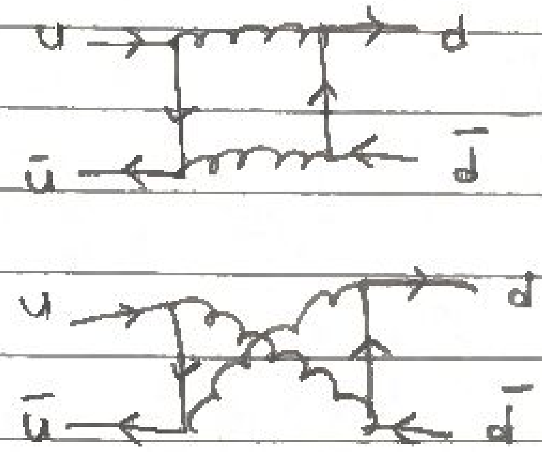
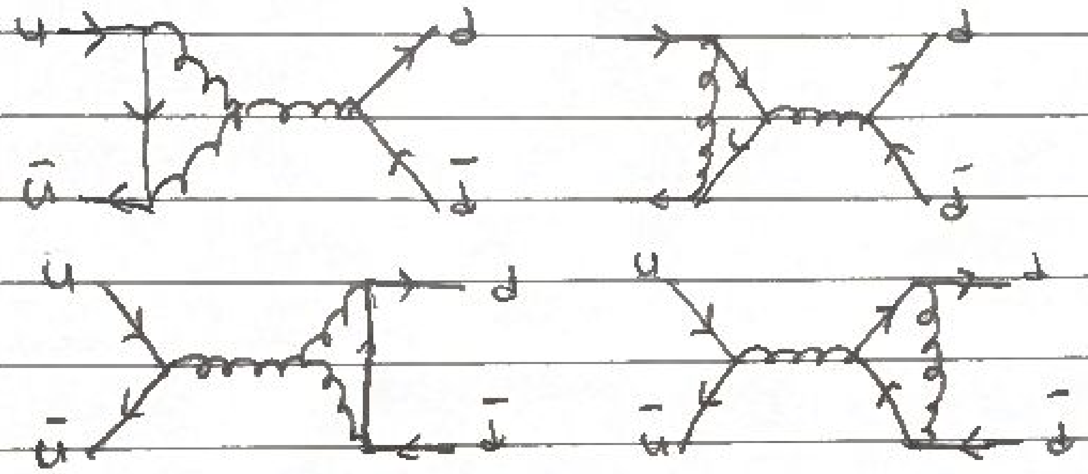
1. 1-loop QCD corrections to $u\bar{u} \to d\bar{d}$

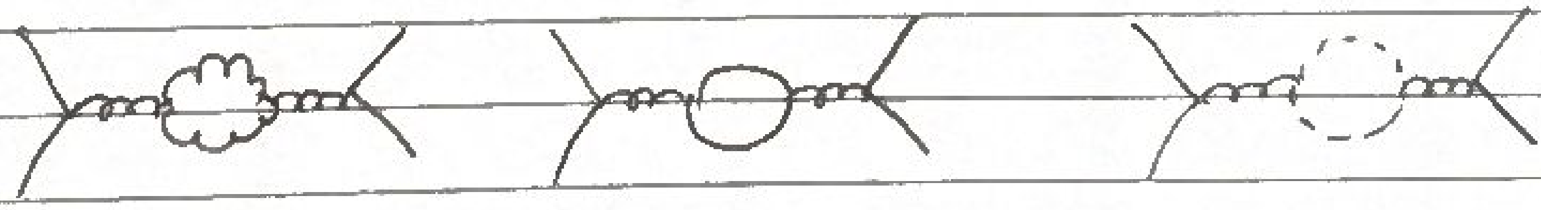
- box topology



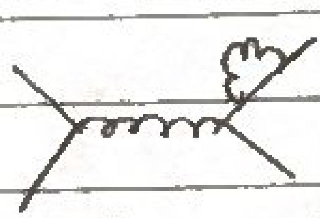
- triangle topology



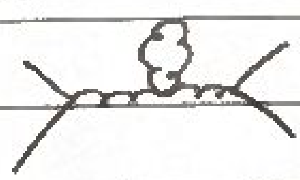
- bubble topology



Note: there are some scaleless contributions, for example:



or



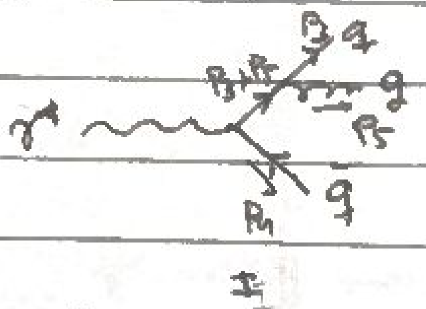
which are vanishing in dimensional regularisation

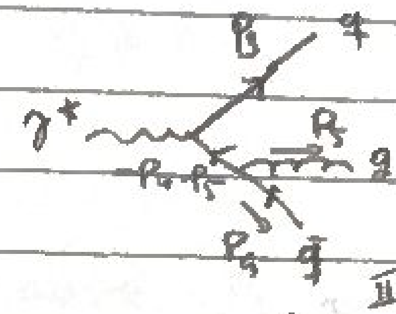
(2) A Soft gluon emission in $\gamma^* \to q\bar{q}$

we consider the following process

$$\gamma^* \to q(p_3) + \bar{q}(p_4) + g(p_5)$$

where $p_5 \to 0$





$$\mathcal{M}^{\mu}_{\gamma^*\to q\bar{q}g} = \bar{u}_3 \, i g_s t^a_{ij} \not{\epsilon}^*(p_5) \, i \frac{\not{p}_3+\not{p}_5}{(p_3+p_5)^2} e i Q_q \gamma^\mu v_4$$

$$+ \bar{u}_3 \, i e Q_q \gamma^\mu \, i \frac{-\not{p}_4-\not{p}_5}{(p_4+p_5)^2} \, i g_s t^a_{ij} \not{\epsilon}^*(p_5) v_4$$

$$= -i e g_s Q_q t^a_{ij} \epsilon^*_\nu(p_5)$$

$$\times \bar{u}_3 \left\{ \gamma^\nu \frac{\not{p}_3+\not{p}_5}{(p_3+p_5)^2}\gamma^\mu - \gamma^\mu \frac{\not{p}_4+\not{p}_5}{(p_4+p_5)^2}\gamma^\nu \right\} v_4$$

$p_5 \to 0$

$$= -i e g_s Q_q t^a_{ij} \epsilon^*_\nu(p_5)$$

$$\cdot \bar{u}_3 \left\{ \gamma^\nu \frac{\not{p}_3}{2 p_3\cdot p_5}\gamma^\mu - \gamma^\mu \frac{\not{p}_4}{2 p_4\cdot p_5}\gamma^\nu \right\} v_4$$

$$\gamma^\nu \not{p}_3 = 2p_3^\nu - \not{p}_3\gamma^\nu \qquad \bar{u}_3\not{p}_3 = 0$$

$$\not{p}_4\gamma^\nu = 2p_4^\nu - \gamma^\nu \not{p}_4 \qquad \not{p}_4 v_4 = 0$$

} Dirac equation

$$= -i e g_s Q_q \, \epsilon^*_\nu(p_5) \cdot \bar{u}_3 \gamma^\mu v_4 \underbrace{\left\{ \frac{p_3^\nu}{p_3\cdot p_5} - \frac{p_4^\nu}{p_4\cdot p_5} \right\} t^a_{ij}}_{\text{soft current } J^{a\nu}}$$

$$= i e Q_q \epsilon^*_\nu(p_5) \bar{u}_3 \gamma^\mu v_4 \cdot (-1) g_s J^{a\nu}$$

$$= -\mathcal{M}_0^\mu \, g_s \epsilon^*_\nu(p_5) J^{a\nu}$$

$$\sum_{\substack{\text{color}\\\text{spin}}} |M_{\gamma^*\to q\bar{q}g}|^2 \overset{p_5\to 0}{=} \sum |M_0|^2\, g_s^2 (-g_{\nu\sigma})\, \overbrace{t^a_{ij} t^a_{ji}}^{N_c C_F} \quad , C_F = \frac{N_c^2 - 1}{2N_c}$$

$$\times \left\{ \frac{p_3^\nu}{p_3\cdot p_5} - \frac{p_4^\nu}{p_4\cdot p_5} \right\} \left\{ \frac{p_3^\sigma}{p_3\cdot p_5} - \frac{p_4^\sigma}{p_4\cdot p_5} \right\}$$

$$= \sum |M_{\gamma^*\to q\bar{q}}|^2\, \frac{2 p_3\cdot p_4}{p_3\cdot p_5\, p_4\cdot p_5}\, C_F\, g_s^2$$

(b) $\gamma^* \to q\bar{q}g$ matrix element where p_3 is collinear to p_5 ($p_3 \parallel p_5$)

From the two diagrams contributing to $\gamma^* \to q\bar{q}g$ only the first diagram contributes due to only $\frac{1}{(p_3+p_5)^2} = \frac{1}{2p_3\cdot p_5}$ propa[gator] blows up in the $p_3 \parallel p_5$ collinear limit

$$M_1^\mu = -ie\, g_s\, Q_q\, t^a_{ij}\, \epsilon^*_\nu(p_5)\, \bar{u}_3 \gamma^\nu \frac{\not{p}_3 + \not{p}_5}{2p_3\cdot p_5} \gamma^\mu v_4$$

Sudakov parametrisation

$$p_3^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2p\cdot n}$$

$$p_5^\mu = (1-z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z} \frac{n^\mu}{2p\cdot n}$$

$p^2 = 0$

$n^2 = 0$

$p\cdot k_\perp = 0$

$n \cdot k_\perp = 0$

$$\underline{p}^\mu = p_3^\mu + p_5^\mu = p^\mu - \frac{k_\perp^2}{2p\cdot n}\left(\frac{1}{z} + \frac{1}{1-z}\right) n$$

$$p_3\cdot p_5 = -\frac{k_\perp^2}{2z(1-z)}$$

collinear limit $\underline{p}^\mu = p^\mu$

$\not{p} = \sum_{\text{spin}} u(p)\bar{u}(p)$

$$\sum |M_1|^2 = e^2 g_s^2 Q_q^2 C_F \underbrace{(-g_{\mu\rho})}_{\gamma^*\text{ polarisation sum}} \underbrace{\left(-g_{\nu\sigma} + \frac{p_5^\nu n^\sigma + p_5^\sigma n^\nu}{p_5\cdot n}\right)}_{\text{gluon pol sum}}$$

$$\times\, \mathrm{tr}\left[\not{p}_3 \gamma^\nu (\not{p}_3 + \not{p}_5) \gamma^\mu \not{p}_4 \gamma^\rho (\not{p}_3 + \not{p}_5) \gamma^\sigma \right] \frac{1}{4(p_3\cdot p_5)^2}$$

(from calculation done in Mathematica)

$$\sum |M_3|^2 = e^2 g_s^2 Q_q^2 C_F \frac{d-2}{4(p_3\cdot p_5)^2}\Bigg[8(d-2)\, p_3\cdot p_4\, p_4\cdot p_5$$

$$+ \frac{m\, p_3\cdot p_5}{p_5\cdot n}\Big(-p_4\cdot n\, p_3\cdot p_5 + p_3\cdot n\, p_4\cdot p_5$$

$$+ p_3\cdot p_4 \left(2 p_3\cdot n + p_5\cdot n \right)\Big)\Bigg]$$

using the Sudakov parametrisation, the dot products are

$$☆ \begin{cases} p_3\cdot p_5 = -\dfrac{k_\perp^2}{2z(1-z)} \\ p_5\cdot n = (1-z)\, p\cdot n \\ p_3\cdot n = z\, p\cdot n \end{cases}$$

$$p_3\cdot p_4 = p_4\cdot k_\perp + z\, p\cdot p_4 - \frac{p_4\cdot n\, k_\perp^2}{2z\, p\cdot n}$$

$$p_4\cdot p_5 = \left(-p_4\cdot k_\perp + (1-z)\, p\cdot p_4 - \frac{p_4\cdot n\, k_\perp^2}{2(1-z)\, p\cdot n} \right)$$

$\boxed{\text{Born term}}$ $\quad \gamma^* \to q(p) + \bar{q}(p_4)$ (reduced tree level)

$$\sum |M_{\gamma^*\to q\bar{q}}|^2 = e^2 Q_q^2\, \mathrm{tr}\left(\not{p}\, \gamma^\mu \not{p}_4 \gamma^\nu \right) (-g_{\mu\nu})$$

$$= 4 e^2 Q_q^2 N_c (d-2)\, p\cdot p_4$$

Using ☆ and keep term up to $k_\perp^2$

$$\sum |M_{\gamma^*\to q\bar{q}g}|^2 \overset{p_3 \parallel p_5}{=} \sum |M_{\gamma^*\to q\bar{q}}|^2 \frac{2}{s_{35}}\, g_s^2$$

$$\times \underbrace{C_F\left[\frac{1+z^2}{1-z} - \epsilon (1-z) \right]}_{\text{Altarelli Parisi splitting function for } q\to qg}$$

$$= g_s^2 \frac{2}{s_{35}} P_{qq}(z) \sum |M_{\gamma^*\to q\bar{q}}|^2$$

[3] (a) Basis of master integrals

$$g_1 = \epsilon I_{20}$$
$$g_2 = \epsilon \lambda I_{21} \quad \text{with} \quad \lambda = \sqrt{s(s-4m^2)}$$

$$I_{n_1 n_2} = e^{\epsilon \gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{[k^2-m^2]^{\nu_1}[(k-p)^2-m^2]^{\nu_2}}$$

$$p^\mu \frac{\partial}{\partial p^\mu} = 2s \frac{\partial}{\partial s}$$

$D_1 = k^2 - m^2$
$D_2 = (k-p)^2 - m^2$

$$p^\mu \frac{\partial}{\partial p^\mu} I_{\nu_1 \nu_2} = e^{\epsilon \gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \underbrace{p^\mu \frac{\partial}{\partial p^\mu} \frac{1}{D_1^{\nu_1} D_2^{\nu_2}}}_{-\nu_2 \frac{1}{D_1^{\nu_1} D_2^{\nu_2}} \cdot \underbrace{\frac{\partial D_2}{\partial p^\mu}}_{2(p-k)_\mu}}$$

$$= e^{\epsilon \gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \frac{(-\nu_2)\, 2\,(p^2 - p\cdot k)}{D_1^{\nu_1} D_2^{\nu_2+1}}$$

$$2p\cdot k = D_1 - D_2 + s \quad \Rightarrow \quad p^2 = s$$

$$p^2 - p\cdot k = \frac{1}{2}(s - D_1 + D_2)$$

$$= -\nu_2\, e^{\epsilon \gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \frac{(s - D_1 + D_2)\, 2}{D_1^{\nu_1} D_2^{\nu_2+1}} \frac{1}{2}$$

$$= -\nu_2 \left[s I_{\nu_1, \nu_2+1} - I_{\nu_1-1, \nu_2+1} + I_{\nu_1 \nu_2} \right]$$

$$\frac{\partial I_{\nu_1 \nu_2}}{\partial s} = \frac{1}{2s} p^\mu \frac{\partial}{\partial p^\mu} I_{\nu_1 \nu_2}$$

$$\frac{\partial I_{\nu_1 \nu_2}}{\partial s} = -\frac{\nu_2}{2s} \left[s I_{\nu_1, \nu_2+1} - I_{\nu_1-1, \nu_2+1} + I_{\nu_1 \nu_2} \right]$$

$$\frac{\partial I_{20}}{\partial s} = 0$$

$$\frac{\partial I_{21}}{\partial s} = -\frac{1}{2s}\left[s I_{22} - I_{12} + I_{21} \right]$$

$$\Rightarrow \frac{\partial g_1}{\partial s} = 0$$

$$\frac{\partial g_2}{\partial s} = \epsilon \lambda \frac{\partial I_{21}}{\partial s} + \epsilon I_{21} \frac{\partial \lambda}{\partial s}$$

$$\lambda = \left(s(s-4m^2) \right)^{1/2} = \left(s^2 - 4sm^2 \right)^{1/2}$$

$$\frac{d\lambda}{ds} = \frac{1}{2}\left(s^2-4sm^2\right)^{-1/2} \cdot \left(2s - 4m^2\right) = \frac{2s-4m^2}{2\lambda}$$

$$\frac{\partial g_2}{\partial s} = \epsilon \lambda \left(-\frac{1}{2s}\right)\left[s I_{22} - I_{12} + I_{21} \right] + \epsilon \frac{2s-4m^2}{2\lambda} I_{21}$$

Using IBP reduction rules provided in the problem set and perform reduction to I_{10} and I_{11}

$$\frac{\partial g_2}{\partial s} = \frac{(d-4)(d-2)\epsilon I_{01} s}{[s(s-4m^2)]^{3/2}} + \frac{(d-4)(d-3)\epsilon I_{11} \sqrt{s(s-4m^2)}}{2(s-4m^2)^2}$$

Using 2^{nd} & 4^{th} IBP reduction rules, we can write I_{11} and I_{01} in terms of I_{21} and I_{20}, which can further be written in terms of g_1 and g_2

$$I_{01} = \frac{m^2}{(d-2)\epsilon} g_1, \quad I_{11} = \frac{g_1}{(d-3)\epsilon} - \frac{s-4m^2}{\epsilon(d-3)\sqrt{s(s-4m^2)}} g_2$$

This gives us

$$\frac{\partial g_2}{\partial s} = \frac{4-d}{2\sqrt{s(s-4m^2)}}\, g_1 - \frac{4-d}{2(s-4m^2)}\, g_2$$

$$\frac{d}{ds} = \frac{dy}{ds}\frac{d}{dy} \qquad \frac{ds}{dy} = m^2\left(-1+\frac{1}{y^2}\right)$$

$$s = -m^2\frac{(1-y)^2}{y}$$

$$\frac{\partial g_2}{\partial s} = \frac{\partial y}{\partial s}\frac{\partial g_2}{\partial y} \rightarrow \frac{\partial g_2}{\partial y} = m^2\left(-1+\frac{1}{y^2}\right)\frac{\partial g_2}{\partial s}$$

$$= \epsilon\left[\frac{1}{y}\, g_1 + \left(\frac{1}{y} - \frac{2}{1+y}\right) g_2\right]$$

→ the change of variable from $(s, m^2) \rightarrow y$ rationalises the square-root λ !!

(b) Let's now solve the DE for g_2

First, write the solution of g_1 as

$$g_1 = (m^2)^{-\epsilon}\left[g_1^{(0)} + \epsilon g_1^{(1)} + \epsilon^2 g_1^{(2)} + \mathcal{O}(\epsilon^3)\right] \qquad (*)$$

where $g_1^{(0)} = 1$, $g_1^{(1)} = 0$, $g_1^{(2)} = \frac{\pi^2}{12}$

for

$$g_2 = (m^2)^{-\epsilon}\left[g_2^{(0)} + \epsilon g_2^{(1)} + \epsilon^2 g_2^{(2)} + \mathcal{O}(\epsilon^3)\right] \qquad (**)$$

Substitute $(*)$ and $(**)$ to DE for g_2, i.e. $\frac{\partial g_2}{\partial y}$

and work order by order in ϵ

$$\frac{\partial}{\partial y}\left(g_2^{(0)} + \epsilon g_2^{(1)} + \epsilon^2 g_2^{(2)} + \dots\right)$$

$$= \epsilon\left[\frac{1}{y}\left(g_1^{(0)} + \overset{=0}{\epsilon g_1^{(1)}} + \epsilon^2 g_1^{(2)} + \dots\right) + \left(\frac{1}{y} - \frac{2}{1+y}\right)\left(g_2^{(0)} + \epsilon g_2^{(1)} + \epsilon^2 g_1^{(2)} + \dots\right)\right]$$

at ϵ^0

$$\frac{\partial g_2^{(0)}}{\partial y} = 0 \rightsquigarrow g_2^{(0)} = \text{const.}$$

Boundary condition $g_2 = 0$ at $y = 1$

thus $g_2^{(0)} = 0$

ϵ^1 terms

$$\frac{\partial g_2^{(1)}}{\partial y} = \frac{1}{y}\overset{=1}{g_1^{(0)}} + \left(\frac{1}{y} - \frac{2}{1+y}\right)\overset{=0}{g_2^{(0)}}$$

$$\frac{\partial g_2^{(1)}}{\partial y} = \frac{1}{y} \rightsquigarrow g_2^{(1)} = \int \frac{dy}{y}$$

$$= \ln(y) + \text{const}$$

or in terms of Goncharov polylogarithm

$$g_2^{(1)} = G(0;y) + \text{const}$$

at $y=1$, $g_2^{(1)}=0$ so

$$g_2^{(1)} = G(0;y)$$

ϵ^2 term

$$\frac{\partial g_2^{(2)}}{\partial y} = \left(\frac{1}{y} - \frac{2}{1+y}\right) g_2^{(1)} \quad + G(0;y)$$

$$g_2^{(2)} = \int \frac{dy}{y} G(0;y) - 2\int \frac{dy}{1+y} G(0;y) + \text{const}$$

from the definition of Goncharov polylogs,

$$\int \frac{dy}{y-z_1} G(z_2,\ldots,z_n;y) = G(z_1,z_2,\ldots,z_n;y)$$

$$g_2^{(2)} = G(0,0;y) - 2G(-1,0,y) + \text{const}$$

at $y=1$

$$g_2^{(2)} = G(0,0;1) - 2G(-1,0;1) + \text{const}$$

$$\overset{\shortparallel}{0} \qquad \overset{\shortparallel}{0} \qquad \overset{\shortparallel}{-\frac{\pi^2}{12}}$$

$$\text{const} = -\frac{\pi^2}{6}$$

so, collecting everything

$$g_2 = (m^2)^{-\epsilon}\left\{ \epsilon\, G(0;y) + \epsilon^2\left[G(0,0;y) - 2G(-1,0;y) - \frac{\pi^2}{6}, O(\epsilon^3)\right]\right\}$$