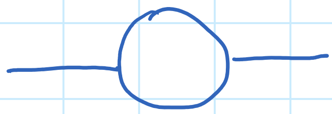
PROBLEM SET 2

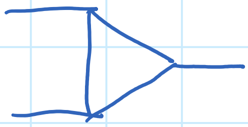
1. From the QCD Feynman rules given in the lecture, draw all Feynman diagrams contributing to 1-loop amplitude for $u + \bar{u} \to d + \bar{d}$ scattering.

Hint: organise the Feynman diagrams according to the 1-loop "topology":

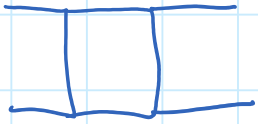
bubble



triangle



box



then assign the appropriate particle type in the loop and external line. Note that the external line in the 1-loop topology may consist of more than one external particle

$\boxed{2}$ Real matrix elements in the soft and collinear limits

Consider the following real emission process of $\gamma^* \to q\bar{q}$

$$\gamma^* \to q(p_3) + \bar{q}(p_4) + g(p_5)$$

(a) Show that in the soft limit, $p_5 \to 0$, the matrix element can be written as:

$$\overline{\sum} |\mathcal{M}_{\gamma^* \to q\bar{q}g}|^2 \overset{p_5 \to 0}{=} g_s^2 \, C_F \underbrace{\frac{2 p_3 \cdot p_4}{p_3 \cdot p_5 \; p_4 \cdot p_5}}_{\text{eikonal factor}} \overline{\sum} |\mathcal{M}_{\gamma^* \to q\bar{q}}|^2$$

(b.) Show that when p_3 and p_5 become collinear the matrix element takes the following form:

$$\overline{\sum} |\mathcal{M}_{\gamma^* \to q\bar{q}g}|^2 \overset{p_3 \parallel p_5}{=} \frac{2 g_s^2}{s_{35}} C_F \underbrace{\left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right]}_{\text{Altarelli-Parisi splitting function for } q \to qg} \overline{\sum} |\mathcal{M}_{\gamma^* \to q\bar{q}}|^2$$

Hint: use Sudakov parameterization

$$p_3^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2 p \cdot n}$$

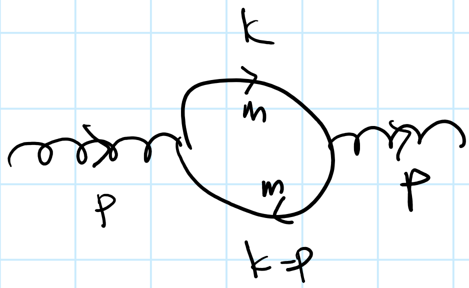
$$p_5^\mu = (1-z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z} \frac{n^\mu}{2 p \cdot n}$$

$k_\perp \to 0$

$\rightsquigarrow$ collinear limit

3 Differential equations for Feynman integrals.

The computation of the following one-loop vacuum polarization diagram (fermionic contribution)



$p^2 = s$

involves two master integrals

$$f_1 = I_{01}$$
$$f_2 = I_{11}$$

where $I_{\nu_1\nu_2} = e^{\epsilon \gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{[k^2-m^2]^{\nu_1}\,[(k-p)^2-m^2]^{\nu_2}}$

γ_E : Euler-Mascheroni constant

(a.) Let's now choose a new basis of master integrals

$$g_1 = \epsilon I_{20}$$
$$g_2 = \epsilon \lambda I_{21} \quad \text{with } \lambda = \sqrt{s(s-4m^2)}$$

First, compute the derivatives of g_1 and g_2 with respect to s

and show that

$$\frac{\partial g_1}{\partial s} = 0$$

$$\frac{\partial g_2}{\partial s} = \frac{4-d}{2\sqrt{s(s-4m^2)}} g_1 - \frac{4-d}{2(s-4m^2)} g_2$$

Hint: $p^\mu \frac{\partial}{\partial p^\mu} = p^\mu \frac{\partial s}{\partial p^\mu} \frac{\partial}{\partial s} = 2s\frac{\partial}{\partial s}$

Employ the following IBP reduction identities

$$I_{12} = \frac{2-d}{2m^2(s-4m^2)} I_{01} + \frac{d-3}{4m^2-s} I_{11}$$

$$I_{21} = \frac{2-d}{2m^2(4m^2-s)} I_{01} + \frac{d-3}{4m^2-s} I_{11} \qquad (*)$$

$$I_{22} = -\frac{(d-2)(-6m^2+2dm^2-s)}{m^2(4m^2-s)^2 s} I_{01} + \frac{(d-3)(4m^2-6s+ds)}{s(s-4m^2)^2} I_{11}$$

$$I_{20} = \frac{d-2}{2m^2} I_{01} \qquad (**)$$

to compute $\frac{\partial g_i}{\partial s}$ in terms of I_{01} and I_{11}

using $(*)$ and $(**)$ write I_{11} and I_{01} in terms of I_{20} and I_{21}, then write $\frac{\partial g_i}{\partial s}$ in terms of g_1 and g_2

Perform change of variable:

$$S = -m^2 \frac{(1-y)^2}{y}$$

and set $d = 4-2\epsilon$ and compute

$$\frac{\partial g_1}{\partial y} \quad \text{and} \quad \frac{\partial g_2}{\partial y}.$$

Show that the resulting differential eqs are

$$\frac{\partial}{\partial y}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & 0 \\ \frac{1}{y} & \frac{1}{y} - \frac{2}{1+y} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad (a.1)$$

(b.) From the DEs we see that g_1 is constant in y, we can't use DE to find its solution. Here we compute g_1 directly using text book method to get

$$g_1 = (m^2)^{-\epsilon}\, \Gamma(1+\epsilon)\, e^{\epsilon \gamma_E}$$

$$= (m^2)^{-\epsilon} \left[1 + \epsilon^2 \frac{\pi^2}{12} + \mathcal{O}(\epsilon^3)\right]$$

To solve g_2 we first write $g_2(\epsilon)$ as expansion in ϵ

$$g_2(\epsilon) = (m^2)^{-\epsilon}\left[g_2^{(0)} + \epsilon g_2^{(1)} + \epsilon^2 g_2^{(2)} + \mathcal{O}(\epsilon^3)\right] \qquad (a.2)$$

Substitute (a.2) to (a.1) and solve g_2 order by order in ϵ

Write the solution in terms of "Goncharov Polylogarithm" defined by

$$\int \frac{dy}{y - z_1} G(z_2, \ldots, z_k; y) = G(z_1, z_2, \ldots, z_k; y) + \text{const}$$

To determine the constants, use g_1 and demand that I_{21} is regular as $s \to 0$

$s = 0 \Rightarrow y = 1$

$$g_2 = \epsilon \sqrt{s(s - 4m^2)}\, I_{21} = \epsilon\, m^2 \left(\frac{1}{y} - y\right) I_{21}$$

$g_2 = 0$ as $y = 1$

Use the following identities for Goncharov polylogs:

$$G(\underbrace{0, \ldots, 0}_{k}; y) = \frac{1}{k!} \log^k(y)$$

$$G(-1, 0; 1) = -\frac{\pi^2}{12}$$

Show the the final result for g_2 is

$$g_2 = (m^2)^{-\epsilon} \Big\{ \epsilon\, G(0; y) + \epsilon^2 \Big[G(0,0;y) - 2\, G(-1,0;y) - \frac{\pi^2}{6} + O(\epsilon^3) \Big\}$$