QCD Lecture 3

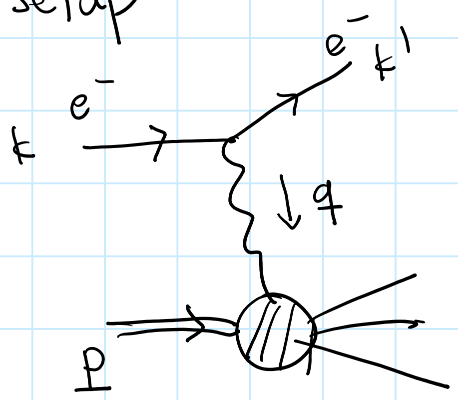
We will now consider scattering process involving initial state hadrons $\rightarrow$ $e^- p \rightarrow e^- +$ hadrons

a.k.a <u>Deep Inelastic Scattering</u>

DIS was measured in SLAC (1960's)

↳ measure the cross section for an electron to break up the proton and scatter to a given final energy and momentum. The final state resulting from the disruption of the proton is ignored.

Kinematic setup



$$q = k - k'$$

invariants:

$$Q^2 = -q^2 \qquad x = \frac{Q^2}{2P\cdot q} \qquad y = \frac{2P\cdot q}{2P\cdot k}$$

y is the fraction of electron's energy, in the lab frame, that is transferred to proton. $\Rightarrow$ $0 \leq y \leq 1$

For $k^0 \gg m_p$

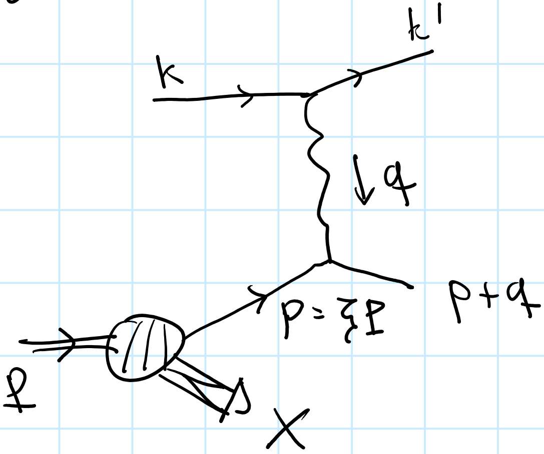
$$(P+q)^2 > 0 \rightarrow 2P\cdot q - Q^2 > 0 \Rightarrow 0 < Q^2 < 2P\cdot q$$

also $\quad S = (P + k)^2 = 2P \cdot k$

so $\quad Q^2 = x\, 2P \cdot q = x\, y\, 2P \cdot k$
$\quad\quad = x y S$

In the parton model, proton is made up of pointlike constituents, collectivelly called "parton" which consist of quarks and gluons.

The simplest process contributing to DIS is the scattering of elctron from a quark:



quark momentum : $p = \xi P \qquad 0 < \xi < 1$

$\downarrow$ longitudinal fraction

Parton distribution function $\quad d\xi\, f(\xi)$
$\Rightarrow$ probability to find a parton (quark) inside a hadron (proton) with momentum fraction between ξ and $\xi + d\xi$

Ignoring masses

$(p+q)^2 = 0$ (on-shell condition for final state quark)

$2p\cdot q + q^2 = 0 \Rightarrow 2\xi P\cdot q - Q^2 = 0$

$$\underbrace{\xi}_{\text{measurable}} = \frac{Q^2}{2P\cdot q} = x$$

Cross section: $\dfrac{d\sigma}{d\cos\theta_{cm}} = \dfrac{1}{2s}\dfrac{1}{16\pi}\overline{\sum}|M|^2$

$$\overline{\sum}|M|^2 = 4e^4 Q_f^2 \frac{s^2+u^2}{t^2}$$

where

$s = (p+k)^2 = 2p\cdot k = 2\xi P\cdot k = xS$

$t = q^2 = -Q^2$

$u = -s-t = -(1-y)xS$

$$\frac{d\sigma_{eq}}{d\cos\theta_{cm}} = \frac{1}{2xS}\frac{1}{16\pi}\frac{2}{4}4e^4Q_f^2\left(\frac{xS}{Q^2}\right)^2\left[1+(1-y)^2\right]$$

$$= \pi\alpha^2 Q_f^2 \frac{1}{Q^4} xS\left[1+(1-y)^2\right]$$

$dt = -dQ^2 = \frac{1}{2}s\, d\cos\theta_{cm} = \frac{1}{2}xS\, d\cos\theta_{cm}$

$$\frac{d\sigma_{eq}}{dQ^2} = \frac{2\pi\alpha^2 Q_f^2}{Q^4}\left[1+(1-y)^2\right]$$

$t = -\frac{s}{2}(1-\cos\theta)$

ep cross section is

$$\sigma(ep \to e + hadrons) = \int dQ^2 \int d\xi \sum_f f_f(\xi) \frac{2\pi\alpha^2 Q_f^2}{Q^4}\left[1+(1-y)^2\right]$$

now $\xi = x$ and $Q^2 = xyS$

$$d\xi \, dQ^2 = dx\,dy \times S$$

$$\frac{d\sigma}{dx\,dy}(ep \to e \underset{\substack{\text{any} \\ \text{hadrons}}}{X}) = \underbrace{\left[\sum_f Q_f^2 \times f_f(x)\right]}_{\substack{F_2(x) \\ \text{Structure} \\ \text{function}}} \times \left(\frac{2\pi\alpha^2 S}{Q^4}\left[1+(1-y)^2\right]\right)$$

"Bjorken scaling" → F_2 depends on x but no Q^2

experimentally, however, this scaling is not exact as

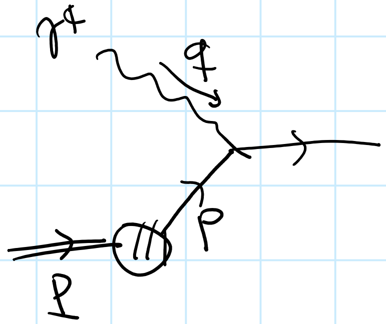
$$\frac{\partial F_2(x,Q^2)}{\partial \log Q^2} \neq 0$$

→ "scaling violation" due to evolution of $f(x)$ stemming from parton radiation/splitting.

Let us now compute QCD corrections to $ep \to eX$ process. As in $e^+e^- \to$ hadrons, let us consider simplified version of the process, by removing the electron line. We will consider

$$\gamma^*(q) + p(P) \to X$$

"virtual Compton scattering". In the parton model

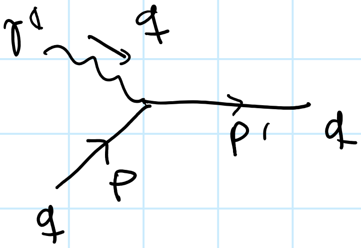


$$p = \xi P \qquad 0 < \xi < 1$$

$$q = p' - p$$

$$q^2 = -2p' \cdot p$$

partonic matrix element



$$M^\mu = \bar{u}(p')\, i e Q_f\, \gamma^\mu u(p)\, \delta_{ij}$$

$$\sum |M|^2 = N_c e^2 Q_f^2 \, \mathrm{tr}\left[\not p' \gamma^\mu \not p \gamma^\nu\right] \left(-g_{\mu\nu} + \cancel{\frac{q_\mu q_\nu}{q^2}}\right)$$

$$= -e^2 Q_f^2 N_c \, \mathrm{tr}\left[\not p' \underbrace{\gamma^\mu \not p \gamma_\mu}_{= (2-d)\not p}\right]$$

$$= -e^2 Q_f^2 N_c \, \mathrm{tr}\left[\not p' \not p\right](2-d)$$

$$= -e^2 Q_f^2 N_c \, 4 p' \cdot p \, (2-d)$$

$$= 4 e^2 Q_f^2 N_c (-q^2)(1-\epsilon)$$

Averaging over initial state spin and colour

$$\overline{\sum} = \frac{1}{2}\frac{1}{N_c}\sum$$

$$\overline{\sum}|M|^2 = 2e^2Q_f^2(-q^2)(1-\epsilon)$$

1-body phase-space

$$\int d\Phi_1 = \int \frac{d^dp'}{(2\pi)^d} 2\pi\delta_+(p'^2)(2\pi)^d\delta^{(d)}(p+q-p')$$

$$= 2\pi\delta_+((p+q)^2) = 2\pi\delta_+(2p\cdot q - Q^2)$$

$$= 2\pi\delta(\xi 2P\cdot q - Q^2)$$

$$= 2\pi\delta(\xi 2P\cdot q - x2P\cdot q)$$

$$= \frac{2\pi}{2P\cdot q}\delta(\xi - x)$$

cross section

$$\sigma(\gamma^* q \to q) = \frac{1}{\text{flux}}\int d\Phi_1 \overline{\sum}|M|^2$$

$$\text{flux} = 4\sqrt{(p\cdot q)^2 - p^2q^2}$$

$$= 4p\cdot q$$

$$\sigma_0(\gamma^* q) = 8\pi^2\alpha Q_f^2 \cdot \frac{x}{Q^2}\delta(\xi - x)(1-\epsilon)$$

The hadronic cross section is then

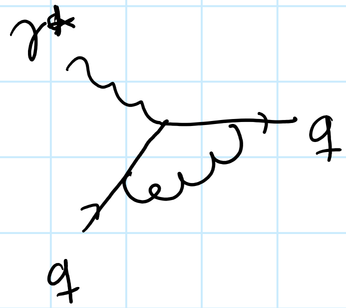
$$\sigma_0(\gamma^* p \to X) = \int d\xi \sum_f f_f(\xi)\, \sigma_0(\gamma^* q)$$

$$= \sum_f f_f(x)\, Q_f^2\, \frac{x}{Q^2}\, 8\pi^2 \alpha\, (1-\epsilon)$$

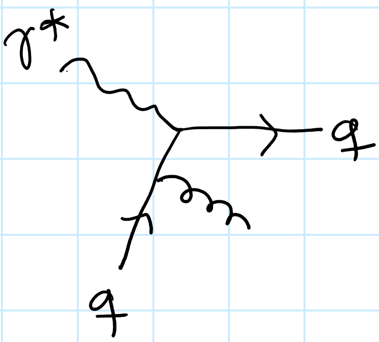
$$= F_2(x)\, \frac{8\pi^2 \alpha}{Q^2}\, (1-\epsilon)$$

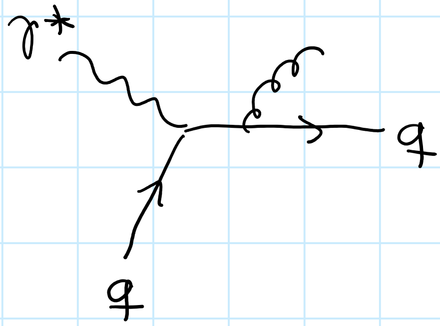
QCD corrections to $\gamma^* q \to q$ consist of virtual and real contributions:

virtual :

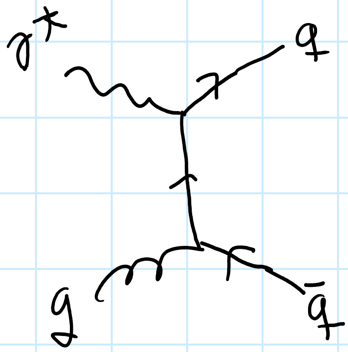


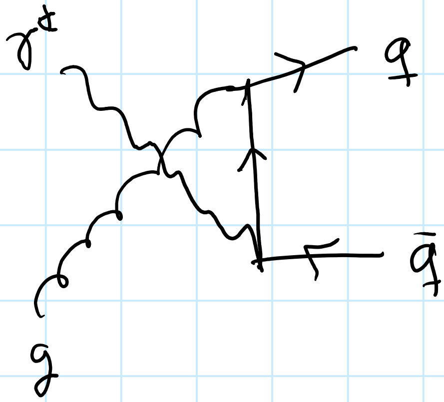
real :





$(\gamma^* q)$





$(\gamma^* g)$

The computation of virtual correction is the same as for $\gamma^* \to q\bar{q}$ process. The one-loop amplitude is identical except for $\left(\frac{4\pi\mu^2}{-s}\right)^\epsilon \to \left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon$

$$= \left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon$$

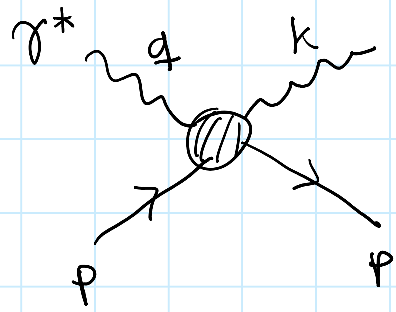
hence there is no need for analytic continuation

The one-loop cross section reads

$$\sigma_1(\gamma^* q \to q) = \sigma_0(\gamma^* q \to q) \times \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon$$

$$\times \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + O(\epsilon)\right)$$

Let's now compute the real correction, first the $\gamma^* q \to qg$ process



$$(p+q)^2 = (p'+k)^2 = 2p'\cdot k > 0$$

so $2p\cdot q - Q^2 > 0$

let $z = \frac{Q^2}{2p\cdot q}$, $0 < z < 1$

$$\sigma_0(\gamma^* q \to qg) = \frac{1}{4p\cdot q}\int d\Phi_2 \overline{\sum} |M|^2$$

d-dimensional 2-body phase space from $\gamma^* \to q\bar{q}$ calculation

$$\int d\Phi_2 = \frac{1}{(2\pi)^{d-2}} \frac{1}{4} \int dE_{p'} \frac{E_{p'}^{d-2}}{E_{p'} E_k} \delta(E_{p'} + E_k - E_p - E_q) \int d\Omega_{d-1}$$

write 2-body phase-space in terms of cm scattering angle θ_*. Also, in cm frame

$$p + q = (\sqrt{s}, 0, 0, 0)$$

from $\delta^{(d-1)}$ integration $\Rightarrow \vec{p}' = -\vec{k}$

so $E_p = E_k$

$$\int d\Phi_2 = \frac{1}{(2\pi)^{d-2}} \frac{1}{4} \int dE_{p'} E_{p'}^{d-4} \delta(2E_{p'} - \sqrt{s}) \int d\Omega_{d-1}$$

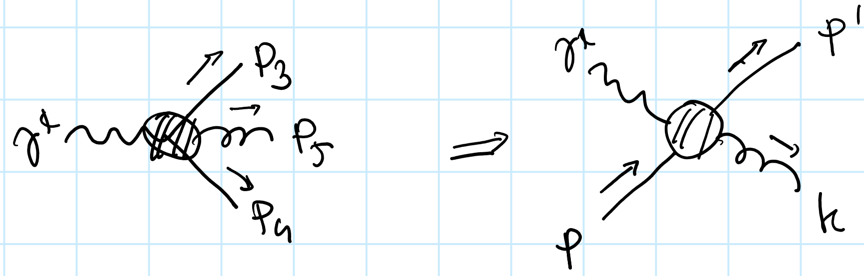
$$= \frac{1}{(2\pi)^{d-2}} \frac{1}{4 \cdot 2} \left(\frac{\sqrt{s}}{2}\right)^{d-4} \int d\Omega_{d-2} \int d\cos\theta_* [\sin\theta_*]^{d-4}$$

$$= \frac{(4\pi)^{\epsilon}}{32\pi^2} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{2\pi^{(d-2)/2}}{\Gamma\left(\frac{d-2}{2}\right)} \int d\cos\theta_* \left[1 - \cos^2\theta_*\right]^{-\epsilon}$$

$$= \frac{1}{16\pi} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_{-1}^{1} d\cos\theta_* \left[\frac{1-\cos^2\theta_*}{4}\right]^{-\epsilon}$$

For the matrix element, we can recycle the results from

$\gamma^* \to q\bar{q}g$ computation



$$p_3 \to p'$$
$$p_4 \to -p$$
$$p_5 \to k$$

$$\sum |M|^2 = 8 \left\{ (1-\epsilon)^2 \left[\frac{2p\cdot k}{2p'\cdot k} + \frac{2p'\cdot k}{2p\cdot k} - 2\frac{q^2\, p\cdot p'}{2p\cdot k\, 2p'\cdot k} \right] - 2\epsilon(1-\epsilon)\frac{q^2\, 2p\cdot p' - 2p\cdot k\; 2p'\cdot k}{2p\cdot k\;\; 2p'\cdot k} \right\}$$
$$\times\, e^2 Q_f^2 g_s^2 \frac{4}{3} \times N_c$$

Just like in $\gamma^* \to q\bar{q}$, we now do same change of kinematic variables. In CM frame

$$q = (E_q, 0, 0, q_3) \qquad \text{with } q_3 = E_p$$
$$p = (E_p, 0, 0, -E_p)$$

$$(p+q)^2 = s = 2p\cdot q - Q^2 = 2(E_q E_p + E_p^2) - Q^2$$
$$= 2E_p(E_q + E_p) - Q^2$$
$$= 2E_p\sqrt{s} - Q^2$$

$$E_p = \frac{S+Q^2}{2\sqrt{S}} \quad , \quad E_q = \sqrt{S} - E_p = \frac{2S - S - Q^2}{2\sqrt{S}}$$

$$= \frac{S - Q^2}{2\sqrt{S}}$$

$$(p+q)^2 = 2p\cdot q - Q^2 = S \rightsquigarrow 2p\cdot q = S + Q^2$$

$$z = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{S+Q^2} \rightsquigarrow S = Q^2 \frac{1-z}{z}$$

$$E_p = q_3 = \frac{S+Q^2}{2\sqrt{S}} = \frac{Q^2 \frac{1-z}{z} + Q^2}{2\left(Q^2 \frac{1-z}{z}\right)^{1/2}}$$

$$= \frac{Q^2}{2z}\left(\frac{z}{Q^2(1-z)}\right)^{1/2} = \frac{\sqrt{Q^2}}{2} \frac{1}{\sqrt{z(1-z)}}$$

$$E_{p'} = E_k = \frac{\sqrt{S}}{2} = \sqrt{Q^2} \sqrt{\frac{1-z}{z}}$$

then

$$2p\cdot q = \frac{Q^2}{z} \qquad 2p'\cdot k = 2p\cdot q - Q^2$$

$$= \frac{Q^2}{z} - Q^2 = \frac{Q^2}{z}(1-z)$$

$$2p\cdot p' = 2E_pE_{p'} - |\vec{p}||\vec{p}'| \cos\theta_*$$

$$= 2E_pE_{p'}(1-\cos\theta_*) = \frac{Q^2}{z} \frac{1-\cos\theta_*}{2}$$

$$2p\cdot k = \frac{Q^2}{z} \frac{1+\cos\theta_*}{2} \qquad (\vec{k} = -\vec{p}')$$

Let $\omega = \frac{1-\cos\theta_*}{2}$ and $1-\omega = \frac{1+\cos\theta_*}{2}$

$$d\omega = \frac{d\cos\theta_*}{2}$$

$$\sum |M|^2 = 8(1-\epsilon)\Bigg\{(1-\epsilon)\left(\frac{1-\omega}{1-z} + \frac{1-z}{1-\omega} + 2\frac{z\omega}{(1-z)(1-\omega)}\right.$$

$$\left. + 2\epsilon\left(\frac{z\omega + (1-z)(1-\omega)}{(1-z)(1-\omega)}\right)\right\}$$

$$\times\; e^2 Q_f^2\, g_s^2\, \frac{4}{3} \times N_c$$

recall that $\overline{\sum} = \frac{1}{2N_c}\sum$

$$\sigma_0(\gamma^* q \to qg) = \frac{8\pi^2\alpha\, Q_f^2}{Q^2}\cdot 2\cdot \frac{4}{3}\,\frac{\alpha_s}{2\pi}\,\frac{1-\epsilon}{\Gamma(1-\epsilon)}\left[\frac{Q^2(1-z)}{4\pi\mu^2 z}\right]^{-\epsilon}$$

$$\int_0^1 d\omega\,[\omega(1-\omega)]^{-\epsilon}\Bigg\{(1-\epsilon)\left[\frac{1+z^2}{(1-\omega)(1-z)} - \frac{2z}{1-z} + \frac{1-\omega}{1-z}\right] + 2\epsilon\,\frac{z\omega}{(1-z)(1-\omega)} + 2\epsilon\Bigg\}$$

we need to compute ω integrals in terms of Γ functions

$$\int_0^1 d\omega\,[\omega(1-\omega)]^{-\epsilon}\,\frac{1}{1-\omega} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(-\frac{1}{\epsilon}\right)$$

$$\int_0^1 dw\,[w(1-w)]^{-\epsilon} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\int_0^1 dw\,[w(1-w)]^{-\epsilon}(1-w) = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{1}{2}$$

$$\int_0^1 dw\,[w(1-w)]^{-\epsilon}\,\frac{w}{1-w} = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(-\frac{1}{\epsilon}-1\right)$$

$$\sigma_0(\gamma^* q \to qg) = \frac{8\pi^2\alpha Q_f^2}{Q^2}\, z\, \frac{4}{3}\,\frac{\alpha_s}{2\pi}\,\frac{(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{4\pi\mu^2 z}{Q^2(1-z)}\right]^{\epsilon}$$

$$\times\left\{-\frac{1}{\epsilon}\,\frac{1+z^2}{1-z} + \frac{3/2-4z+z^2}{1-z} + \epsilon\,\frac{5/2-6z}{1-z} + \dots\right\}$$

To obtain hadronic cross section, need to combine with $f(\xi)$ and integrate over ξ

$$x = \frac{Q^2}{2\underline{P}\cdot q} = \frac{Q^2}{2p\cdot q}\,\xi = z\xi$$

$$\int d\xi\, f(\xi) = \int dz\,\frac{x}{z^2}\, f\!\left(\frac{x}{z}\right)$$

$$\sigma_0(\gamma^* p \to X) = 8\pi\alpha\, Q_f^2 \frac{x}{Q^2}(1-\epsilon)$$

$$\times \Big\{ f_q(x) \Big[1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\times \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \mathcal{O}(\epsilon) \right) \Big]$$

$$+ \int_x^1 \frac{dz}{z} f_q\left(\frac{x}{z}\right) \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\cdot \Big[\Big(-\frac{1}{\epsilon}\frac{1+z^2}{1-z} + \frac{3/2 - 4z + z^2}{1-z}$$

$$+ \epsilon \frac{5/2 - 6z}{1-z} + \dots \Big) z^{\epsilon}(1-z)^{-\epsilon} \Big]$$

$$+ \text{terms from } \gamma^* g \to q\bar{q} \Big\}$$

$$\int_x^1 \frac{dz}{z} f_q\left(\frac{x}{z}\right) z^{\epsilon} (1-z)^{-1-\epsilon} \Big[-\frac{1}{\epsilon}(1+z)^2 + \frac{3}{2} - 4z + z^2$$

$$+ \epsilon\left(\frac{5}{2} - 6z\right) + \dots \Big]$$

$$(1-z)^{-1-\epsilon} = -\frac{1}{\epsilon}\delta(1-z) + \frac{1}{[1-z]_+} - \epsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + \dots$$

"+" distribution is defined by

$$\int_0^1 dz \frac{f(z)}{[1-z]_+} = \int_0^1 dz \frac{f(z) - f(1)}{1-z}$$

and $\frac{1}{[1-z]_+} = \frac{1}{1-z}$ for $z \neq 0$

the other plus function

$$\int_0^1 dz\, f(z) \left[\frac{\ln^n(1-z)}{1-z}\right]_+ \equiv \int_0^1 dz\, (f(x)-f(1)) \frac{\ln^n(1-z)}{1-z}$$

with $\left[\frac{\ln^n(1-z)}{1-z}\right]_+ = \frac{\ln^n(1-z)}{1-z}$ for $z \neq 1$

$$\sigma_0(\gamma^* p \to X) = 8\pi\alpha Q_f^2 \frac{x}{Q^2}(1-\epsilon)$$

$$\times \left\{ f_q(x) \left[1 - \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \text{finite}\right)\right]\right.$$

$$+ \int_x^1 \frac{dz}{z} f_q\left(\frac{x}{z}\right) \frac{4}{3}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\cdot \left\{ \delta(1-z)\left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \text{finite}\right] - \frac{1}{\epsilon}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] + \text{finite}\right\}\left.\vphantom{\frac{1}{1}}\right\}$$

$+$ terms from $\gamma^* g \to q\bar{q}$

We see that the poles in $\delta(z-1)$ term cancels out. There is remaining divergency of the form

$$-\frac{1}{\epsilon}\int \frac{dz}{z} f_q\left(\frac{x}{z}\right)\frac{\alpha_s}{2\pi}\underbrace{\frac{4}{3}\left(\frac{1+z^2}{[1-z]_+}+\frac{3}{2}\delta(1-z)\right)}_{\text{regularised Altarelli-Parisi splitting function for } q\to qg}$$

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z}{[1-z]_+}+\frac{3}{2}\delta(1-z)\right)$$

$$\int_0^1 dz\, P_{qq}(z) = 0 \Leftarrow$$

PDF renormalisation $\Rightarrow$ absorb this divergencies in the PDF. In the $\overline{\text{MS}}$ scheme:

$$f_i^{\text{bare}}(x) = f_i^{\overline{\text{MS}}}(x,\mu_F) + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}e^{-\epsilon\gamma_E}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}$$

$$\times \int \frac{dz}{z}\sum_j P_{ij}(z)\, f_j\left(\frac{x}{z}\right)$$

where $j \in \{q\bar{q}, gg, \bar{q}g, gq\}$

After PDF redefinition, expanding the $\left(\frac{1-z}{z}\right)^{-\epsilon}$ term and keep only terms contributing to $\mathcal{O}(1)$ (including those of $\frac{1}{\epsilon}\cdot\epsilon$ terms), we have

$$\sigma_0(\gamma^* p \to X) = 8\pi\alpha \frac{x}{Q^2}(1-\epsilon)$$

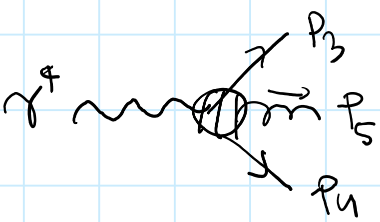
$$\times \sum_q \Big\{ f_q^{\overline{MS}}(x) \left[1 + \frac{\alpha_s}{2\pi} \cdot \text{finite}\right]$$

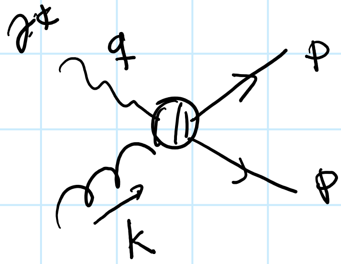
$$+ \int_x^1 \frac{dz}{z} f_q^{\overline{MS}}\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \cdot \text{finite} \Big\} e_q^2$$

$$+ \text{ terms from } \gamma^* g \to q\bar{q}$$

Now, let's compute the $\gamma^* g \to q\bar{q}$ real emission cross section. The ingredients are similar to those of $\gamma^* q \to gq$, except for crossing of external momenta.

For the matrix element squared we can use the $\gamma^* \to q\bar{q}g$ result with external momentum rearrangement





so $p_3 \to p$
$p_4 \to p'$
$p_5 \to -k$

Be careful, we use different kinematic setup for the two processes:

$$\gamma^*(q) + q(p) \to q(p') + g(k)$$

$$\gamma^*(q) + g(k) \to q(p) + \bar{q}(p')$$

Now the spin summed matrix element squared for $\gamma^* g \to q\bar{q}$ is

$$\Sigma |M|^2 = 8 \left\{ (1-\epsilon)^2 \left(\frac{2p'\cdot k}{2p\cdot k} + \frac{2p\cdot k}{2p'\cdot k} + \frac{2q^2\, 2p\cdot p'}{2p\cdot k\, 2p'\cdot k} \right) + 2\epsilon(1-\epsilon)\left[\frac{q^2\, 2p\cdot p'}{2p\cdot k\, 2p'\cdot k} - 1 \right] \right\}$$

Rederiving map from dot products to Q^2, z, w variables:

$$q^2 = -Q^2 \qquad , s = Q^2 \frac{1-z}{z}$$

$$2k\cdot q = \frac{Q^2}{z} \qquad 2p\cdot p' = \frac{Q^2}{z}(1-z)$$

$$2p\cdot k = \frac{Q^2}{z} w \qquad 2p'\cdot k = \frac{Q^2}{z}(1-w)$$

$$\Sigma |M|^2 = 8(1-\epsilon) \left\{ (1-\epsilon)\left(\frac{1-w}{w} + \frac{w}{1-w} - 2\frac{z(1-z)}{w(1-z)} \right) - 2\epsilon \left(\frac{z(1-z)}{w(1-w)} + 1 \right) \right\}$$

The phase space integration will involve

$$\int_0^1 dw \, [w(1-w)]^{-\epsilon} \quad \text{integration}$$

which results in Gamma functions

The averaging factor is also different now

due to initial state gluon

$$\overline{\sum} = \frac{1}{2} \frac{1}{N_c^2 - 1} \sum$$

The $\gamma^* g \to q\bar{q}$ cross section is now

$$\sigma_0(\gamma^* g \to q\bar{q}) = 8\pi^2 \alpha \, Q_f^2 \frac{z}{Q^2} (1-\epsilon) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2 z}{Q^2(1-z)} \right)^{\epsilon}$$

$$\cdot 2 \cdot \left[-\frac{1}{\epsilon} \, \frac{1}{2} \left(z^2 + (1-z)^2 \right) + \mathcal{O}(\epsilon) \right]$$

the contribution to $\sigma(\gamma^* p \to X)$ is

$$\sigma(\gamma^* p \to X) = 8\pi^2 \alpha \, Q_f^2 \frac{x}{Q^2} (1-\epsilon)$$

$$\cdot \int_x^1 \frac{dz}{z} f_g\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2 z}{Q^2(1-z)} \right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\cdot 2 \cdot \left[-\frac{1}{\epsilon} \frac{1}{2} \left[z^2 + (1-z)^2 \right] + \ldots \right]$$

The divergent term is

$$-\frac{1}{\epsilon} \, \frac{1}{2} \left[z^2 + (1-z)^2 \right] \to -\frac{1}{\epsilon} P_{qg}(z)$$

Renormalise f_g → subtract the $\frac{1}{\epsilon}$ pole

then (note, no + distribution for $\gamma^* g \to q\bar{q}$

$$\sigma_0(\gamma^* p \to X) = 8\pi\alpha\, Q_f^2 \frac{x}{Q^2}(1-\epsilon)$$

$$\cdot \sum_q \Big\{ f_q^{\overline{MS}}(x) \left[1 - \frac{\alpha_s}{2\pi} \cdot \text{finite} \right]$$

$$+ \int_x^1 \frac{dz}{z} f_q^{\overline{MS}}\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \cdot \text{finite} \Big\} \cdot Q_q^2$$

$$+ \sum_q Q_q^2 \int_x^1 \frac{dz}{z} f_g^{\overline{MS}}\left(\frac{x}{z}\right) \frac{\alpha_s}{2\pi} \cdot \text{finite}$$

DGLAP equations

Dokshitzer – Gribov – Lipatov – Altarelli – Parisi

From the expression of renormalised PDF

$$f_i^{\overline{MS}}(x,\mu_F) = f_i(x) - \frac{\alpha_s}{2\pi} e^{-\epsilon\gamma_E}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}$$

$$\cdot \int \frac{dz}{z} \sum_j P_{ij}(z) f_j\left(\frac{x}{z}\right)$$

$$a^{\epsilon} = 1 + \epsilon \log a + \dots$$

$$f_i^{\overline{MS}}(x,\mu_F) = f_i(x) - \frac{\alpha_s}{2\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log 4\pi - \log\frac{\mu_F^2}{\mu^2}\right]$$

$$\cdot \int \frac{dz}{z} \sum_j P_{ij}(z) f_j\left(\frac{x}{z}\right)$$

$$\frac{\partial f_i^{\overline{MS}}(x,\mu_F)}{\partial \ln \mu_F^2} = \frac{\alpha_s}{2\pi} \int \frac{dz}{z} \sum_j P_{ij}(z) f_j\left(\frac{x}{z}\right)$$

$$f_j\left(\frac{x}{z}\right) = f_j^{\overline{MS}}\left(\frac{x}{z}\right) + O(\alpha_s)$$

$$= \frac{\alpha_s}{2\pi} \int \frac{dz}{z} \sum_j P_{ij}(z) f_j^{\overline{MS}}\left(\frac{x}{z}\right)$$

Obtain PDF evolution at scale μ_2 from μ_1 by solving DGLAP equations

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