the cross-section multiplied by the relative velocity, averaged over the dark matter velocity distribution, then:

$$\frac{dn}{dt} + 3Hn = -\frac{n^2}{2}\langle \sigma v \rangle \times 2 + (n\text{-independent contribution from dark-matter production processes})\,. \tag{4}$$

The first factor of 1/2 avoids double-counting pairs of annihilating particles, while the second factor of 2 reflects the fact that two dark-matter particles are removed per annihilation. The term involving $\langle \sigma v \rangle$ is the annihilation rate. In equilibrium the left-hand side vanishes and $n = n_{\rm eq}$, so the production term must equal $\langle \sigma v \rangle n_{\rm eq}^2$. Thus the Boltzmann equation can be written as

$$\frac{dn}{dt} + 3Hn = -\langle \sigma v \rangle \left(n^2 - n_{\rm eq}^2\right). \tag{5}$$

The equilibrium density $n_{\rm eq}$ is set by the Boltzmann distribution. Parametrically, if T is the temperature (we assume here the interactions are sufficiently fast to keep the dark and visible matter at the same temperature), and m_χ denotes the dark matter mass, there are two limiting cases:

$$n_{\rm eq} \sim \begin{cases} (m_\chi T)^{3/2} e^{-m_\chi/T}, & T \ll m_\chi \quad \text{(non-relativistic)}, \\ T^3, & T \gg m_\chi \quad \text{(relativistic)}. \end{cases} \tag{6}$$

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When $\langle\sigma v\rangle \to 0$, the number density simply evolves as $n \propto a^{-3}$. When $\langle\sigma v\rangle$ is large, annihilations and inverse production processes force n to remain exponentially close to $n_{\rm eq}$. The crossover, which we call "freeze-out", occurs when

$$\langle\sigma v\rangle n^2 \sim Hn, \tag{7}$$

or equivalently

$$n\langle\sigma v\rangle \sim H. \tag{8}$$

Thus freeze-out occurs when the inverse timescale for annihilation becomes comparable to the inverse timescale for expansion.

For a full solution, one solves the Boltzmann equation in the FLRW background, including the detailed evolution of the Standard Model temperature (which as discussed last time can be modified from the baseline redshift scaling due to annihilations, interactions, etc). But as a useful approximation, we can define the freeze-out temperature T_f by

$$n\langle\sigma v\rangle = H \quad \text{at } T = T_f. \tag{9}$$

The late-time comoving density is then approximately the comoving density at freeze-out,

$$n_{\rm late} a_{\rm late}^3 \sim n_f a_f^3, \tag{10}$$

and since freeze-out corresponds to where the density first diverges from the equilibrium density, this means the late-time comoving density is approximately the comoving *equilibrium* density at freeze-out.

II. HOT RELICS

The first case to consider is a hot relic: dark matter freezes out while still relativistic. In this case,

$$n_f a_f^3 \sim T_f^3 a_f^3, \tag{11}$$

and therefore the late-time density is

$$n_{\rm late} \sim \frac{(T_f a_f)^3}{a_{\rm late}^3}. \tag{12}$$

As we discussed last time, the Standard Model temperature satisfies $T \propto 1/a$, up to modifications due to interactions (e.g. Standard Model particles becoming non-relativistic and annihilating into the radiation bath). These modifications are generally modest and can be ignored for an order-of-magnitude estimate. Therefore $T_f \sim 1/a_f$ in the sense that $T_f a_f$ is nearly constant, and in this case (relativistic freezeout) the late-time freeze-out density is nearly independent of the freeze-out temperature.

As a consequence, the late-time number density of a hot relic is comparable to the photon number density (unless there are a large number of currently-unknown particles that annihilate and heat up the photons relative to the dark matter between freeze-out and today). This makes hot thermal relics highly constrained: their number density is essentially fixed, so their mass must be small enough not to overclose the Universe, while their free-streaming must not erase too much structure. There are roughly 1.6×10^9 photons for every baryon in the universe, so if there are a comparable number of dark matter particles, given there is about $5\times$ as much mass in dark matter as in baryons, this means the dark matter mass must be smaller than the baryon mass by a factor of $5/(1.6 \times 10^9) \approx 3 \times 10^{-9}$, i.e. ~ 3 eV (this number can change at the $\mathcal{O}(1)$ level depending on whether dark matter is a fermion or a boson, and its number of internal degrees of freedom). This is far below the Tremaine-Gunn bound we discussed last time, so the dark matter would have to be bosonic, but more importantly, it also implies a free-streaming scale that is far too large.

III. COLD RELICS

The second case is a cold relic: dark matter freezes out after becoming non-relativistic. At freeze-out,

$$H \sim \left[(m_\chi T)^{3/2} e^{-m_\chi/T}\right] \langle\sigma v\rangle, \tag{13}$$

where the bracketed factor is parametrically $n_{\rm eq} \sim n$ at freeze-out.
During the radiation-dominated epoch, from the Friedmann equation we have:

$$H^2 \propto \rho \propto T^4, \tag{14}$$

so

$$T \propto H^{1/2} \propto t^{-1/2}. \tag{15}$$

It is convenient to write

$$H = H(m_\chi)x^{-2}, \qquad x \equiv \frac{m_\chi}{T}, \tag{16}$$

where $H(m_\chi)$ denotes the Hubble rate at $T = m_\chi$. Then the freeze-out condition gives parametrically

$$H(m_\chi)x_f^{-2} \sim (m_\chi^2)^{3/2} x_f^{-3/2} e^{-x_f} \langle\sigma v\rangle, \tag{17}$$

where $x_f = T_f/m_\chi$, or

$$e^{-x_f} \sim \frac{x_f^{-1/2}}{m_\chi^3\langle\sigma v\rangle/H(m_\chi)}. \tag{18}$$

An approximate solution is therefore

$$x_f \sim \ln\left(\frac{m_\chi^3\langle\sigma v\rangle}{H(m_\chi)}\right), \tag{19}$$

up to logarithmic corrections. Thus x_f has only logarithmic dependence on the dark-matter mass and annihilation cross section.
The number density of dark matter at freeze-out is

$$n_\chi \sim m_\chi^3 x_f^{-3/2} e^{-x_f} \sim \frac{H(m_\chi)x_f^{-2}}{\langle\sigma v\rangle}. \tag{20}$$

By comparison, the photon number density at freeze-out is

$$n_\gamma \sim T_f^3 \sim \frac{m_\chi^3}{x_f^3}. \tag{21}$$

Both n_χ and n_γ then scale approximately as a^{-3} after freeze-out, ignoring injections of additional particles/heat. Hence

$$\frac{n_\chi}{n_\gamma} \sim \left[\frac{H(m_\chi)}{m_\chi^2}\right]\left(\frac{x_f}{m_\chi}\right)\frac{1}{\langle\sigma v\rangle} \Rightarrow \rho_\chi \sim \left[\frac{H(m_\chi)}{m_\chi^2}\right]\frac{x_f}{\langle\sigma v\rangle}n_\gamma \tag{22}$$

The factor in square brackets is independent of m_χ provided freeze-out occurs during radiation domination, since then $H \propto T^2$. Thus the present-day dark-matter mass density (i.e. ρ_χ evaluated in the present day), and hence Ω_χ, has only a logarithmic dependence on m_χ (from x_f) and scales primarily as

$$\rho_\chi \propto \frac{1}{\langle\sigma v\rangle}. \tag{23}$$

More explicitly, during radiation domination we can write $H \sim T^2/m_{\rm Pl}$, where $m_{\rm Pl}$ is the Planck mass and arises from the factor of Newton's constant G on the right-hand side of the Friedmann equation. Thus we obtain:

$$\frac{m_\chi n_\chi}{n_\gamma} \sim \frac{x_f}{\langle\sigma v\rangle}\frac{1}{m_{\rm Pl}}. \tag{24}$$

To estimate the numerical value, compare to the observed dark-matter mass density. The dark-matter mass density is approximately five times the baryon mass density, so

$$\rho_\chi \sim 5\rho_b \sim 5 \text{ GeV} \times n_b. \tag{25}$$

Using the baryon-to-photon ratio $n_b/n_\gamma \sim 6 \times 10^{-10}$, this gives

$$\rho_\chi \sim 5\, n_\gamma\, (6 \times 10^{-10}) \text{ GeV}. \tag{26}$$

Equivalently, the observed abundance requires

$$\frac{m_\chi n_\chi}{n_\gamma} \sim 3 \times 10^{-9} \text{ GeV}. \tag{27}$$

Matching this to the freeze-out estimate gives

$$3 \times 10^{-9} \text{ GeV} \sim \frac{x_f}{\langle\sigma v\rangle}\, 10^{-19} \text{ GeV}^{-1}, \tag{28}$$

so

$$\langle\sigma v\rangle \sim x_f \times 10^{-10.5} \text{ GeV}^{-2}. \tag{29}$$

As a first guess, since x_f is only logarithmically sensitive to parameters, we could take $x_f \sim 1$. This would give

$$\langle\sigma v\rangle \sim 10^{-10.5} \text{ GeV}^{-2}. \tag{30}$$

If the annihilation cross section is of the schematic form

$$\langle\sigma v\rangle \sim \frac{\alpha^2}{m_\chi^2}, \tag{31}$$

with $\alpha \sim 10^{-2}$, this points to

$$m_\chi \sim 10^3 \text{ GeV}. \tag{32}$$

A better estimate uses

$$x_f \sim \ln\left(m_\chi m_{\rm Pl}\langle\sigma v\rangle\right) \sim \ln\left(10^{22} \text{ GeV}^2 \times 10^{-10.5} \text{ GeV}^{-2}\right) \sim \ln(10^{11.5}) \sim 25. \tag{33}$$

Substituting this back into the required annihilation rate gives

$$\langle\sigma v\rangle \sim 10^{-9} \text{ GeV}^{-2} \sim 10^{-26} \text{ cm}^3/\text{s}. \tag{34}$$

A more careful calculation gives

$$\langle\sigma v\rangle \sim 2\text{–}3 \times 10^{-26} \text{ cm}^3/\text{s}, \tag{35}$$

nearly independent of mass. This is the famous thermal relic cross section.

There is another simple way to see the same scaling, which also generalizes easily to n-body annihilation with $n > 2$. As before,

$$\frac{dn}{dt} + 3Hn = -\langle\sigma v\rangle\left(n^2 - n_{\rm eq}^2\right). \tag{36}$$

At freeze-out,

$$H_f \simeq \langle \sigma v \rangle n_f. \tag{37}$$

At matter-radiation equality, $\rho_\chi \sim \rho_{\text{rad}}$, ignoring baryons, and the photon bath temperature is T_{MRE}. Because the energy density of radiation scales as T^4, we can write $\rho_{\text{rad}} \sim T_{\text{MRE}}^4$. If the freeze-out happens during radiation domination, the evolution of the frozen-out abundance between freeze-out and matter-radiation equality is just governed by the cosmic expansion, so $n_{\text{MRE}} = \rho_{\text{MRE}}/m_\chi \sim T_{\text{MRE}}^4/m_\chi$, and we can set this equal to $n_f(T_{\text{MRE}}/T_f)^3$, obtaining:

$$n_f m_\chi \sim T_{\text{MRE}} T_f^3. \tag{38}$$

Taking $T_f \sim m_\chi$ for this order-of-magnitude estimate, noting that during radiation domination $H \sim T^2/m_{\text{Pl}}$, and imposing the freezeout condition $n_f\langle \sigma v \rangle = H(T_f)$, we can write

$$n_f \sim T_{\text{MRE}} m_\chi^2, \quad \frac{T_f^2}{m_{\text{Pl}}} \sim \langle \sigma v \rangle n_f \sim \langle \sigma v \rangle T_{\text{MRE}} m_\chi^2. \tag{39}$$

Taking $T_f \sim m_\chi$ again, this gives

$$\langle \sigma v \rangle \sim \frac{1}{m_{\text{Pl}} T_{\text{MRE}}}, \tag{40}$$

which is set entirely by known cosmological parameters. With

$$T_{\text{MRE}} \sim 1 \text{ eV} \sim 10^{-9} \text{ GeV}, \qquad m_{\text{Pl}} \sim 10^{19} \text{ GeV}, \tag{41}$$

we find

$$\langle \sigma v \rangle \sim \frac{1}{(100 \text{ TeV})^2} \sim \left(\frac{10^{-2}}{\text{TeV}}\right)^2. \tag{42}$$

This suggests a TeV-scale dark matter mass for a perturbative coupling $\alpha \sim 10^{-2}$, with $m_\chi \lesssim 100$ TeV. The natural scale thus lines up with electroweak masses and couplings, which is suggestive and has given rise to the term "Weakly Interacting Massive Particle" or WIMP for these scenarios. That said, the basic mechanism also works at lower masses and up to the 100 TeV scale (we will say a bit more about this upper limit later, but e.g. a careful calculation finds an upper limit of order a few hundred TeV for thermal relic dark matter [1]).

IV. VARIATIONS ON THE SIMPLE FREEZE-OUT STORY

There are many variations on the simplest freeze-out picture. Some examples are:

- The annihilation cross section can be velocity-dependent. For example, it may be p-wave suppressed, with $\langle \sigma v \rangle \propto v^2$, or Sommerfeld-enhanced, with $\langle \sigma v \rangle \propto 1/v$ or even $\propto 1/v^2$ over some range of velocities.
- Coannihilation can be important, with multiple species involved in determining the relic abundance.
- Dark matter can annihilate into the dark sector rather than directly into Standard Model particles.
- Forbidden or impeded dark matter can annihilate to kinematically forbidden states, with the abundance set by Boltzmann-suppressed processes [2, 3].
- There are scenarios where it is 3-body or higher interactions that set the abundance, rather than 2-body annihilation (one such class of models is called SIMPs = strongly interacting massive particles) [4, 5].
- Dark-matter scattering with the Standard Model can decouple before annihilation freezes out, so that the two sectors can have different temperatures. One class of such scenarios is sometimes called ELDERs [6].

- Freeze-in scenarios never reach complete thermal equilibrium: dark-matter–Standard-Model interactions create the abundance but do not deplete it. These scenarios typically require much smaller cross sections than thermal freeze-out.

The classic example of this last case is the sterile neutrino: a new neutrino-like state coupling to Standard Model particles similarly to active neutrinos, but with interactions suppressed by a small mixing angle θ. The mixing angle governs both the decay rate, which is constrained by astrophysical observations, and the production rate. In the simplest case, sterile neutrinos are produced from the Standard Model plasma with abundance proportional to $\theta^2 m_{\mathrm{DM}}$ [7]. This baseline scenario is now ruled out by a combination of bounds on the free-streaming length and the decay rate, but many variations may still have open parameter space, such as resonant production or decays of heavier states rather than oscillations from the Standard Model plasma.

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