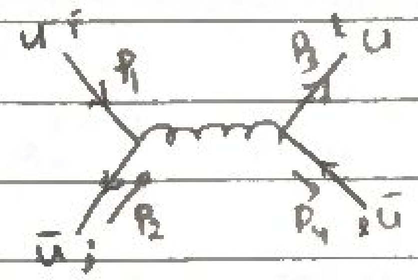
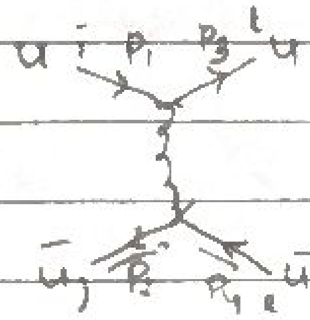
BONUS SOLUTION ($u\bar u \to gg$ starts on page 5)

$u(p_1) + \bar u(p_2) \to u(p_3) + \bar u(p_4)$



I



II

$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2 = (p_2-p_4)^2$$
$$u = (p_1-p_4)^2 = (p_2-p_3)^2$$

$$M_I = \bar v(p_2)\, i g_s \gamma^\mu t^a_{ji}\, u(p_1) \times \frac{-i g_{\mu\nu}}{(p_1+p_2)^2} \times \bar u(p_3)\, i g_s \gamma^\nu t^a_{k\ell}\, v(p_4)$$

$$= t^a_{ji} t^a_{k\ell}\; i g_s^2 \frac{1}{(p_1+p_2)^2}\, \bar v_2 \gamma^\mu u_1\, \bar u_3 \gamma_\mu v_4$$

$$= \frac{i g_s^2}{2}\left(\delta_{j\ell}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{k\ell}\right)\frac{1}{s}\, \bar v_2 \gamma^\mu u_1\, \bar u_3 \gamma_\mu v_4$$

$$= \frac{i g_s^2}{2}\left[\delta_{j\ell}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{k\ell}\right] A_I$$

$$A_I = \frac{1}{s}\, \bar v_2 \gamma^\mu u_1\, \bar u_3 \gamma_\mu v_4$$

rearrangement of fermion line

$$M_{II} = -\,\bar v(p_2)\, i g_s \gamma^\mu t^a_{j\ell}\, v(p_4) \times \frac{-i g_{\mu\nu}}{(p_1-p_3)^2} \times \bar u(p_3)\, i g_s \gamma^\nu t^a_{ki}\, u(p_1)$$

$$= i\frac{g_s^2}{2}\left[\delta_{ji}\delta_{\ell k} - \frac{1}{N_c}\delta_{j\ell}\delta_{ki}\right] A_{II}$$

$$A_{II} = -\frac{1}{t}\, \bar v_2 \gamma^\mu v_4\, \bar u_3 \gamma_\mu u_1$$

Helicity configurations

$\big(u(p_1,h_1) + \bar u(p_2,h_2) \to u(p_3,h_3) + \bar u(p_4,h_4)\big) \Rightarrow A_i^{h_1 h_2 h_3 h_4}$

Diagram I : $\boxed{A^{+-+-}}$, A^{+--+}, A^{-++-}, A^{-+-+}

Diagram II : $\boxed{A^{+-+-}}$, A^{++++}, A^{-+-+}, A^{----}

$$A_I^{+-+-} = \frac{1}{s}\, \bar{v}_{2-}\gamma^\mu u_{1+} \quad \bar{u}_{3+}\gamma_\mu v_{4-}$$

$$= \frac{1}{s}\, [2\gamma^\mu 1\rangle \quad [3\gamma_\mu 4\rangle = 2\,\frac{\langle 14\rangle[32]}{s}$$

$$A_I^{+--+} = A_I^{+-+-}\Big|_{p_3\leftrightarrow p_4} = 2\,\frac{\langle 13\rangle[42]}{s}$$

$$A_I^{-+-+} = A_I^{+-+-}\Big|_{\langle ij\rangle\leftrightarrow[ji]} = 2\,\frac{[41]\langle 23\rangle}{s}$$

$$A_I^{-++-} = A_I^{+--+}\Big|_{\langle ij\rangle\leftrightarrow[ji]} = 2\,\frac{[31]\langle 24\rangle}{s}$$

$$A_{II}^{+-+-} = -\frac{1}{t}\, \bar{v}_{2-}\gamma^\mu v_{4-} \quad \bar{u}_{3+}\gamma_\mu u_{1+}$$

$$= -\frac{1}{t}\, [2\gamma^\mu 4\rangle \quad [3\gamma_\mu 1\rangle = -2\,\frac{\langle 14\rangle[23]}{t}$$

$$A_{II}^{++++} = -\frac{1}{t}\, \langle 2\gamma^\mu 4] \, [3\gamma_\mu 1\rangle = -2\,\frac{\langle 12\rangle[43]}{t} = -A_{II}^{+-+-}\Big|_{p_2\leftrightarrow -p_4}$$

$$A_{II}^{-+-+} = A_{II}^{+-+-}\Big|_{\langle\rangle\leftrightarrow[]} = -2\,\frac{[41]\langle 32\rangle}{t}$$

$$A_{II}^{----} = A_{II}^{++++}\Big|_{\langle\rangle\leftrightarrow[]} = -2\,\frac{[21]\langle 34\rangle}{t}$$

Squared amplitude

$$M = \frac{ig^2}{2}\left\{ \underbrace{\left(\delta_{jl}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{kl}\right)}_{C_I} A_I + \underbrace{\left(\delta_{ji}\delta_{kl} - \frac{1}{N_c}\delta_{jl}\delta_{ki}\right)}_{C_{II}} A_{II} \right\}$$

$$|C_I|^2 = \left(\delta_{jl}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{kl}\right)\left(\delta_{lj}\delta_{ik} - \frac{1}{N_c}\delta_{ij}\delta_{lk}\right)$$

$$= N_c^2 - \frac{1}{N_c}N_c - \frac{1}{N_c}N_c + \frac{1}{N_c^2}N_c^2 = N_c^2 - 1$$

$$|C_{II}|^2 = |C_I|^2 = N_c^2 - 1$$

$$C_I C_{II}^* = \left(\delta_{il}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{kl}\right)\left(\delta_{ij}\delta_{kl} - \frac{1}{N_c}\delta_{kj}\delta_{il}\right)$$

$$= \cancel{N_c} - \cancel{\frac{1}{N_c} N_c^2} - \frac{1}{N_c} N_c^2 + \frac{1}{N_c^2} N_c$$

$$= -N_c + \frac{1}{N_c}$$

$$\sum_{col} |M|^2 = \frac{g_s^4}{4}\left\{ (N_c^2 - 1)\left[|A_I|^2 + |A_{II}|^2\right] + \left(-N_c + \frac{1}{N_c}\right)\left[A_I^* A_{II} + A_{II}^* A_I\right]\right\}$$

$$u = (P_1 - P_4)^2 = -2P_1 \cdot P_4$$

$$u = (P_2 - P_3) = -2P_2 \cdot P_3$$

$$|A_I^{+-+-}|^2 = |A_I^{-+-+}|^2 = 4\,\frac{2P_1\cdot P_4\; 2P_2\cdot P_3}{s^2}$$

$$= 4\,\frac{u^2}{s^2}$$

$$|A_I^{+--+}|^2 = |A_I^{-++-}|^2 = 4\,\frac{2P_1\cdot P_3\; 2P_2\cdot P_4}{s^2} = 4\,\frac{t^2}{s^2}$$

$$|A_{II}^{+-+-}|^2 = 4\,\frac{u^2}{t^2} = |A_{II}^{-+-+}|^2$$

$$|A_{II}^{++++}|^2 = |A_{II}^{----}|^2 = 4\,\frac{s^2}{t^2}$$

$$\left(A_I^{+-+-}\right)^* A_{II}^{+-+-} = +4\,\frac{2P_1\cdot P_4\; 2P_2\cdot P_3}{st} = 4\,\frac{u^2}{st}$$

$$\left(A_I^{-+-+}\right)^* A_{II}^{-+-+} = 4\,\frac{u^2}{st}$$

$$\left(A_I^{+--+}\right)^* A_{II}^{+--+} = 0$$

$$\left(A_I^{-++-}\right)^* A_{II}^{-++-} = 0$$

$$\left(A_{II}^{\vec{h}}\right)^* A_I^{\vec{h}} = \left(A_I^{\vec{h}}\right)^* A_{II}^{\vec{h}}$$

$$\sum_{col} |M^{+-+-}|^2 = \frac{g_s^4}{4} \left\{ (N_c^2-1) \left(4\frac{u^2}{s^2} + 4\frac{u^2}{t^2} \right) \right.$$

$$\left. \left(-N_c + \frac{1}{N_c}\right)\left(8\frac{u^2}{st}\right) \right\}$$

$$\sum_{col} |M^{-+-+}|^2 = \sum_{col} |M^{+-+-}|^2$$

$$\sum_{col} |M^{+--+}|^2 = \frac{g_s^4}{4} \left\{ (N_c^2-1)\, 4\frac{t^2}{s^2} \right\}$$

$$= \sum_{col} |M^{-++-}|^2$$

$$\sum_{col} |M^{++++}|^2 = \frac{g_s^4}{4} \left\{ (N_c^2-1)\, 4\frac{s^2}{t^2} \right\} = \sum_{col} |M^{----}|^2$$

$$\sum_{spin}\sum_{col} |M|^2 = \frac{g_s^4}{4} \left\{ (N_c^2-1) \left[8\frac{u^2}{s^2} + 8\frac{u^2}{t^2} + 8\frac{t^2}{s^2} + 8\frac{s^2}{t^2} \right] \right.$$

$$\left. + \left(-N_c + \frac{1}{N_c}\right)\left[16\frac{u^2}{st}\right] \right\}$$

$$= 2g_s^4 \left\{ (N_c^2-1)\left[\frac{u^2+t^2}{s^2} + \frac{u^2+s^2}{t^2} \right] \right.$$

$$\left. + \left(-N_c + \frac{1}{N_c}\right) 2\frac{u^2}{st} \right\}$$

$N_c = 3$

$$= 2g_s^4 \left\{ 8\left[\frac{u^2+t^2}{s^2} + \frac{u^2+s^2}{t^2} \right] - \frac{16}{3}\frac{u^2}{st} \right\} \qquad \frac{16}{36} = \frac{4}{9}$$

$$= 16 g_s^4 \left\{ \frac{u^2+t^2}{s^2} + \frac{u^2+s^2}{t^2} - \frac{2}{3}\frac{u^2}{st} \right\}$$

$$\overline{\sum} = \frac{1}{2}\frac{1}{2}\frac{1}{3}\frac{1}{3}\sum = \frac{1}{36}\sum$$

$$\overline{\sum}|M|^2 = \frac{4}{9} g_s^4 \left\{ \frac{u^2+t^2}{s^2} + \frac{u^2+s^2}{t^2} - \frac{2}{3}\frac{u^2}{st} \right\} //$$

Comment

✓ A_{II} and A_I are actually related

$$A_{II}^{+-+-} = (-1)\underbrace{\Big[-A_I^{+-+-}\Big|_{p_2 \leftrightarrow -p_3}\Big]}_{}$$

rearranging fermion line

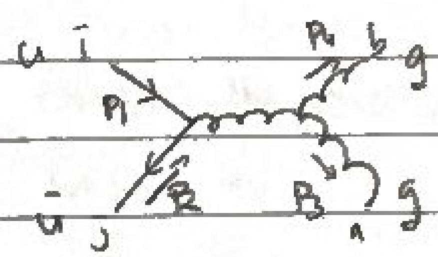
$$s = (p_1+p_2)^2 \to (p_1-p_3)^2 = t$$

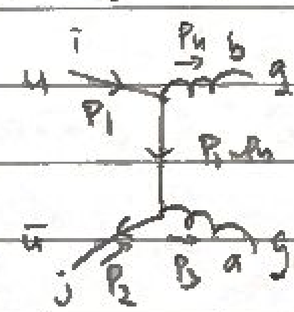
$$= +2\,\frac{\langle 14\rangle[32]}{s}\Big|_{p_2\leftrightarrow -p_3} = -2\,\frac{\langle 14\rangle[23]}{t}$$

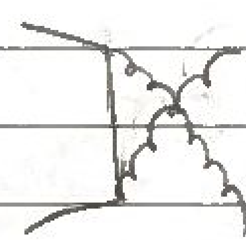
$$\Big(\,|-p\rangle = i|p\rangle \quad |-p] = i|p]\,\Big)$$

Sometimes it is more convenient to work using "all outgoing" or "all incoming" momenta (i.e. $0 \to u(p_1) + \bar{u}(p_2) + u(p_3) + \bar{u}(p_4)$) to avoid analytic continuation of spinor products, when crossing initial and final state momenta

$\boxed{4b}$ $u(p_1) + \bar{u}(p_2) \to g(p_3) + g(p_4)$







A_s A_t

we have derived the color decomposition $u\bar{u} \to gg$

$$M = t^a_{jk}\, t^b_{ki}\, A_{ab} + t^b_{jk}\, t^a_{ki}\, A_{ba}$$

Helicity configurations : $u(p_1,h_1) + \bar{u}(p_2,h_2) \to g(p_3,h_3) + g(p_4,h_4)$

$\Rightarrow A^{h_1h_2h_3h_4}$

$A^{+-++} = 0$ (at tree level)

$\boxed{A^{+-+-} \quad A^{+--+}}$ → independent helicity configuration

$A^{+---} = 0$ (at tree level)

$A^{++++}, A^{++++} = 0$

A^{-+-+}, A^{-++-}

From lecture notes $\qquad s_{14} = (p_1 - p_4)^2$

$$\tilde{A}_{ab} = A_{s,1} + A_t$$

$$A_{s,1} = -i\, g_s^2 \frac{1}{s_{12}} \epsilon^*_\mu(p_3)\, \epsilon^*_\nu(p_4)\; \bar{v}_2 \gamma_\rho u_1$$

$$\times \left[g^{\mu\nu}(-p_3+p_4)^\rho + g^{\nu\rho}(-p_3-2p_4)^\mu + g^{\rho\mu}(2p_3+p_4)^\nu \right]$$

$$A_{s,1}^{+-h_3h_4} = -i\, g_s^2 \frac{1}{s_{12}} \left[2\, \gamma_\rho\, 1\right\rangle \times \Big\{ \epsilon^*_{h_3}(p_3)\cdot\epsilon^*_{h_4}(p_4)\,(-\not{p}_3+\not{p}_4)^\rho$$

$$+ \epsilon^{\rho *}_{h_4}(p_4)\,(-p_3-2p_4)\cdot\epsilon^*_{h_3}(p_3)$$

$$+ \epsilon^{\rho *}_{h_3}(p_3)\,(2p_3+p_4)\cdot\epsilon^*_{h_4}(p_4) \Big\}$$

$$A_t = -i g_s^2\, \epsilon^*_\mu(p_3)\,\epsilon^*_\nu(p_4)\; \bar{v}_2 \gamma^\mu \frac{(\not{p}_1-\not{p}_4)}{(p_1-p_4)^2} \gamma^\nu u(p_1)$$

$$A_t^{+-h_3h_4} = -i g_s^2\, \epsilon^*_{\mu h_3}(p_3)\,\epsilon^*_{\nu h_4}(p_4) \left[2\, \gamma^\mu (\not{p}_1-\not{p}_4)\gamma^\nu 1\right\rangle \frac{1}{s_{14}}$$

choose the reference vectors : $r_3 = p_4$, $r_4 = p_3$ $\rightarrow$ last 2 terms in $A_{s,1}$ vanish due to their choice

we have :

$$r_i \cdot \epsilon^*_\pm(p_i, r_i) = 0$$

$$p_i \cdot \epsilon^*_\pm(p_i, r_i) = 0$$

$$\epsilon_+^{*\mu}(p_i, r_i) = \frac{1}{\sqrt{2}} \frac{\langle r_i \gamma^\mu p_i]}{\langle r_i p_i\rangle}, \quad \epsilon_-^{*\mu}(p_i, r_i) = -\frac{1}{\sqrt{2}} \frac{[r_i \gamma^\mu p_i\rangle}{[r_i p_i]}$$

$$\not{\epsilon}^*_+(p_i, r_i) = \frac{\sqrt{2}}{\langle r_i p_i\rangle}\Big(p_i]\langle r_i + r_i\rangle[p_i \Big), \quad \not{\epsilon}^*_-(p_i, r_i) = -\frac{\sqrt{2}}{[r_i p_i]}\Big(p_i\rangle[r_i + r_i]\langle p_i \Big)$$

$$\epsilon^*_+(p_i, r_i)\cdot\epsilon^*_+(p_j, r_j) = \frac{\langle r_i r_j\rangle[p_i p_j]}{\langle r_i p_i\rangle\langle r_j p_j\rangle} \; ; \quad \epsilon^*_+(p_i, r_i)\cdot\epsilon^*_-(p_j, r_j) = -\frac{\langle r_i p_j\rangle[r_j p_i]}{\langle r_i p_i\rangle[r_j p_j]}$$

$$\epsilon^*_-(p_i, r_i)\cdot\epsilon^*_+(p_j, r_j) = -\frac{[r_i p_j]\langle r_j p_i\rangle}{[r_i p_i]\langle r_j p_j\rangle}$$

$\boxed{+-++}$ $p_4 = -p_1 - p_2 - p_3$

$$A_{s,1}^{+-++} = -i g_s^2 \frac{1}{s_{12}} [2(-\cancel{p_3}+\cancel{p_4})1\rangle \cdot \frac{\cancel{\langle 43\rangle}[34]}{\cancel{\langle 43\rangle}\langle 34\rangle}$$

$$= -i g_s^2 \frac{1}{s_{12}} [2(\cancel{-p_1}\cancel{-p_2}-2p_3)1\rangle \cdot \frac{[34]}{\langle 34\rangle}$$

$s_{12} = s_{34}$

$\langle 12\rangle[21] = \langle 34\rangle[43]$

$$= 2i g_s^2 \frac{1}{\underset{\langle 34\rangle[43]}{\langle 12\rangle[21]}} [23]\langle 31\rangle \frac{[34]}{\langle 34\rangle} = -2i g_s^2 \frac{[23]\langle 31\rangle}{\langle 34\rangle^2}$$

$$A_t^{+-++} = -i \underset{(-\langle 23\rangle[32])}{g_s^2} [2\, \epsilon_+^*(p_3,p_4)(\cancel{p_1}-\cancel{p_4})\epsilon_+^*(p_4,p_3)\, 1\rangle$$

$s_{14} = (p_1-p_4)^2 = -2p_1 p_4$

$= -2p_2 p_3 =$

$= -\langle 23\rangle[32]$

$$= \frac{i g_s^2}{\langle 23\rangle[32]} \frac{2}{\langle 34\rangle\langle 43\rangle} [2(3]\langle 4)(\cancel{p_1}-\cancel{p_4})(4]\langle 31\rangle$$

$\langle 14\rangle[41] = \langle 23\rangle[32]$

$$= -2i g_s^2 \frac{1}{\langle 23\rangle[23]\langle 34\rangle^2} [23]\langle 41\rangle[14]\langle 31\rangle$$

$$= -2i g_s^2 \frac{\cancel{[23]}\cancel{\langle 23\rangle}[32]\langle 31\rangle}{\cancel{\langle 23\rangle}\cancel{[23]}\langle 34\rangle^2}$$

$$= -2i g_s^2 \frac{[32]\langle 31\rangle}{\langle 34\rangle^2} = +2i g_s \frac{[23]\langle 31\rangle}{\langle 34\rangle^2}$$

$$A_{ab}^{+-++} = A_{s,1}^{+-++} + A_t^{+-++} = 0$$

$\boxed{+-+-}$

$$A_{s,1}^{+-+-} = -i g_s^2 \frac{1}{s_{12}} [2(-2p_3)1\rangle\, (-1) \frac{\langle 44\rangle[33]}{\langle 43\rangle[34]} = 0$$

$$A_t^{+-+-} = i \frac{g_s^2\; 2}{\langle 23\rangle[32]} [2(3]\langle 4)(\cancel{p_1}-\cancel{p_4}) \frac{(-1)}{\langle 43\rangle[34]} (3]\langle 4)1\rangle$$

$$= 2i \frac{g_s^2}{\langle 23\rangle[\cancel{23}]} [\cancel{23}]\langle 41\rangle[13]\langle 41\rangle \frac{1}{\langle 34\rangle[43]} \times \frac{\langle 21\rangle}{\langle 21\rangle}$$

$$\langle 21\rangle[13] = \langle 2\, p_1\, 3] = \langle 2(-\cancel{p_2}+\cancel{p_3}+p_4)3] = \langle 24\rangle[43]$$

$$A_t^{+-+-} = 2i \, \frac{g_s^2 \langle 41\rangle^2 \langle 24\rangle \cancel{[43]}}{\langle 23\rangle\langle 34\rangle \cancel{[43]} \langle 21\rangle}$$

$$= +2i \, g_s^2 \, \frac{\langle 14\rangle^3 \langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$\boxed{+--+}$

$$A_{s,1}^{+--+} = 0 \quad \text{(same reasoning as } +-+-\text{)}$$

$$A_t^{+--+} = \frac{i g_s^2}{\langle 23\rangle[32]} \, \frac{(-2)}{[43]\langle 34\rangle} \, [2\big(4]\langle 3\big)(\not{p}_1 - \not{p}_4)\big(4]\langle 3\big)1\rangle$$

$$= -2i \, g_s^2 \, \frac{[24]\langle 31\rangle[14]\langle 31\rangle}{\langle 23\rangle[32][43]\langle 34\rangle} \, \frac{\langle 12\rangle\langle 41\rangle}{\langle 12\rangle\langle 41\rangle}$$

$$[24]\langle 41\rangle = +[2\not{p}_4 1\rangle = -[23]\langle 31\rangle$$

$$\langle 12\rangle[14] = -\langle 21\rangle[14] = -\langle 23\rangle[34]$$

$$= -2i \, g_s^2 \, \frac{\langle 31\rangle^3 \cancel{[23]} \langle 23\rangle \cancel{[34]}}{\langle 23\rangle \cancel{[32]} \cancel{[43]} \langle 34\rangle\langle 12\rangle\langle 41\rangle}$$

$$= 2i \, g_s^2 \, \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Recap

$$A_{ab}^{+-+-} = 2i g_s^2 \, \frac{\langle 14\rangle^3 \langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A_{ab}^{+--+} = 2i g_s^2 \, \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$\langle ij\rangle \leftrightarrow [ji]$

$$A_{ab}^{-+-+} = 2i g_s^2 \, \frac{[41]^3 [42]}{[21][32][43][14]}$$

$\langle ij\rangle \leftrightarrow [ji]$

$$A_{ab}^{-++-} = 2i g_s^2 \, \frac{[31]^3 [32]}{[21][32][43][14]}$$

$$A_{ba}^{+-+-} = A_{ab}^{+--+}\Big|_{p_3 \leftrightarrow p_4}$$

$$= 2ig_s^2 \frac{\langle 14\rangle^3 \langle 24\rangle}{\langle 12\rangle \langle 24\rangle \langle 43\rangle \langle 31\rangle}$$

$$A_{ba}^{+--+} = A_{ab}^{+-+-}\Big|_{p_3 \leftrightarrow p_4}$$

$$= 2ig_s^2 \frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle \langle 24\rangle \langle 43\rangle \langle 31\rangle}$$

$$A_{ba}^{-+-+} = 2ig_s^2 \frac{[41]^3 [42]}{[21][42][34][13]}$$

$$A_{ba}^{++--} = 2ig_s^2 \frac{[31]^3 [32]}{[21][42][34][13]}$$

Squaring the amplitude

$$M = t^a_{jk} t^b_{ki} A_{ab} + t^b_{jk} t^a_{ki} A_{ba}$$

$$\sum_{\text{color}} |M|^2 = \underbrace{t^a_{jk} t^b_{ki}\, t^a_{\ell j}\, t^b_{i\ell}}_{C_1} \left[|A_{ab}|^2 + |A_{ba}|^2 \right]$$

$$+ \underbrace{t^a_{jk} t^b_{ki}\, t^b_{\ell j}\, t^a_{i\ell}}_{C_2} \left[A_{ab}^* A_{ba} + A_{ab} A_{ba}^* \right]$$

$$C_1 = \frac{1}{2}\left(\delta_{jj}\delta_{\ell k} - \frac{1}{N_c}\delta_{jk}\delta_{\ell j} \right) \frac{1}{2}\left(\delta_{k\ell}\delta_{ii} - \frac{1}{N_c}\delta_{ki}\delta_{i\ell} \right)$$

$$= \frac{1}{4}\left(N_c - \frac{1}{N_c} \right)\delta_{\ell k} \left(N_c - \frac{1}{N_c} \right)\delta_{k\ell} = \frac{N_c}{4}\left(N_c - \frac{1}{N_c} \right)^2 = \frac{1}{4N_c}\left(N_c^2 - 1 \right)^2$$

$$C_2 = \frac{1}{4}\left(\delta_{j\ell}\delta_{ik} - \frac{1}{N_c}\delta_{jk}\delta_{i\ell} \right)\left(\delta_{kj}\delta_{\ell i} - \frac{1}{N_c}\delta_{ki}\delta_{\ell j} \right)$$

$$= \frac{1}{4}\left[\cancel{N_c} - \frac{1}{\cancel{N_c}} N_c^2 - \frac{1}{N_c} N_c^2 + \frac{1}{N_c^2} N_c \right] = -\frac{1}{4}\left(N_c - \frac{1}{N_c} \right) = -\frac{1}{4N_c}\left(N_c^2 - 1 \right)$$

$$\sum_{col} |A_{ab}^{+-+-}|^2 = 4g_s^4 \frac{(2P_1\cdot P_4)^3(2P_3\cdot P_4)}{(2P_1\cdot P_2)(2P_2\cdot P_3)(2P_3\cdot P_4)(2P_1\cdot P_4)}$$

$$= 4g_s^4 \frac{(-t)^3(u)}{s^2(-t)^2} = 4g_s^4 \frac{ut}{s^2} =$$

$s = (P_1+P_2)^2 = 2P_1\cdot P_2$
$= (P_3+P_4)^2 = 2P_3\cdot P_4$
$t = (P_1-P_4)^2 = -2P_1\cdot P_4$
$= (P_2-P_3)^2 = -2P_2\cdot P_3$
$u = (P_1-P_3)^2 = -2P_1\cdot P_3$
$= (P_2-P_4)^2 = -2P_2\cdot P_4$

$$\sum_{col} |A_{ab}^{+--+}|^2 = 4g_s^4 \frac{(-u)^3(-t)}{s^2(-t)^2} = \frac{u^3}{s^2 t} 4g_s^4 = \sum_{col} |A^{-++-}|^2$$

$$\sum_{col} |A_{ba}^{+-+-}|^2 = 4g_s^4 \frac{(-t)^3(-u)}{s^2(-u)^2} = 4g_s^4 \frac{t^3}{s^2 u} = \sum_{col} |A_{ba}^{-+-+}|^2$$

$$\sum_{col} |A_{ba}^{+--+}|^2 = \sum_{col} |A_{ba}^{-++-}|^2 = 4g_s^4 \frac{(-u)^3(-t)}{s^2(-u)^2} = 4g_s^2 \frac{ut}{s^2}$$

cross term

$$\left(A_{ab}^{+-+-}\right)^* A_{ba}^{+-+-} = 4g_s^4 \frac{[41]^3[42]}{[21][32][43][14]} \cdot \frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$

$$= 4g_s^4 \frac{(-t)^3(-u)}{s(-s)[3 \not{P}_2 \not{P}_4 \not{P}_1 3\rangle}$$

$$[3 \not{P}_2 \not{P}_4 \not{P}_1 3\rangle = [3 \not{P}_2 (+\not{P}_1 + \not{P}_2 - \not{P}_3)\not{P}_1 3\rangle$$

$$= -[3 \not{P}_2 \not{P}_3 \not{P}_1 3\rangle = -2P_2\cdot P_3 [3 \not{P}_1 3\rangle + \cancel{[3 \not{P}_3 \not{P}_2 \not{P}_1 3\rangle}^{=0}$$

$$= -2P_2\cdot P_3 \cdot 2P_1\cdot P_3$$

$$= -ut$$

$$= 4g_s^4 \frac{ut^3}{s^2 ut} = 4g_s^4 \frac{t^2}{s^2}$$

$$A_{ab}^{+-+-} \left(A_{ba}^{+-+-}\right)^* = 4g_s^4 \frac{t^2}{s^2} \quad \text{also for } -+-+$$

$$\left(A_{ab}^{+--+}\right)^* A_{ba}^{+--+} = 4g_s^4 \frac{[31]^3[32]}{[21][32][43][14]} \frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$

$$= 4g_s^4 \frac{(-u)^3(-t)}{s^2 ut} = 4g_s^4 \frac{u^2}{s^2}$$

$$A_{ab}^{+--+} \left(A_{ba}^{+--+}\right)^* = 4g_s^4 \frac{u^2}{s^2} \quad \text{also for } -++-$$

Putting everything together

$$\sum_{\text{col}}\sum_{\text{hel}} |M|^2 = \frac{(N_c^2-1)^2}{4N_c} g_s^4 \left\{ 8\frac{ut}{s^2} + 8\frac{u^3}{s^2 t} + 8\frac{t^3}{s^2 u} + 8\frac{ut}{s^2} \right\}$$

$$- \frac{N_c^2-1}{4N_c} g_s^4 \left\{ 16\frac{t^2}{s^2} + 16\frac{u^2}{s^2} \right\}$$

$$= 8g_s^4 \frac{N_c^2-1}{4N_c} \left\{ (N_c^2-1)\frac{u^2t^2 + u^4 + t^4 + u^2t^2}{s^2 tu} - 2\frac{t^2+u^2}{s^2} \right\}$$

$$= 8g_s^4 \frac{N_c^2-1}{4N_c} \left\{ (N_c^2-1)\frac{(u^2+t^2)^2}{s^2 tu} - 2\frac{t^2+u^2}{s^2} \right\}$$

using $s+t+u=0$

$s^2 = (t+u)^2 = t^2+u^2+2ut$

$$-2(N_c^2-1) - 2$$
$$-2N_c^2 + 2 - 2$$
$$-2N_c^2$$

$$\frac{t^2+u^2}{tu}\,\frac{t^2+u^2}{s^2} = \left(\frac{t}{u}+\frac{u}{t}\right)\left(\frac{s^2-2tu}{s^2}\right)$$

$$= \frac{t}{u} + \frac{u}{t} - 2\frac{tu}{s^2}\left(\frac{t}{u}+\frac{u}{t}\right)$$

$$= \frac{t}{u} + \frac{u}{t} - 2\frac{\cancel{tu}}{s^2}\left(\frac{t^2+u^2}{\cancel{ut}}\right) = \frac{t}{u} + \frac{u}{t} - 2\frac{t^2+u^2}{s^2}$$

$$\sum_{\text{col}}\sum_{\text{hel}} |M|^2 = 8g_s^4 \frac{N_c^2-1}{4N_c} \left\{ (N_c^2-1)\left(\frac{t}{u}+\frac{u}{t}\right) - 2N_c^2\,\frac{t^2+u^2}{s^2} \right\}$$