(2) Massless vector boson polarisation can be written in terms of spinor products / strings as follows

$$\varepsilon_+^{*\mu}(p_i, r) = \frac{1}{\sqrt{2}} \frac{\langle r \gamma^\mu i]}{\langle r i \rangle}$$

$$\varepsilon_-^{*\mu}(p_i, r) = -\frac{1}{\sqrt{2}} \frac{[r \gamma^\mu i\rangle}{[r i]}$$

where r is an arbitrary massless reference vector.

a.) $$\varepsilon_+^{*}(p_i, r) \cdot p_i = \frac{1}{\sqrt{2}} \frac{\langle r \gamma^\mu i]}{\langle r i \rangle} p_{i\mu}$$

$$= \frac{1}{\sqrt{2}} \frac{\langle r \not{p}_i i]}{\langle r i \rangle}$$

$$= \frac{1}{\sqrt{2}\langle r i \rangle} \frac{\langle r \left(i\rangle[i + i]\langle i \right) i]}{\langle r i \rangle}$$

$$= \frac{1}{\sqrt{2}\langle r i \rangle} \frac{\langle r i\rangle[i i] + \langle r i]\langle i i]}{\langle r i \rangle}$$

$$= 0$$

$$\varepsilon_-^{*}(p_i, r) \cdot p_i = -\frac{1}{\sqrt{2}} \frac{[r \gamma^\mu i\rangle}{[r i]} p_{i\mu}$$

$$= -\frac{1}{\sqrt{2}} \frac{[r i]\langle i i\rangle + [r i\rangle[i i\rangle}{[r i]}$$

$= 0$

$$\varepsilon_+^{*\mu}(p_i, r)\, r_\mu = \frac{1}{\sqrt{2}} \frac{\langle r \gamma^\mu i]}{\langle r i\rangle} r_\mu$$

$$= \frac{1}{\sqrt{2}} \frac{\langle r \not{r} i]}{\langle r i\rangle} = 0$$

$$\varepsilon_-^{*\mu}(p_i, r)\, r_\mu = -\frac{1}{\sqrt{2}} \frac{[r \gamma^\mu i\rangle}{[r i]} r_\mu$$

$$= -\frac{1}{\sqrt{2}} \frac{[r \not{r} i\rangle}{[r i]} = 0$$

b.) $$\varepsilon_+^{*\mu}(p_1, r_1) \cdot \varepsilon_{+\mu}^{*}(p_2, r_2)$$

$$= \frac{1}{\sqrt{2}} \frac{\langle r_1 \gamma^\mu 1]}{\langle r_1 1\rangle} \frac{1}{\sqrt{2}} \frac{\langle r_2 \gamma_\mu 2]}{\langle r_2 2\rangle}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}\langle r_1 r_2\rangle [2 1]}{\langle r_1 1\rangle \langle r_2 2\rangle}$$ (using Fierz identity)

$$\varepsilon_-^{*\mu}(p_1, r_1) \cdot \varepsilon_{-\mu}^{*}(p_2, r_2)$$

$$= \left(-\frac{1}{\sqrt{2}}\right) \frac{[r_1 \gamma^\mu 1\rangle}{[r_1 1]} \left(-\frac{1}{\sqrt{2}}\right) \frac{[r_2 \gamma_\mu 2\rangle}{[r_2 2]}$$

$$= \frac{[r_1 r_2]\langle 2 1\rangle}{[r_1 1][r_2 2]}$$

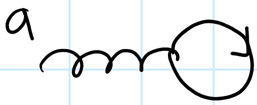
$$\epsilon_+^{*\mu}(p_1,r_1)\,\epsilon^*_{-\mu}(p_2,r_2) = \frac{1}{\sqrt{2}}\frac{\langle r_1\gamma^\mu 1]}{\langle r_1 1\rangle}\left(-\frac{1}{\sqrt{2}}\right)\frac{[r_2\gamma_\mu 2\rangle}{[r_2 2]}$$

$$= -\frac{\langle r_1 2\rangle [1 r_2]}{\langle r_1 1\rangle [r_2 2]}$$

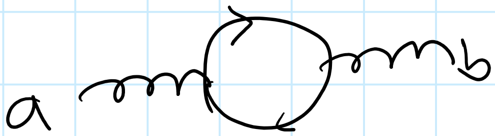
$$\epsilon_-^{*\mu}(p_1,r_1)\,\epsilon^*_{+\mu}(p_2,r_2) = \left(-\frac{1}{\sqrt{2}}\right)\frac{[r_1\gamma_\mu 1\rangle}{[r_1 1]}\frac{1}{\sqrt{2}}\frac{\langle r_2\gamma^\mu 2]}{\langle r_2 2\rangle}$$

$$= -\frac{[r_1 2]\langle r_2 1\rangle}{[r_1 1]\langle r_2 2\rangle}$$

3

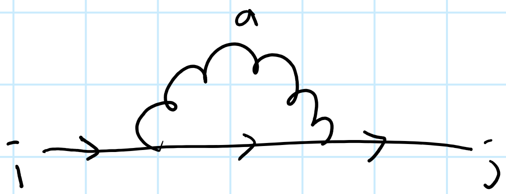


$$t^a_{ii} = \mathrm{tr}(t^a) = 0$$



$$t^a_{ij}\,t^b_{ji} = \mathrm{tr}(t^a t^b)$$

$$= \frac{1}{2}\delta^{ab}$$



$$\Rightarrow t^a_{jm}t^a_{mi}$$

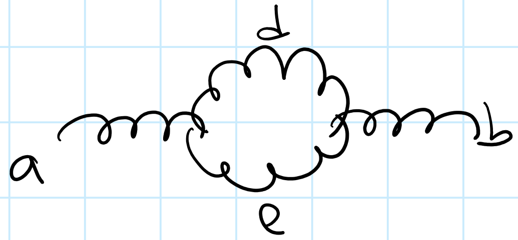
$$= \frac{1}{2}\left[\delta_{ji}\delta_{mm} - \frac{1}{N_c}\delta_{jm}\delta_{mi}\right]$$

$$= \frac{1}{2}\left[N_c - \frac{1}{N_c}\right]\delta_{ji}$$

$$= C_F\,\delta_{ji}$$

$$\boxed{C_F = \frac{N_c^2-1}{2N_c}}$$

"quadratic Casimir operator for $SU(N_c)$ in fundamental reps"



$$\to f^{ade} f^{bde}$$

$$t^d [t^a, t^b] = i f^{abc} t^c t^d$$

$$\mathrm{tr}\left(t^d [t^a, t^b]\right) = i f^{abc} \, \mathrm{tr}(t^c t^d)$$

$$2\, \mathrm{tr}\left(t^d [t^a, t^b]\right) = i f^{abd}$$

$$f^{abc} = -2i \, \mathrm{tr}\left(t^c [t^a, t^b]\right)$$

$$f^{ade} f^{bde} = -4 \, \mathrm{tr}\left(t^e [t^a, t^d]\right) \cdot \mathrm{tr}\left(t^e [t^b, t^d]\right)$$

$$= -4 \; t^e_{ij} [t^a, t^d]_{ji} \, t^e_{kl} [t^b, t^d]_{lk}$$

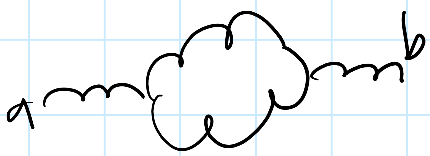
$$= -4 \left(t^e_{ij} t^a_{jm} t^d_{mi} - t^e_{ij} t^d_{jm} t^a_{mi}\right)\left(t^e_{kl} t^b_{ln} t^d_{nk} - t^e_{kl} t^d_{ln} t^b_{nk}\right)$$

```
In[26]:= tmp = -4 * (t[e, i, j] * t[a, j, m] * t[d, m, i] - t[e, i, j] * t[d, j, m] * t[a, m, i]) *
           (t[e, k, l] * t[b, l, n] * t[d, n, k] - t[e, k, l] * t[d, l, n] * t[b, n, k])

Out[26]= -4 (-t[a, m, i] × t[d, j, m] × t[e, i, j] + t[a, j, m] × t[d, m, i] × t[e, i, j])
          (-t[b, n, k] × t[d, l, n] × t[e, k, l] + t[b, l, n] × t[d, n, k] × t[e, k, l])

In[34]:= FierzId = {
           t[a_, i_, j_] * t[a_, k_, l_] :→ (1 / 2) * (d[i, l] * d[k, j] - 1 / Nc * d[i, j] * d[k, l])
         };

In[35]:= Contract = {
           d[i1_, i2_] * d[i2_, i3_] :→ d[i1, i3],
           t[a_, i1_, i2_] * d[i2_, i3_] :→ t[a, i1, i3],
           t[a_, m_, m_] :→ 0,
           d[i1_, i1_] :→ Nc,
           t[a_, i1_, i2_] * t[b_, i2_, i1_] :→ trT[a, b]
         };

In[36]:= (Expand[tmp] //. FierzId // Expand) //. Contract /. trT[a_, b_] :→ (1 / 2) * d[a, b]

Out[36]= Nc d[a, b]
```

→ repeated use of Fierz identity

$$t^a_{ij}\, t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

and $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$



$$= f^{ade} f^{bde}$$

$$= N_c\, \delta^{ab}$$

$$" \; C_A$$