# QCD Lecture 4

## I. Renormalisation, running coupling and scale dependence

We have performed one-loop amplitude computation in the context of $\gamma^* \to q\bar{q}$ QCD corrections, and we have seen that loop amplitudes suffer from UV and IR singularities. We have employed dimensional regularisation scheme to regulate such divergencies, where the singularities manifest as poles in $\epsilon$ (dimensional regulator, $d = 4-2\epsilon$).
In our $\gamma^* \to q\bar{q}$ 1-loop example, the UV singularities don't appear in the final result while the remaining IR poles are cancelled by the real corrections.

In a generic QCD process, UV singularities will be present in the final result and we employ the "renormalisation" procedure to cancel them.

To see how this works, recall QCD Lagrangian written down in Lecture 1. We will call this "BARE" Lagrangian where all the fields, coupling and mass parameters are 'bare' quantities (labelled by "0" subscript)

$\Rightarrow A^a_{\mu 0}\,,\ \psi_0\,,\ c^a_0\,,\ g_{s,0}\,,\ m_0$

To renormalise the theory, we introduce renormalised fields, mass and coupling constants:

$$\psi_i,\ A^a_\mu,\ c^a,\ g$$

The bare and renormalised fields are related by renormalisation constants

$$\psi_0 = \sqrt{Z_2}\,\psi \qquad m_0 = Z_m m$$
$$A^a_{0\mu} = \sqrt{Z_3}\,A^a_\mu$$
$$c^a_0 = \sqrt{Z_2^c}\,c^a$$

The QCD bare Lagrangian in terms of these renormalised fields is

$$\mathcal{L}_b = iZ_2\bar{\psi}_i \not\partial \psi_i - Z_2 Z_m m\bar{\psi}_i\psi_i + g_0 Z_2 Z_3^{1/2}\bar{\psi}_i\gamma^\mu t^a_{ij}A_\mu\psi_j$$
$$+\frac{Z_3}{2}A^a_\mu\left[g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right]A^a_\nu + \frac{Z_3}{2\lambda}A^a_\mu\partial^\mu\partial^\nu A^a_\nu$$
$$-\frac{1}{2}g_0 Z^{3/2} f^{abc}A^b_\mu A^c_\nu\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)$$
$$-\frac{1}{4}g_0^2 Z_3^2 f^{abc}f^{amn}A^b_\mu A^c_\nu A^{m\mu}A^{n\nu}$$
$$-Z_2^c\,\bar{c}^a\Box c^a + g_0 Z_2^c Z_3^{1/2} f^{abc}\left[(\bar{c}^a\partial^\mu c^b)A^c_\mu\right.$$
$$\left.+(\partial^\mu A^c_\mu)(\bar{c}^a c^b)\right]$$

Now we splite the bare Lagrangian into "renormalised" part and "counter-term" part.

$$\mathcal{L}_b = \mathcal{L}_{ren} + \mathcal{L}_{CT}$$

↳ your bare Lagrangian but with ALL bare quantities replaced by the renormalised ones

$$\mathcal{L}_{ren} = i\bar{\psi}_i \partial\!\!\!/ \psi_i - m\bar{\psi}_i\psi_i + g\bar{\psi}_i \gamma^\mu t^a_{ij} A^a_\mu \psi_j$$
$$+ \frac{1}{2} A^a_\mu \left[ g^{\mu\nu}\Box - \left(\frac{1}{\alpha} - 1\right)\partial^\mu \partial^\nu \right] A^a_\nu$$
$$- \frac{1}{2} g f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)$$
$$- \frac{1}{4} g^2 f^{abc} f^{amn} A^b_\mu A^c_\nu A^{m\mu} A^{n\nu} - \bar{c}^a \Box c^a$$
$$+ g f^{abc} \left[ \left(\bar{c}^a \gamma^\mu c^b\right) A^c_\mu + \partial^\mu A^c_\mu \left(\bar{c}^a c^b\right) \right]$$

$$\mathcal{L}_{CT} = i\delta_2 \bar{\psi}_i \partial\!\!\!/ \psi_i - (\delta_2 + \delta_m)\bar{\psi}_i\psi_i + g\delta_1 \bar{\psi}_i \gamma^\mu t^a_{ij} A^a_\mu \psi_j$$
$$+ \frac{\delta_3}{2} A^a_\mu \left[ g^{\mu\nu}\Box - \left(\frac{1}{\alpha} - 1\right)\partial^\mu \partial^\nu \right] A^a_\nu$$
$$- \frac{\delta_1^{3g}}{2} g f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)$$
$$- \frac{\delta_1^{4g}}{4} g^2 f^{abc} f^{amn} A^b_\mu A^c_\nu A^{m\mu} A^{n\nu} - \delta_2^c \bar{c}^a \Box c^a$$
$$+ g\delta_1^c f^{abc} \left[ \left(\bar{c}^a \gamma^\mu c^b\right) A^c_\mu + \partial^\mu A^c_\mu \left(\bar{c}^a c^b\right) \right]$$

where

$$\delta_2 = Z_2 - 1$$

$$\delta_m = Z_m - 1$$

$$\delta_1 = \left(\frac{g_0}{g}\right) Z_2 Z_3^{1/2} - 1 \quad \circledast$$

$$\delta_3 = Z_3 - 1$$

$$\delta_1^{3g} = \frac{g_0}{g} Z_3^{3/2} - 1 \quad \circledast$$

$$\delta_1^{4g} = \left(\frac{g_0}{g}\right)^2 Z_3^2 - 1 \quad \circledast$$

$$\delta_2^c = Z_2^c - 1$$

$$\delta_1^c = \left(\frac{g_0}{g}\right) Z_2^c Z_3^{1/2} - 1 \quad \circledast$$

From $\circledast$ we can derive relations among counter terms (at 1-loop)

$$\frac{g_0}{g} = 1 + \delta_1 - \delta_2 - \frac{1}{2}\delta_3$$

$$\frac{g_0}{g} = 1 + \delta_1^{3g} - \frac{3}{2}\delta_3$$

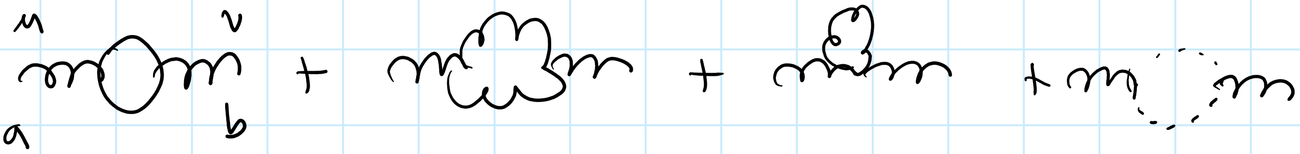
$$\frac{g_0}{g} = 1 - \delta_3 + \frac{\delta_1^{4g}}{2}$$

$$\frac{g_0}{g} = 1 + \delta_1^c - \delta_2^c + \frac{\delta_3}{2}$$

How do we compute these renormalisation constants at one loop?

→ perform 1-loop computation for the relevant propagator or vertex and extract the UV singularities.

$\boxed{\delta_3}$




$+$ 

$$\Pi^{ab}_{\mu\nu}(k)^2 = (k_\mu k_\nu - g_{\mu\nu}k^2)\,\delta^{ab}\,\Pi(k^2)$$

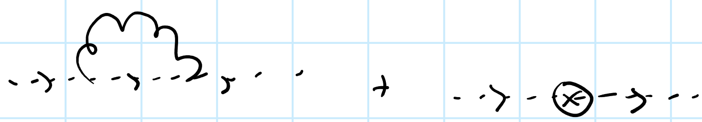
$$\Pi(k^2) = -\frac{g_s^2}{16\pi^2\epsilon}\left(10 - \frac{4}{3}n_f\right) + \delta_3 + \text{finite}$$

MS scheme: $$\delta_3 = \frac{g_s^2}{16\pi^2\epsilon}\left(5 - \frac{2}{3}n_f\right)$$

$\boxed{\delta_1}$


$$\delta_1 = \underbrace{\frac{g_{s,0}}{g_s}Z_2 Z_3^{1/2}}_{Z_1} - 1 = -\frac{g_s^2}{16\pi^2\epsilon}\,\frac{13}{3}$$

$\boxed{\delta_2^c}$

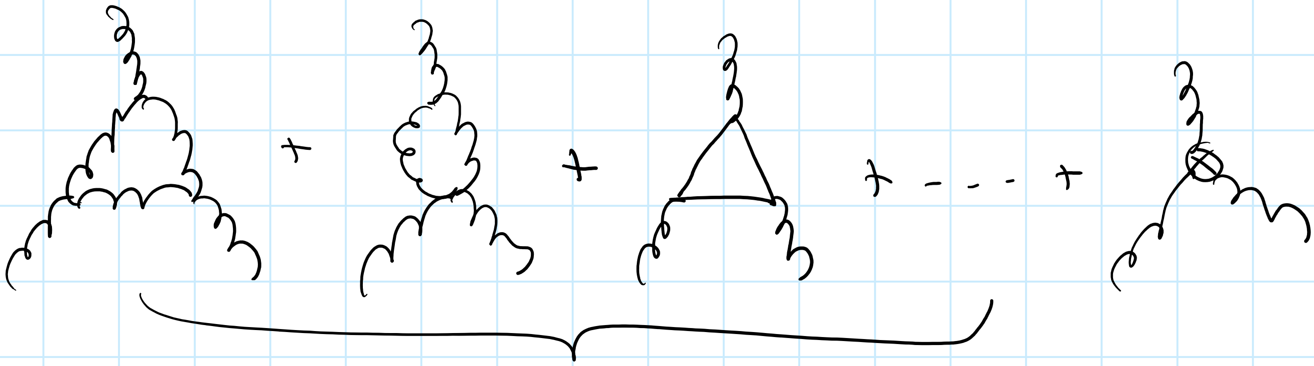



$$\delta_2^c = \frac{3}{32}\,\frac{g_s^2}{\pi^2\epsilon}$$

$\boxed{\delta_1^c}$

$$\delta_1^c = -\frac{3}{32}\frac{g_s^2}{\pi^2 \epsilon}$$

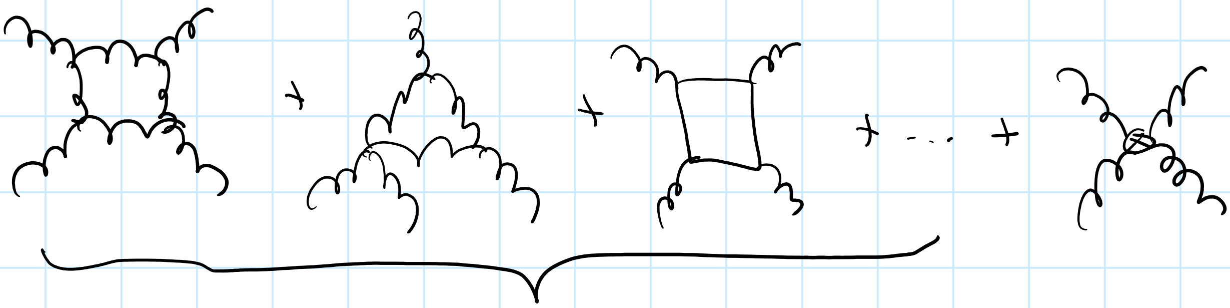
$\boxed{\delta_1^{3g}}$



8 diagrams

$$\delta_1^{3g} = -\frac{g_s^2}{2\pi^2\epsilon}\left(\frac{1}{4} - \frac{n_f}{12}\right)$$

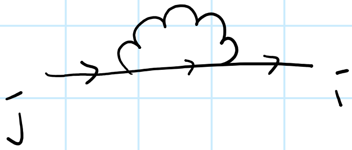
$\boxed{\delta_1^{4g}}$



24 diagrams

$$\delta_1^{4g} = -\frac{g_s^2}{2\pi^2\epsilon}\left(\frac{1}{8} + \frac{n_f}{12}\right)$$

## $\delta_2$ and $\delta_m$

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$$\delta^{ij}\left(-i\delta_m + i\not{p}\,\delta_2\right)$$

$$\Sigma_{ij} = \delta_{ij}\left[\frac{g_s^2}{12\pi^2\epsilon}\left(-\not{p} + 4m\right) - \delta_2\not{p} + m(\delta_2+\delta_m)\right] + \text{finite}$$

$$\delta_2 = -\frac{g_s^2}{12\pi^2\epsilon}\,, \qquad \delta_m = -\frac{g_s^2}{4\pi^2\epsilon}$$

From these renormalisation constants we can compute the renormalisation factor for $\alpha_s$

$$g_{s,0} = \underbrace{\frac{Z_1}{Z_2 Z_3^{1/2}}}_{\sqrt{Z_{\alpha_s}}}\, g_s$$

$$\boxed{\alpha_s = \frac{g_s^2}{4\pi}}$$

$$= \left(1 + \delta_1 - \delta_2 - \frac{\delta_3}{2}\right) g_s$$

$$\boxed{\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f}$$

$$= \left[1 - \frac{g_s^2}{16\pi^2\epsilon}\,\frac{13}{3} + \frac{g_s^2}{12\pi^2\epsilon} - \frac{g_s^2}{32\pi^2\epsilon}\left(5 - \frac{2}{3}n_f\right)\right] g_s$$

$$= \left[1 - \left(11 - \frac{2}{3}n_f\right)\frac{g_s^2}{32\pi^2\epsilon}\right] g_s$$

or

$$= \left[1 - \frac{\beta_0}{2}\,\frac{\alpha_s}{4\pi}\,\frac{1}{\epsilon}\right] g_s$$

## II. Running coupling from counter-terms

Including the $\mu$ factor in the $g_s$ renormalisation

$$g_{s0} = Z_g \mu^{\epsilon} g_s$$

$$\Downarrow$$

$$\alpha_{s,0} = \underbrace{Z_g^2}_{Z_{\alpha_s}} \mu^{2\epsilon} \alpha_s$$

$$Z_{\alpha_s} = \left(1 - \frac{\beta_0}{2}\frac{\alpha_s}{4\pi\epsilon}\right)^2$$

$$= 1 - \beta_0 \frac{\alpha_s}{4\pi\epsilon} + \mathcal{O}(\alpha_s^2)$$

The bare strong coupling does not depend on $\mu$

$$\frac{d\alpha_{s,0}}{d\ln\mu^2} = 0$$

$$\ln\alpha_{s,0} = \ln\alpha_s + \epsilon\ln\mu^2 + \ln Z_{\alpha_s}$$

$$\frac{d\ln\alpha_{s,0}}{d\ln\mu^2} = \frac{d\ln\alpha_s}{d\ln\mu^2} + \epsilon + \frac{d\ln Z_{\alpha_s}}{d\ln\mu^2}$$

$$= \epsilon + \frac{1}{\alpha_s}\frac{d\alpha_s}{d\ln\mu^2} + \frac{\partial\ln Z}{\partial\alpha_s}\frac{d\alpha_s}{d\ln\mu^2}$$

Define
$$\beta(\alpha_s) = \frac{d\alpha_s}{d\ln\mu^2}$$

then
$$0 = \epsilon + \frac{\beta(\alpha_s)}{\alpha_s} + \beta(\alpha_s)\,\frac{\partial \ln Z_{\alpha_s}}{\partial \alpha_s}$$

$$\ln Z_{\alpha_s} = \ln\left(1 - \frac{\alpha_s}{4\pi}\frac{\beta_0}{\epsilon} + O(\alpha_s^2)\right)$$

$$= -\frac{\alpha_s}{4\pi}\frac{\beta_0}{\epsilon} + O(\alpha_s^2)$$

$$\frac{\partial \ln Z_{\alpha_s}}{\partial \alpha_s} = -\frac{1}{4\pi}\frac{\beta_0}{\epsilon} + O(\alpha_s)$$

$$0 = \epsilon\alpha_s + \beta(\alpha_s) - \beta(\alpha_s)\frac{\alpha_s}{4\pi}\frac{\beta_0}{\epsilon} + O(\alpha_s^2)$$

$$\beta(\alpha_s) = -\epsilon\alpha_s\left[1 + \frac{\beta_0\alpha_s}{4\pi\epsilon} + \cdots\right]$$

$$\frac{d\alpha_s}{d\ln\mu^2} = -\epsilon\alpha_s - \frac{\alpha_s^2}{4\pi}\beta_0 + O(\alpha_s^3)$$

taking $\epsilon \to 0$

$$\boxed{\frac{d\alpha_s}{d\ln\mu^2} = -\frac{\alpha_s^2}{4\pi}\beta_0 + O(\alpha_s^3)}$$

Solving this one-loop renormalization group equation:

$$\frac{d\alpha_s}{\alpha_s^2} = -\frac{\beta_0}{4\pi}\, d\ln\mu^2$$

$$-\frac{1}{\alpha_s(\mu)} + \frac{1}{\alpha_s(\mu_0)} = -\frac{\beta_0}{4\pi}\left(\ln\mu^2 - \ln\mu_0^2\right)$$

$$\alpha_s(\mu) = \frac{1}{\frac{1}{\alpha_s(\mu_0)} + \frac{\beta_0}{4\pi}\ln\frac{\mu^2}{\mu_0^2}}$$

$$= \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)\frac{\beta_0}{4\pi}\ln\frac{\mu^2}{\mu_0^2}}$$

$\Lambda_{QCD}$ = location of Landau pole

$$1 + \alpha_s(\mu_0)\frac{\beta_0}{4\pi}\ln\frac{\mu^2}{\mu_0^2} = 0 \quad \text{for } \mu = \Lambda_{QCD}$$

$$1 + \alpha_s(\mu_0)\frac{\beta_0}{4\pi}\ln\frac{\Lambda_{QCD}^2}{\mu_0^2} = 0$$

$$-\frac{\beta_0}{4\pi}\ln\frac{\Lambda_{QCD}^2}{\mu_0^2} = \frac{1}{\alpha_s(\mu_0)}$$

$$\boxed{\alpha_s(\mu) = \frac{1}{\beta_0/4\pi\,\ln\mu^2/\Lambda_{QCD}^2}}$$

## III $\mu_R$ dependence on (QCD) observables.

Let's return to our first example in computing QCD corrections: the R-ratio

$$R = \frac{\sigma(\gamma^* \to q\bar{q})}{\sigma(\gamma^* \to \mu^+\mu^-)}$$

R can be expanded in series of $\alpha_s$

$$R = R^{(0)} + \frac{\alpha_s}{4\pi} R^{(1)} + \frac{\alpha_s}{(4\pi)^2} R^{(2)} + \dots$$

RG equation for R

$$R = R(\mu^2, \alpha_s(\mu))$$

Physical observable - like R ratio - can't depend on $\mu$

$$\mu^2 \frac{dR}{d\mu^2} = 0$$

$$\mu^2 \frac{\partial R}{\partial \mu^2} + \boxed{\mu^2 \frac{d\alpha_s}{d\mu^2}} \frac{\partial R}{\partial \alpha_s} = 0$$

$$\Rightarrow \mu^2 \frac{\partial R}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial R}{\partial \alpha_s} = 0$$

with $\beta(\alpha_s) = -\frac{\beta_0}{4\pi}\alpha_s^2 - \frac{\beta_1}{(4\pi)^2}\alpha_s^3 + \dots$

$\mu$ dependence at $O(\alpha_s)$

$$\frac{\partial R}{\partial \alpha_s} = \frac{1}{4\pi}R^{(1)} + \frac{2\alpha_s}{(4\pi)^2}R^{(2)} + \frac{3\alpha_s^2}{(4\pi)^3}R^{(3)} + \dots$$

$$\mu^2 \frac{\partial R}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial R}{\partial \alpha_s} = 0$$

$$\mu^2\left[\frac{\partial R^{(0)}}{\partial \mu^2} + \frac{\alpha_s}{4\pi}\frac{\partial R^{(1)}}{\partial \mu^2} + \left(\frac{\alpha_s}{4\pi}\right)^2\frac{\partial R^{(2)}}{\partial \mu^2} + O(\alpha_s^3)\right]$$

$$+\left(-\frac{\alpha_s^2}{4\pi}\beta_0 - \frac{\alpha_s^3}{(4\pi)^2}\beta_1 + O(\alpha_s^4)\right)$$

$$\times\left(\frac{1}{4\pi}R^{(1)} + \frac{2\alpha_s}{(4\pi)^2}R^{(2)} + \frac{3\alpha_s^2}{(4\pi)^3}R^{(3)} + O(\alpha_s^4)\right) = 0$$

Take $O(\alpha_s^0)$

$$\mu^2\frac{\partial R^{(0)}}{\partial \mu^2} = 0 \rightarrow \text{no } \mu \text{ dependence at LO}$$

now take $O(\alpha_s)$

$$\mu^2\frac{\partial R^{(1)}}{\partial \mu^2} = 0 \rightarrow \text{also no } \mu \text{ dependence at NLO}$$

consider the $\mathcal{O}(\alpha_s^2)$ terms

$$\frac{\mu^2}{(4\pi)^2}\frac{\partial R^{(2)}}{\partial \mu^2} - \frac{\beta_0}{(4\pi)^2} R^{(1)} = 0$$

$R^{(1)}$ does not depend on $\mu$

$$R^{(2)} = \beta_0 R_0^{(1)} \int \frac{\partial \mu^2}{\mu^2}$$

$$= \beta_0 R_0^{(1)} \ln(\mu^2) + R_0^{(2)}$$

$\mathcal{O}(\alpha_s^3)$ terms

$$\frac{\mu^2}{(4\pi)^3}\frac{\partial R^{(3)}}{\partial \mu^2} - \frac{\beta_1}{(4\pi)^3} R^{(1)} - \frac{2\beta_0}{(4\pi)^3} R^{(2)} = 0$$

$$\mu^2 \frac{\partial R^{(3)}}{\partial \mu^2} - \beta_1 R_0^{(1)} - 2\beta_0 \left( R_0^{(2)} + \beta_0 R_0^{(1)} \ln\mu^2 \right) = 0$$

$$R^{(3)} = R_0^{(3)} + \beta_1 R_0^{(1)} \ln\mu^2 + 2\beta_0 R_0^{(2)} \ln\mu^2$$

$$+ \beta_0^2 R_0^{(1)} \ln^2\mu^2$$

$$= R_0^{(3)} + \left[\beta_1 R_0^{(1)} + 2\beta_0 R_0^{(2)}\right] \ln\mu^2$$

$$+ \beta_0^2 R_0^{(1)} \ln^2\mu^2$$

For a generic QCD cross section

$$\hat{\sigma} = (4\pi\alpha_s)^n \left[ \hat{\sigma}^{(0)} + \frac{\alpha_s}{4\pi}\hat{\sigma}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{\sigma}^{(2)} + \dots \right]$$

RG equation

$$\mu^2 \frac{d\hat{\sigma}}{d\mu^2} + \beta(\alpha_s) \frac{\partial \hat{\sigma}}{\partial \alpha_s} = 0$$

doing the same as in R ratio,

$$\hat{\sigma}(\mu) = (4\pi\alpha_s(\mu))^n \Bigg[ \hat{\sigma}^{(0)}$$

$$+ \frac{\alpha_s(\mu)}{4\pi}\left( \hat{\sigma}_0^{(1)} + n\beta_0 \hat{\sigma}^{(0)} \ln\mu^2 \right)$$

$$+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \Big\{ \hat{\sigma}_0^{(2)} + \left( (n+1)\beta_0 \hat{\sigma}^{(1)} + n\beta_1 \hat{\sigma}^{(0)} \right) \ln\mu^2$$

$$+ \frac{n(n+1)}{2} \beta_0^2 \hat{\sigma}^{(0)} \ln^2\mu^2 \Big\} + \mathcal{O}(\alpha_s^3) \Bigg]$$

where $\sigma_0^{(i)}$ is the scale independent cross section evaluated at $\mu = 1$.

Consider NLO cross section

$$\hat{\sigma}_{NLO} = (4\pi\alpha_s)^n \left[ \hat{\sigma}^{(0)} + \frac{\alpha_s}{4\pi}\left( \hat{\sigma}_0^{(1)} + n\beta_0 \hat{\sigma}^{(0)} \ln\mu^2 \right) \right]$$

$$\frac{d\hat{\sigma}_{NLO}}{d\ln\mu^2} \sim \alpha_s^n \alpha_s^2 + \dots$$

$\rightarrow$ scale variations to estimate missing higher-order terms.

Naively, in the fixed order calculation truncating perturbative expansion at $\mathcal{O}(\alpha^{n+p})$ results in residual $\mu_R$-dependence $\sim \mathcal{O}(\alpha_s^{n+p+1})$

→ In hadronic collision, $\mu_F$ dependence should be taken into account as well. → DGLAP equations

$\sim \ln \mu_F^2 \cdot P_{ij} \otimes \hat{\sigma}_{jk}^{(n)} + \ldots$

+ Caveat : - new channels open up at higher order
- large logarithms

scale variations to estimate missing higher order terms sometimes not an optimal proxy