DM1: Cosmology

(a) Assume the universe is either dominated by radiation, matter, or dark energy, with negligible contributions to the energy density from the other components. In each case, solve the Friedmann equation to determine the relationship between redshift $1 + z$ and elapsed coordinate time t. For each choice, does the expansion of the universe speed up or slow down with increasing time?

(b) Using the Planck 2018 cosmological parameters, estimate (1) the redshift of matter-radiation equality, when the contributions to the energy density from matter and radiation are equal, (2) the redshift of matter-dark energy equality, when the contributions to the energy density from matter and dark energy are equal.

(c) Using your results from (a) and (b), estimate the age of the universe at Big Bang nucleosynthesis ($T \sim 1$ MeV), matter-radiation equality, recombination ($z \sim 1000$), and matter-dark energy equality. You may assume the present-day temperature of the CMB is 2.725 K ($\sim 2 \times 10^{-4}$ eV), and neglect changes in the number of relativistic degrees of freedom (although if you wish to include them and know how to do it, that is also fine).

Solutions

(a) Note $H(z) = (1/a)da/dt = (1+z)d/dt(1+z)^{-1} = -\frac{1}{1+z}\frac{dz}{dt}$.

Radiation domination: $H^2 = \Omega_{\rm rad} H_0^2 (1+z)^4 \Rightarrow \left(\frac{dz}{dt}\right)^2 = H_0^2 \Omega_{\rm rad}(1+z)^6$, i.e. $-(1+z)^{-3}dz = H_0\sqrt{\Omega_{\rm rad}}dt \Rightarrow (1/2)/(1+z)^2 = \sqrt{\Omega_{\rm rad}}t \Rightarrow t = \frac{1}{H_0}\frac{1}{2(1+z)^2\sqrt{\Omega_{\rm rad}}} \Rightarrow 1+z = \sqrt{\frac{1}{H_0 t \times 2\sqrt{\Omega_{\rm rad}}}}$.

Matter domination: $H^2 = \Omega_m H_0^2 (1+z)^3 \Rightarrow \left(\frac{dz}{dt}\right)^2 = H_0^2 \Omega_m (1+z)^5$, i.e. $-(1+z)^{-2.5}dz = H_0\sqrt{\Omega_m}dt \Rightarrow (2/3)/(1+z)^{1.5} = \sqrt{\Omega_m}t \Rightarrow t = \frac{1}{H_0}\frac{2}{3(1+z)^{3/2}\sqrt{\Omega_m}} \Rightarrow 1+z = \left(\frac{1}{H_0 t}\frac{2}{3\sqrt{\Omega_m}}\right)^{2/3}$.

Dark energy domination: $H^2 = \Omega_\Lambda H_0^2 \Rightarrow \left(\frac{dz}{dt}\right)^2 = H_0^2\Omega_\Lambda(1+z)^2$, i.e. $-(1+z)^{-1}dz = H_0\sqrt{\Omega_\Lambda}dt \Rightarrow C - \ln(1+z) = \sqrt{\Omega_\Lambda}t \Rightarrow t = (t_0 H_0\sqrt{\Omega_\Lambda} - \ln(1+z))\frac{1}{H_0\sqrt{\Omega_\Lambda}} \Rightarrow 1+z = e^{-H_0\sqrt{\Omega_\Lambda}(t-t_0)}$.

We observe that the expansion is exponentially accelerating in dark energy domination, and decelerating in the other two cases ($d^2a/dt^2 < 0$).

(b) Matter-radiation equality: $(1+z)\Omega_{\rm rad} = \Omega_m \Rightarrow 1+z \approx \frac{0.32}{10^{-4}} \sim 3000$.

Matter-dark energy equality: $(1+z)^3\Omega_m = \Omega_\Lambda \Rightarrow (1+z)^3 \approx \frac{0.68}{0.32} \Rightarrow 1+z \approx 1.28$.

(c) The first two epochs occur during radiation domination (or at its end), and occur respectively at $z \sim 5\times 10^9$ (since temperature redshifts roughly as $(1+z)$) and $z \sim 1000$. So using our result above for radiation domination, $t = \frac{1}{H_0}\frac{1}{2(1+z)^2\sqrt{\Omega_{\rm rad}}}$, we obtain for BBN $t \approx 2\times 10^{-18}/H_0 \approx 1s$, and for recombination $t \approx 6\times 10^{-6}/H_0 \approx 70,000$ years.

Recombination and matter-DE equality occur during matter domination, so substituting $z = 1000$ and $z = 1.28$ into $t = \frac{1}{H_0}\frac{2}{3(1+z)^{3/2}\sqrt{\Omega_m}}$, we obtain: $t = 4\times 10^{-5}/H_0 \approx 480,000$ years for recombination, $0.8/H_0 \approx 11$ billion years for matter-DE equality.