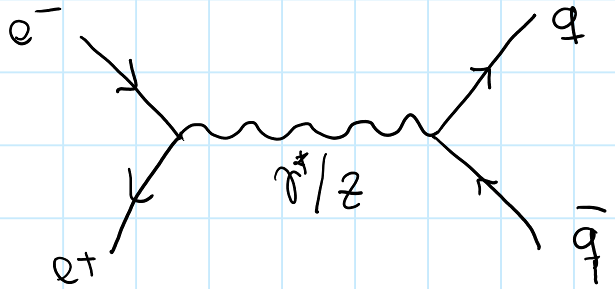
QCD Lecture 2

Many of the basic ideas and properties of perturbative QCD can be illustrated by considering the production of hadrons in electron-positron annihilation.

We will consider the following process

$$e^-(p_1) \rightarrow e^+(p_2) \rightarrow q(p_3) + \bar{q}(p_4)$$

represented by the Feynman diagram below



Even through we measure hadrons in the final state, summing over the accessible quarks gives correct inclusive results away from the resonances regions (like J/ψ, Υ) since hadronization changes the composition of the final state, but not the total probability summed over all hadronic states.

So,

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = \sum_q \sigma(e^+e^- \rightarrow q\bar{q})$$

$\downarrow$

total production rate aka cross section (more later)

We particulary interested in the following ratio:

$$R = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

This ratio offers a clean, direct probe to measure the number of quark colour N_c.

To simplify the technical calculation in this lecture, we will work with $\gamma^* \to$ hadrons and $\gamma^* \to \mu^+\mu^-$ and form the ratio. The leptonic part $(e^+e^- \to \gamma^*)$ is common to both processes

Let's now compute the tree-level matrix element

$$M_q = \bar{u}(p_3)\, iQ_q e\, \delta_{ij}\, \gamma^\mu\, v(p_4)\, \epsilon^*_\mu(p_{34})$$

$$= ieQ_q \delta_{ij}\, \epsilon_\mu^*(p_{34})\, A_q^\mu$$

$$A_q^\mu = \bar{u}(p_3)\gamma^\mu v(p_4)$$

Polarisation sum for off-shell photon

$$\sum_\lambda \epsilon_\lambda^{*\mu}(q)\, \epsilon_\lambda^{\mu}(q) = -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}$$

But the second term will not contribute since $p_{34}^\mu A_{q\,\mu} = 0$

Using spinor helicity formalism

$$A_q^{\mu +-} = [3\gamma^\mu 4\rangle \qquad A_q^{\mu -+} = \langle 3\gamma^\mu 4]$$

$$\sum_{col}\sum_{\lambda} |M_q|^2 = e^2 Q_q^2 \delta_{i\bar{i}} \left(-g_{\mu\nu}\right) A_q^\mu (A_q^\nu)^*$$

$$= -e^2 Q_q^2 N_c \; A_q^\mu (A_{q\mu})^*$$

$$\sum_{col}\sum_{\lambda} |M_q^{+-}|^2 = \sum_{col}\sum_{\lambda} |M_q^{-+}|^2$$

$$= -e^2 Q_q^2 N_c \; \langle 3\gamma_\mu 4]\langle 4\gamma^\mu 3]$$

$$= -2e^2 Q_q^2 N_c \; \langle 34\rangle[34]$$

$$= 2e^2 Q_q^2 N_c \; \underbrace{2p_3\cdot p_4}_{s}$$

$$\sum_{col}\sum_{hel}\sum_{\lambda} |M_q|^2 = 4e^2 Q_q^2 N_c s$$

Averaging over initial state off-shell photon polarization

$$\overline{\sum}|M_q|^2 = \frac{1}{3}\sum_{col}\sum_{hel}\sum_{\lambda} |M_q|^2$$

$$= \frac{4}{3} e^2 Q_q^2 N_c s$$

for the $\gamma^* \to \mu^-\mu^+$ squared matrix element the result differs by the fractional charge Q_i and N_c

$$\overline{\sum_i} |M_e|^2 = \frac{4}{3} e^2 s$$

Now we can compute the production rate / cross section

From scattering amplitudes to cross section

$$d\sigma = \frac{1}{\text{flux}} \overline{\sum} |M|^2 \, d\Phi_n$$

proportional to number of events

incoming flux

n-body Lorentz invariant phase space

where
$$d\Phi_n = \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}\left(P_{ini} - \sum_i P_i\right)$$

In our $\gamma^* \to q\bar{q}$ example, due to the absence of e^+e^- initial state, what we compute is not cross section, but decay rate instead.

$$\Gamma_{\gamma^* \to q\bar{q}} = \frac{1}{2\sqrt{s}} \int d\Phi_2 \overline{\sum_i} |M_q|^2$$

our R ratio is now

$$R = \frac{\Gamma_{\gamma^* \to \text{hadrons}}}{\Gamma_{\gamma^* \to \mu^+\mu^-}} = \frac{\frac{1}{2\sqrt{s}} \int d\Phi_2 \overline{\sum_i} |M_q|^2}{\frac{1}{2\sqrt{s}} \int d\Phi_2 \overline{\sum_i} |M_e|^2}$$

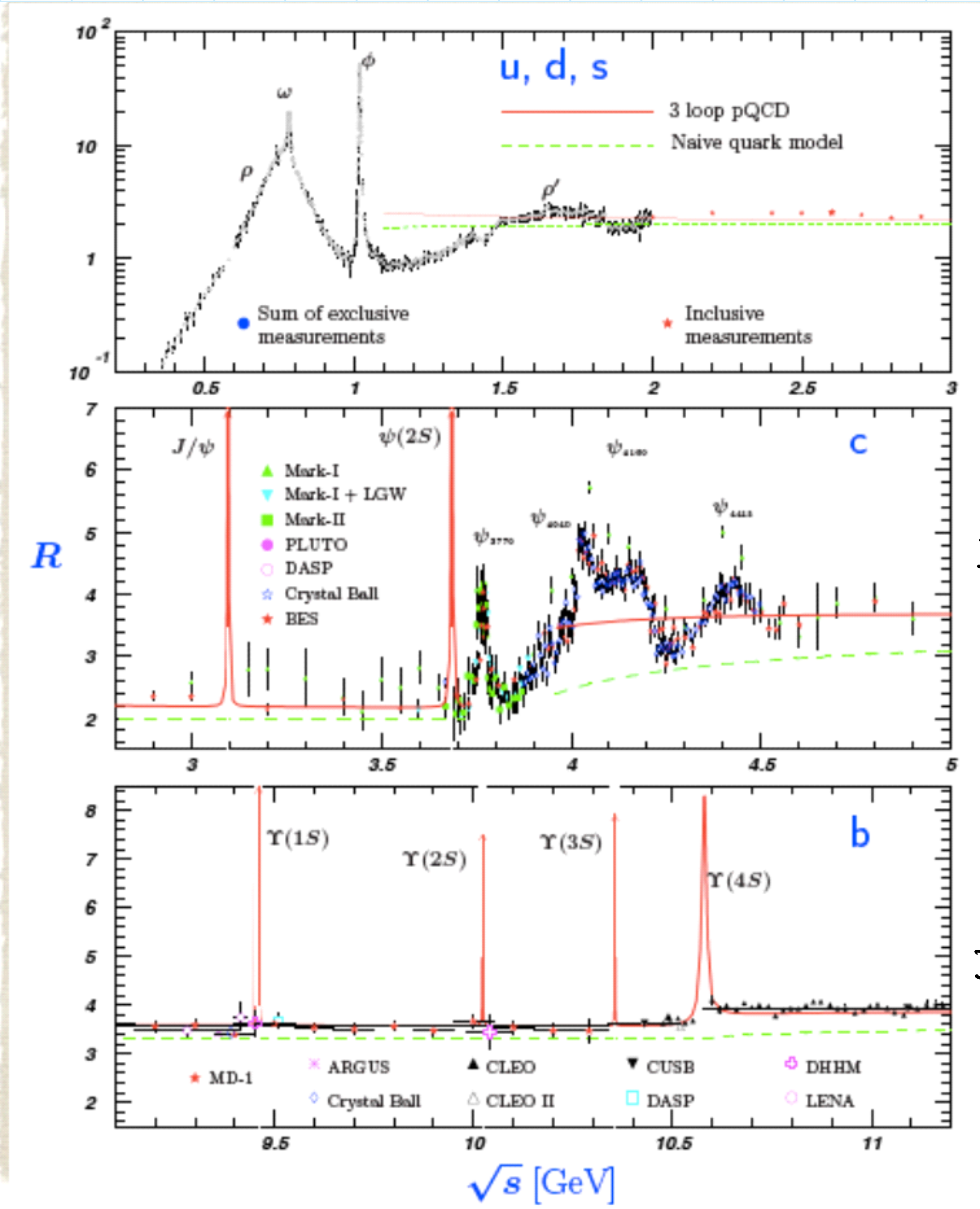
We will not perform the phase space integral explicitly but instead we use the fact that

$$\sum_i |M_q|^2 = Q_q^2 N_c \sum_i |M_e|^2$$

such that

$$R = N_c \sum_q Q_q^2$$

Actual comparison with data for $R = \dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$



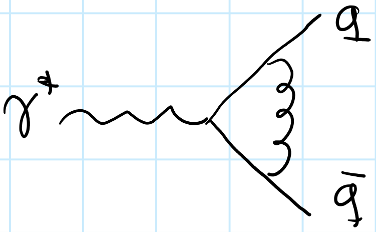
$$3 \times \left[\left(\frac{2}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2\right] = 2$$

$$3 \times \left[2\left(\frac{2}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2\right] = \frac{10}{3}$$

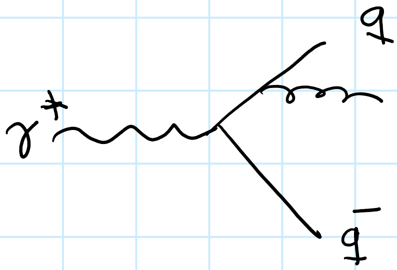
$$3 \times \left[2\left(\frac{2}{3}\right) + 3\left(\frac{1}{3}\right)^2\right] = \frac{11}{3}$$

QCD corrections to R ratio

The R ratio receives higher order QCD corrections from the virtual and real gluon emission in $\gamma^* \to q\bar{q}$ process. Example Feynman diagrams are



virtual



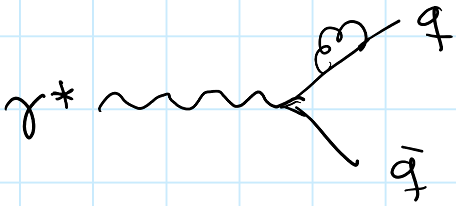
real

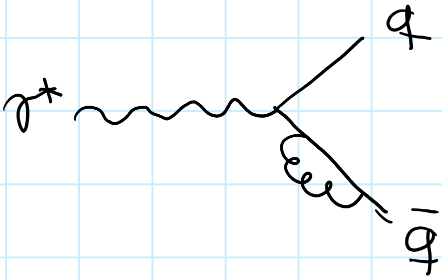
Let's do virtual corrections first.

Virtual QCD corrections

consists of 1-loop vertex and self energy corrections

The self energy corrections



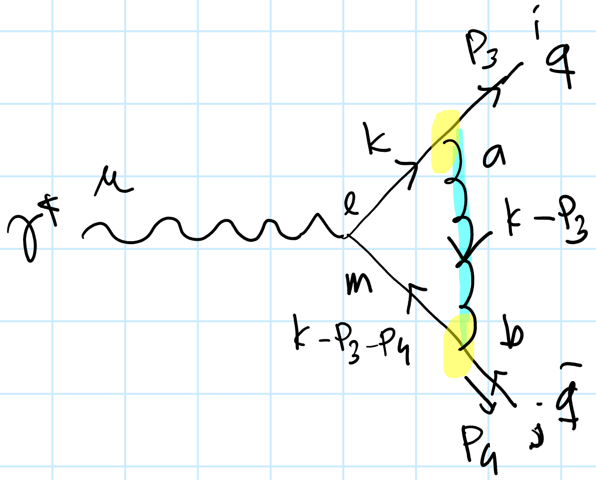


for massless quark the self energy vanishes since it is made up only of scaleless integrals (more later)

$\Rightarrow$ use trace techniques instead of spinor helicity

$\hookrightarrow$ easier to work in d-dimensions for this example

Let's now consider the one-loop vertex correction



$$p = p_3 + p_4$$

$$M_\mu^{(1)} = \int \frac{d^4k}{(2\pi)^4}\, \bar u(p_3)\left[i g_s t^a_{il}\gamma^\rho\right] i\frac{\not k}{k^2}\left[ieQ_q\gamma_\mu\delta_{lm}\right] i\frac{\not k - \not p_3 - \not p_4}{(k-p_3-p_4)^2}$$

$$\times\left[i g_s t^b_{mj}\gamma^\sigma\right] v(p_4) \times \frac{-i\delta^{ab} g_{\rho\sigma}}{(k-p_3)^2}\, \epsilon^*_\mu(p)$$

$$M_\mu^{(1)} = t^a_{il}\,\delta_{lm}\, t^b_{mj}\,\delta^{ab}\; e Q_q g_s^2$$

$$\times \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{\bar u(p_3)\gamma^\rho \not k \gamma_\mu (\not k - \not p_3 - \not p_4)\gamma_\rho v(p_4)}{k^2 (k-p_3)^2 (k-p_3-p_4)^2}}_{\Lambda_\mu}$$

$$= \underbrace{t^a_{im} t^a_{mj}}_{C_F} e Q_q g_s^2 \Lambda^{(1)}_\mu$$

$$\frac{1}{2}\left(\delta_{ij}\delta_{mm} - \frac{1}{N_c}\delta_{im}\delta_{mj}\right) = \overbrace{\left(\frac{N_c}{2} - \frac{1}{2N_c}\right)}^{C_F}\delta_{ij}$$

$$= C_F\,\delta_{ij}\, e Q_q g_s^2 \Lambda^{(1)}_\mu$$

$$\Lambda_\mu^{(1)} = \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p_3)\gamma^\beta \not{k} \gamma_\mu (\not{k} - \not{p}_3 - \not{p}_4)\gamma_\beta v(p_4)}{k^2 (k-p_3)^2 (k-p_3-p_4)^2}$$

this 1-loop vertex correction is IR divergent (as $k \to 0$)

by naive power counting, it's logarithmically UV divergent

$\Rightarrow$ work with dimensional regularization (dimreg)

$\underline{\underline{d = 4 - 2\epsilon}}$ space-time dimensions

So now

$$\Lambda_\mu^{(1)} = \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{\bar{u}(p_3)\gamma^\beta \not{k} \gamma_\mu (\not{k} - \not{p}_3 - \not{p}_4)\gamma_\beta v(p_4)}{k^2 (k-p_3)^2 (k-p_3-p_4)^2}$$

note that the dimensions of g_s is $[g_s] = \epsilon$ in $d = 4 - 2\epsilon$

include $\mu^{2\epsilon}$ to keep g_s dimensionless

$\int d^4x \, g_s \bar{\psi} \not{A} \psi$

$\Rightarrow$ algebra of γ matrices also in d-dimensions

γ matrix relations:

$\gamma^\mu \gamma_\mu = d$

$\gamma^\mu \gamma^\nu = 2g^{\mu\nu} - \gamma^\nu \gamma^\mu$

$$\gamma^\nu \gamma_\mu \gamma_\nu = \gamma^\nu (2g_{\mu\nu} - \gamma_\nu \gamma_\mu) = 2\gamma_\mu - d\gamma_\mu$$

$$= (2-d)\gamma_\mu$$

$\rightarrow$ get rid of β contraction

$$\gamma^\beta \not{k} \gamma_\mu (\not{k} - \not{p}_3 - \not{p}_4)\gamma_\beta = \gamma^\beta \not{k} \gamma_\mu \left[2(k-p_3-p_4)_\beta - \gamma_\beta (\not{k} - \not{p}_3 - \not{p}_4) \right]$$

$$= 2(\not{k} - \not{p}_3 - \not{p}_4)\not{k}\gamma_\mu - \underline{\gamma^\beta \not{k} \gamma_\mu \gamma_\beta (\not{k} - \not{p}_3 - \not{p}_4)} \; *$$

$$* = -\gamma^\rho \not{k}\gamma_\mu\gamma_\rho(\not{k}-\not{p}_3-\not{p}_4)$$

$$= -\left(2k^\rho - \not{k}\gamma^\rho\right)\left(2g_{\mu\rho} - \gamma_\rho\gamma_\mu\right)(\not{k}-\not{p}_3-\not{p}_4)$$

$$= -\left[4k_\mu - 2\not{k}\gamma_\mu - 2\not{k}\gamma_\mu + \not{k}\gamma^\rho\gamma_\rho\gamma_\mu\right](\not{k}-\not{p}_3-\not{p}_4)$$

$$= -\left[4k_\mu - 4\not{k}\gamma_\mu + d\,\not{k}\gamma_\mu\right](\not{k}-\not{p}_3-\not{p}_4)$$

$$= \left[-4k_\mu + (4-d)\not{k}\gamma_\mu\right](\not{k}-\not{p}_3-\not{p}_4)$$

$$\gamma^\rho\not{k}\gamma_\mu(\not{k}-\not{p}_3-\not{p}_4)\gamma_\rho = 2(\not{k}-\not{p}_3-\not{p}_4)\not{k}\gamma_\mu - 4k_\mu(\not{k}-\not{p}_3-\not{p}_4) + (4-d)\not{k}\gamma_\mu(\not{k}-\not{p}_3-\not{p}_4)$$

Form factor projection

$$\Lambda^{(1)}_\mu = \bar{u}(p_3)\gamma_\mu v(p_4) F^{(1)}$$

$$\sum_{\text{spin}}\left[\bar{v}(p_4)\gamma^\mu u(p_3)\right]\Lambda_\mu = F^{(1)}\sum_{\text{spin}}\bar{v}(p_4)\gamma^\mu u(p_3)\bar{u}(p_3)\gamma_\mu v(p_4)$$

$$= F^{(1)}\mathrm{Tr}\left[\not{p}_4\gamma^\mu\not{p}_3\gamma_\mu\right]$$

$$= F^{(1)}\mathrm{Tr}\left[\not{p}_4\gamma^\mu\left(2p_{3\mu} - \gamma_\mu\not{p}_3\right)\right]$$

$$= F^{(1)}\mathrm{Tr}\left[2\not{p}_4\not{p}_3 - d\,\not{p}_4\not{p}_3\right]$$

$$= F^{(1)}\mathrm{Tr}\left[(2-d)\not{p}_4\not{p}_3\right]$$

$$= F^{(1)}(2-d)\,4p_4\cdot p_3$$

$$= 2F^{(1)}(2-d)\,s$$

$$F^{(1)} = \frac{1}{2s(2-d)} \underbrace{\sum_{\text{spin}} [\bar{v}(p_4)\gamma^\mu u(p_3)]\, \Lambda^{(1)}_\mu}_{\star}$$

just consider the loop numerator of $\Lambda^{(1)}_\mu$

$$\mathrm{Num}(\star) = \sum_{\text{spin}} [\bar{v}(p_4)\gamma^\mu u(p_3)]$$
$$\times \bar{u}(p_3)\Big[2(\not{k}-\not{p}_3-\not{p}_4)\not{k}\gamma_\mu - 4k_\mu(\not{k}-\not{p}_3-\not{p}_4)$$
$$+ (4-d)\not{k}\gamma_\mu(\not{k}-\not{p}_3-\not{p}_4)\Big] v(p_4)$$

$$= \mathrm{Tr}\Big[2\not{p}_4\gamma^\mu\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)\not{k}\gamma_\mu$$
$$-4\not{p}_4\not{k}\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)$$
$$+(4-d)\not{p}_4\gamma^\mu\not{p}_3\not{k}\gamma_\mu(\not{k}-\not{p}_3-\not{p}_4)\Big]$$

$$= \mathrm{Tr}\Big[2(2-d)\not{p}_4\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)\not{k}$$
$$-4\not{p}_4\not{k}\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)$$
$$+(4-d)\not{p}_4(2p_3^\mu-\not{p}_3\gamma^\mu)\not{k}\gamma_\mu(\not{k}-\not{p}_3-\not{p}_4)\Big]$$

$$= \mathrm{Tr}\Big[2(2-d)\not{p}_4\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)\not{k}$$
$$-4\not{p}_4\not{k}\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)$$
$$+2(4-d)\not{p}_4\not{k}\not{p}_3(\not{k}-\not{p}_3-\not{p}_4)$$
$$-(4-d)(2-d)\not{p}_4\not{p}_3\not{k}(\not{k}-\not{p}_3-\not{p}_4)\Big]$$

apply trace identity

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

$$\begin{aligned}\mathrm{Num}(\not{A}) = & -4(d-4)(d-2)\, p_3\cdot p_4\, k^2 \\ & -16(d-2)\, k\cdot p_3\, k\cdot p_4 \\ & +8(d-4)(d-2)\, p_3\cdot p_4\, k\cdot p_3 \\ & +16(d-2)\, p_3\cdot p_4\, k\cdot p_4\end{aligned}$$

$$F^{(1)} = \frac{\mu^{2\epsilon}}{2s(2-d)} \int \frac{d^d k}{(2\pi)^d} \frac{\mathrm{Num}(\not{A})}{D_1 D_2 D_3}$$

where $D_1 = k^2$, $D_2 = (k-p_3)^2$, $D_3 = (k-p_3-p_4)^2$

express k^2, $k\cdot p_3$, $k\cdot p_4$ in terms of D_i

$$D_2 - D_1 = k^2 - 2k\cdot p_3 - k^2 = -2k\cdot p_3 \Rightarrow k\cdot p_3 = -\frac{D_2 - D_1}{2}$$

$$\begin{aligned}D_3 - D_2 &= \cancel{k^2} \cancel{-2k\cdot p_3} - 2k\cdot p_4 + 2p_3\cdot p_4 \cancel{-k^2} \cancel{+2k\cdot p_3} \\ &= -2k\cdot p_4 + 2p_3\cdot p_4 \Rightarrow k\cdot p_4 = -\frac{D_3}{2} + \frac{D_2}{2} + p_3\cdot p_4\end{aligned}$$

Notation for Feynman integral

$$I_{n_1 n_2 n_3} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3}}$$

$$
\begin{aligned}
\mathrm{Num}(\not{A}) = &-4(d-4)(d-2)\, p_3\cdot p_4\, D_1 \\
&-16(d-2)\left[-\frac{D_2}{2}+\frac{D_1}{2}\right]\left[-\frac{D_3}{2}+\frac{D_2}{2}+p_3\cdot p_4\right] \\
&+8(d-4)(d-2)\, p_3\cdot p_4 \left(-\frac{D_2}{2}+\frac{D_1}{2}\right) \\
&+16(d-2)\, p_3\cdot p_4 \left(-\frac{D_3}{2}+\frac{D_2}{2}+p_3\cdot p_4\right)
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{Num}(\not{A})}{(2-d)\,D_1D_2D_3} = &\;4\,\underbrace{\cancel{I_{001}}}_{\text{scaleless}} - 4\,\underbrace{\cancel{I_{010}}}_{\text{scaleless}} \\
&+4s\,\underbrace{\cancel{I_{011}}}_{\text{scaleless}} - 4\,I_{1-11} \\
&+4\,\underbrace{\cancel{I_{100}}}_{\text{scaleless}} + 2(d-8)\,s\,I_{101} \\
&+4s\,\underbrace{\cancel{I_{110}}}_{\text{scaleless}} - 4s^2\,I_{111}
\end{aligned}
$$

$$
= -4\,I_{1-11} + 2(d-8)\,s\,I_{101} - 4s^2\,I_{111}
$$

Scaleless integral vanishes in dimensional regularisation

$$
\int d^dk\,\frac{1}{k^2} \sim \frac{1}{\epsilon_{IR}} - \frac{1}{\epsilon_{UV}} = 0
$$

are the remaining 3 integrals independent?

$\Rightarrow$ NO, use Integration-by-parts technique

IBP reduction

IBP identities

$$0 = \int d^d k \, \frac{\partial}{\partial k^\mu} \left(\frac{v^\mu}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} \right)$$

$$v^\mu = \{ k, p_3, p_4 \}$$

1. Reducing I_{111} to I_{101}

$$0 = \int d^d k \, \frac{\partial}{\partial k_\mu} \left[\frac{(k-p_3)^\mu}{D_1 D_2 D_3} \right]$$

$$0 = \int d^d k \, \frac{\overset{\to d}{\partial k^\mu}}{\partial k_\mu} \left[\frac{1}{D_1 D_2 D_3} \right] - \int d^d k \, \frac{(k-p_3)^\mu \cdot \frac{\partial D_1}{\partial k_\mu}}{D_1^2 D_2 D_3}$$

$$- \int d^d k \, \frac{(k-p_3)^\mu \cdot \frac{\partial}{\partial k_\mu} D_2}{D_1 D_2^2 D_3} - \int d^d k \, \frac{(k-p_3)^\mu \frac{\partial}{\partial k_\mu} D_3}{D_1 D_2 D_3^2}$$

$$\frac{\partial}{\partial k_\mu} D_1 = \frac{\partial}{\partial k_\mu} k^2 = \frac{\partial}{\partial k_\mu} k_\mu k^\mu = 2k^\mu$$

$$\frac{\partial}{\partial k_\mu} D_2 = \frac{\partial}{\partial k_\mu} (k-p_3)^2 = \frac{\partial}{\partial k_\mu} (k^2 - 2k \cdot p_3) = 2(k-p_3)^\mu$$

$$\frac{\partial}{\partial k_\mu} D_3 = 2(k-p_3-p_4)^\mu$$

$$0 = d I_{111} - 2\int d^dk \frac{(k-p_3)\cdot k}{D_1^2 D_2 D_3} - 2\int d^dk \frac{(k-p_3)\cdot(k-p_3)}{D_1 D_2^2 D_3}$$

$$- 2\int d^dk \frac{(k-p_3)\cdot(k-p_3-p_4)}{D_1 D_2 D_3^2}$$

$$(k-p_3)\cdot k = k^2 - k\cdot p_3 = D_1 + \frac{D_2 - D_1}{2} = \frac{D_1}{2} + \frac{D_2}{2}$$

$$(k-p_3)^2 = D_2$$

$$(k-p_3)\cdot(k-p_3-p_4) = (k-p_3)^2 - (k-p_3)\cdot p_4$$

$$= D_2 - k\cdot p_4 + p_3\cdot p_4$$

$$= D_2 + \frac{D_3}{2} - \frac{D_2}{2} - \frac{s}{2} + \frac{s}{2}$$

$$= \frac{D_2}{2} + \frac{D_3}{2}$$

$$0 = d J_{111} - I_{111} - I_{201} - 2 I_{111} - I_{102} - I_{111}$$

$$0 = (d-4) I_{111} - I_{201} - I_{102} \qquad \circledast$$

derive IBP relations for I_{102} and I_{201}

$$0 = \int d^dk \frac{\partial}{\partial k_\mu}\left[\frac{k^\mu}{D_1 D_3}\right]$$

$$0 = d\, I_{101} - \int d^dk \, \frac{k^\mu \frac{\partial D_1}{\partial k_\mu}}{D_1^2 D_3} - \int d^dk \, \frac{k^\mu \frac{\partial D_3}{\partial k_\mu}}{D_1 D_3^2}$$

$$0 = d\, I_{101} - 2 \int d^dk \, \frac{\overset{\to D_1}{k \cdot k}}{D_1^2 D_3} - 2 \int d^dk \, \frac{k \cdot (k - p_3 - p_4)}{D_1 D_3^2}$$

$$= d\, I_{101} - 2 \int d^dk \, \frac{1}{D_1 D_3} - \int d^dk \, \frac{D_1 + D_3 - s}{D_1 D_3^2}$$

$$\left[\begin{aligned} k \cdot (k - p_3 - p_4) &= k^2 - k \cdot p_3 - k \cdot p_4 \\ &= D_1 + \cancel{\frac{D_2}{2}} - \frac{D_1}{2} + \frac{D_3}{2} - \cancel{\frac{D_2}{2}} - \frac{s}{2} \\ &= \frac{D_1}{2} + \frac{D_3}{2} - \frac{s}{2} \end{aligned} \right]$$

$$0 = (d-2) I_{101} - I_{101} + s\, I_{102}$$

$$= (d-3) I_{101} + s\, I_{102}$$

$$\Rightarrow \quad I_{102} = -\frac{(d-3)}{s} I_{101}$$

by symmetry $\quad I_{201} = -\dfrac{(d-3)}{s} I_{101}$

$$(d-4)\, I_{111} = I_{201} + I_{102}$$

$$= -\frac{2(d-3)}{s} I_{101}$$

$$\boxed{I_{111} = -\frac{2(d-3)}{s(d-4)} I_{101}}$$

2. Reducing I_{1-11} to I_{101}

done similarly as I_{111}

$$0 = \int d^d k \frac{\partial}{\partial k_\mu}\left[\frac{k^\mu D_2}{D_1 D_3}\right]$$

$$\hookrightarrow \quad 0 = (d-2)\, I_{1-11} + s\, I_{1-12} \qquad (1)$$

$$0 = \int d^d k \frac{\partial}{\partial k_\mu}\left[\frac{(p_3+p_4)^\mu D_2}{D_1 D_3}\right]$$

$$0 = s\, I_{1-12} - s\, I_{2-11} \qquad (2)$$

$$\hookrightarrow \quad I_{1-12} = I_{2-11}$$

$$0 = \int d^d k \frac{\partial}{\partial k_\mu}\left[\frac{p_3^\mu}{D_1 D_3}\right]$$

$$\Rightarrow \quad 0 = -I_{101} + I_{2-11} + I_{1-12} + s\, I_{102} \qquad (3)$$

from ①, ②, ③ and $I_{102} = -\frac{d-3}{s} I_{101}$

we get

$$I_{1-11} = -\frac{s}{2} I_{101}$$

Now substitute I_{111} and I_{1-11} in terms of I_{101}

$$\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Num}(\not{A})}{(2-d)\, D_1 D_2 D_3} = -4 I_{1-11} + 2(d-8) s\, I_{101} - 4s^2 I_{111}$$

$$= \frac{2(16 - 7d + d^2) s}{d-4} I_{101}$$

$$F^{(1)} = \frac{16 - 7d + d^2}{d-4} \mu^{2\epsilon} I_{101}$$

evaluating the "master" integral I_{101}

$$I_{101} = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 (k - p_3 - p_4)^2}$$

Feynman parametrisation

$$I_{101} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{D^2},$$

where

$$D = x(k - p_3 - p_4)^2 + (1-x) k^2$$

$$\mathbb{D} = \cancel{x k^2} + x(P_3+P_4)^2 - 2x k\cdot(P_3+P_4) + k^2 - \cancel{x k^2}$$

$$= k^2 + x(P_3+P_4)^2 - 2x k\cdot(P_3+P_4)$$

shift loop momentum

$$k' = k - x(P_3+P_4) \Rightarrow k = k' + x(P_3+P_4)$$

$$\mathbb{D} = k'^2 + \cancel{2x k'\cdot(P_3+P_4)} + x^2(P_3+P_4)^2$$

$$+ x(P_3+P_4)^2 - \cancel{2x k'\cdot(P_3+P_4)} - 2x^2(P_3+P_4)^2$$

$$= k'^2 + x(1-x)(P_3+P_4)^2$$

$$= k'^2 + x(1-x)s$$

$$I_{101} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k'^2 - \Delta]^2} \quad \text{with } \Delta = x(1-x)(-s)$$

Yellow Note (at the end of this document) #1

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k'^2 - \Delta]^2} = \frac{(-1)^2 i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$d = 4 - 2\epsilon \Rightarrow 2 - \frac{d}{2} = 2 - 2 + \epsilon = \epsilon$$

$$= \frac{i}{(4\pi)^{d/2}} \Gamma(\epsilon) \Delta^{-\epsilon}$$

$$I_{101} = \frac{i}{(4\pi)^2 (4\pi)^{-\epsilon}} \Gamma(\epsilon) (-s)^{-\epsilon} \int_0^1 dx\, x^{-\epsilon} (1-x)^{-\epsilon}$$

Euler Beta function

$$B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\int_0^1 dx\, x^{-\epsilon} (1-x)^{-\epsilon} = \int_0^1 x^{1-\epsilon-1} (1-x)^{1-\epsilon-1} dx$$

$$= B(1-\epsilon, 1-\epsilon)$$

$$= \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$\Gamma(1+1-2\epsilon)$
$=(1-2\epsilon)\Gamma$

Gamma function identity $\Gamma(1+z) = z\Gamma(z)$

$$I_{101} = \frac{i(4\pi)^\epsilon}{16\pi^2} (-s)^{-\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)\Gamma(1-2\epsilon)}$$

$$= i\left(\frac{4\pi}{-s}\right)^\epsilon \frac{1}{16\pi^2} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{1}{\epsilon} + 2 + 4\epsilon + \mathcal{O}(\epsilon^2)\right]$$

$$F^{(1)} = \frac{16-7d+d^2}{d-4} \mu^{2\epsilon} I_{101} = i\left(\frac{4\pi\mu^2}{-s}\right)^\epsilon \frac{1}{16\pi^2} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\times \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right]$$

Recall the expression for 1-loop amplitude

$$M_\mu^{(1)} = C_F\, \delta_{i\bar{j}}\, e\, Q_q\, g_s^2\, \Lambda_\mu^{(1)}$$

$$= C_F\, \delta_{i\bar{j}}\, e\, Q_q\, g_s^2\, \bar{u}(p_3)\gamma_\mu v(p_4)\, F^{(1)}$$

$$= i\, C_F\, \delta_{i\bar{j}}\, e\, Q_q\, g_s^2\, \bar{u}(p_3)\gamma_\mu v(p_4)$$
$$\times \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \frac{1}{16\pi^2} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$
$$\times \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right]$$

$$= C_F\, M_\mu^{(0)}\, \frac{\alpha_s}{4\pi} \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} r_\Gamma \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right)$$

$$M_\mu^{(0)} = i\, \delta_{i\bar{j}}\, e\, Q_q\, \bar{u}(p_3)\gamma_\mu v(p_4)$$

$$\alpha_s = \frac{g_s^2}{4\pi}, \qquad r_\Gamma = \frac{\Gamma^2(1-\epsilon)\,\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

The squared matrix element including one-loop corrections is given by

$$M = M^{(0)} + \alpha_s M^{(1)} + \cdots$$

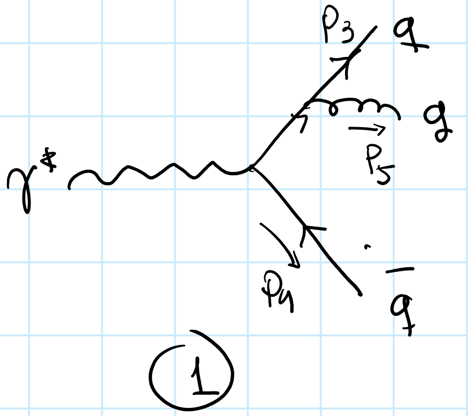
$$|M|^2 = |M^{(0)}|^2 + \alpha_s\left(M^{(0)*} M^{(1)} + M^{(0)} M^{(1)*}\right) + \cdots$$

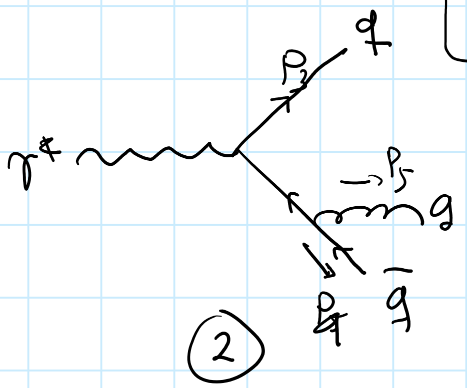
$$= |M^{(0)}|^2 + \alpha_s\, 2\,\mathrm{Re}\, M^{(0)*} M^{(1)} + \cdots$$

Real corrections

The real gluon emission proses is given by the following two Feynman diagrams

$$Q = p_3 + p_4 + p_5$$
$$Q^2 = s$$





$$M^{(0)}_{real} = M^{(0)}_{r1} + M^{(0)}_{r2}$$

$$M^{(0)\mu}_{r1} = \bar{u}(p_3)\, i g_s t^a_{ij} \not{\epsilon}^*(p_5) \cdot i \frac{\not{p}_3 + \not{p}_5}{(p_3+p_5)^2} \cdot ieQ_q \gamma^\mu \cdot v(p_4)$$

$$= -ieQ_q g_s t^a_{ij}\, \bar{u}(p_3) \not{\epsilon}^*(p_5)(\not{p}_3 + \not{p}_5)\gamma^\mu v(p_4) \frac{1}{s_{35}}$$

$$M^{(0)\mu}_{r2} = \bar{u}(p_3)\, ieQ_e \gamma^\mu\, i \frac{-\not{p}_4 - \not{p}_5}{(p_4+p_5)^2} \cdot i g_s t^a_{ij} \not{\epsilon}^*(p_5)\, v(p_4)$$

$$= ieQ_q t^a_{ij}\, \bar{u}(p_3)(\not{p}_4 + \not{p}_5) \not{\epsilon}^*(p_5) v(p_4) \frac{1}{s_{45}}$$

$$|M^{(0)}_{real}|^2 = \overbrace{\left(M^{(0)\mu}_{r_1}\right)^* M^{(0)\nu}_{r_1} \chi_{\mu\nu}}^{\mathrm{I}} + \overbrace{\left(M^{(0)\mu}_{r_2}\right)^* M^{(0)\nu}_{r_2} \chi_{\mu\nu}}^{\mathrm{II}}$$

$$+ \underbrace{\left(M^{(0)\mu}_{r_1}\right)^* M^{(0)\nu}_{r_2} \chi_{\mu\nu}}_{\mathrm{III}} + \underbrace{\left(M^{(0)}_{r_2}\right)^{\mu *} M^{(0)\nu}_{r_1} \chi_{\mu\nu}}_{\mathrm{IV}}$$

where $\chi_{\mu\nu} = -g_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2}$

$$\text{I} = e^2 Q_q^2 g_s^2 \underbrace{t^a_{ij} t^a_{ji}}_{\to N_c C_F} \chi_{\mu\nu} \sum_\lambda \varepsilon^*_\rho(p_5) \varepsilon_\sigma(p_5) \frac{1}{s_{35}^2}$$

$$\times \operatorname{tr}\left[\not{p}_3 \gamma^\rho (\not{p}_3 + \not{p}_5) \gamma^\mu \not{p}_4 \gamma^\nu (\not{p}_3 + \not{p}_5) \gamma^\sigma \right]$$

gluon polarisation sum

$$\sum_\lambda \varepsilon^*_\rho(p_5) \varepsilon_\sigma(p_5) = - g_{\rho\sigma} + \frac{p_{5\rho} n_\nu + n_\rho p_{5\nu}}{p \cdot n}$$

here we choose $n^\mu = p_3^\mu$.

working out the γ matrix algebra (in d dimensions!!) and evaluating the trace we have:

$$\text{I} = e^2 Q_q^2 g_s^2 N_c C_F \; 2(d-2) \left[\frac{2 p_3 \cdot p_4}{s} + \frac{(d_5 - 2) p_4 \cdot p_5}{p_3 \cdot p_5} \right]$$

$$\text{II} = e^2 Q_q^2 g_s^2 N_c C_F \; 2 \; (d-2) \left[\frac{2 p_3 \cdot p_4}{s} + \frac{(d-2) p_3 \cdot p_5}{p_4 \cdot p_5} \right.$$
$$\left. + 4 \frac{p_3 \cdot p_4 \, (p_3 \cdot p_4 + p_3 \cdot p_5)}{p_3 \cdot p_5 \; p_4 \cdot p_5} \right]$$

$$\text{III} = e^2 Q_q^2 g_s^2 N_c C_F \; 2(d-2) \left[d - 4 + 2 p_3 \cdot p_4 \left(\frac{1}{p_3 \cdot p_5} - \frac{1}{s} \right) \right]$$

$$\text{IV} = \text{III}$$

Now we perform change of variables, focusing on energy distribution. Let's define

$$x_3 = \frac{2p_3 \cdot Q}{Q^2} \qquad x_4 = \frac{2p_4 \cdot Q}{Q^2} \qquad x_5 = \frac{2p_5 \cdot Q}{Q^2}$$

remember $Q = p_3 + p_4 + p_5$ and $Q^2 = S$

In the CM frame of (345) - system

$$Q = (Q_0, 0, 0, 0)$$

So

$$x_3 = \frac{E_3}{Q_0/2} \qquad x_4 = \frac{E_4}{Q_0/2} \qquad x_5 = \frac{E_5}{Q_0/2}$$

These variables obey $0 < x_i < 1$

and $x_3 + x_4 + x_5 = 2$

we need to express dot products involving p_3, p_4, p_5 in terms of x_3, x_4, x_5

$$(p_3 + p_4)^2 = (Q - p_5)^2$$

$$2p_3 \cdot p_4 = Q^2 - 2p_5 \cdot Q = Q^2 - 2\,Q_0^2 \frac{E_5}{Q_0}$$

$$= Q^2 (1 - x_5)$$

$$= S(1 - x_5)$$

similarly

$$2p_3 \cdot p_5 = S(1 - x_4)$$

$$2p_4 \cdot p_5 = S(1 - x_3)$$

x_3 and x_4 are integrated over the region

$$x_3, x_4 < 1$$

$$x_5 = 2 - x_3 - x_4 < 1 \rightarrow x_3 + x_4 \geq 1$$

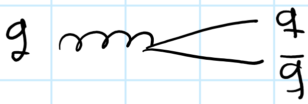
Consider the following limits

$x_3 \to 1$ 

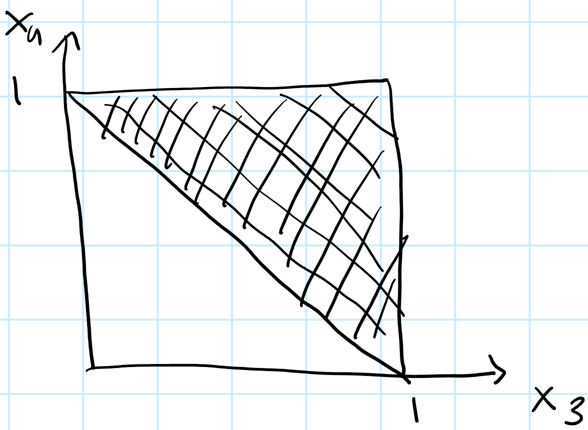
collinear $\bar{q}g$

$x_4 \to 1$ 

collinear qg

$x_5 \to 1$ 

collinear $q\bar{q}$



Back to squared real matrix element computation, after substituting $p_i \cdot p_j$ in terms of x_3, x_4, x_5

$$\sum |\mathcal{M}^{(0)}_{real}|^2 = e^2 Q_q^2 \, g_s^2 \, N_c \, C_F \, 8(1-\epsilon) \, \frac{x_3^2 + x_4^2 - \epsilon\,(2 - x_3 - x_4)^2}{(1-x_3)(1-x_4)}$$

Let's revisit the tree level $\gamma^* \to q\bar{q}$ squared amplitude now computed in d dimensions

$$\mathcal{M}^{(0)} = i e Q_q \delta_{ij} \, \epsilon_\mu^*(p_{34}) \, \bar{u}(p_3) \gamma^\mu v(p_4)$$

$$\sum |M^{(0)}|^2 = e^2 Q_q^2 N_c \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right)$$

$$\times \operatorname{tr}\left(\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu \right)$$

$$= 4 e^2 Q_q^2 \underbrace{2 p_3 \cdot p_4}_{\to s} (1-\epsilon)$$

Let's consider the 2-body phase space in d dimensions

$$\int d\Phi_2 = \int \frac{d^{d-1} p_3}{(2\pi)^{d-1} 2E_3} \frac{d^{d-1} p_4}{(2\pi)^{d-1} 2E_4} (2\pi)^d \delta^{(d)}(p_3 + p_4 - Q)$$

center of mass frame $\quad Q^\mu = (\sqrt{s}, 0, 0, 0)$

$$\delta^{(d)}(p_3 + p_4 - Q) = \delta(E_3 + E_4 - \sqrt{s}) \delta^{(d-1)}(\vec{p}_3 + \vec{p}_4)$$

integrating the $\delta^{(d-1)}(\vec{p}_3 + \vec{p}_4)\, d^{d-1} p_4$, $\quad \vec{p}_3 = -\vec{p}_4$, $\quad E_3 = E_4$

$$\int d\Phi_2 = \int \frac{d^{d-1} p_3}{(2\pi)^{d-2} 4E_3^2} \delta(2E_3 - \sqrt{s})$$

write p_3 integral measure in d-dimensional spherical coordinate

$$d^{d-1} p_3 = p_3^{d-2}\, dp_3\, d\Omega_{d-1}$$

$\rightsquigarrow$ parametrize vector in d dimensions

1 radial variable

$d-2$ polar angles

1 azimuthal angles

Parametrisation of momentum k in d dimensions

$$k^0 = k\cos\theta_1$$
$$k^1 = k\sin\theta_1\cos\theta_2$$
$$\vdots$$
$$k^{d-2} = k\sin\theta_1\ldots\sin\theta_{d-2}\cos\theta_{d-1}$$
$$k^{d-1} = k\sin\theta_1\ldots\sin\theta_{d-2}\sin\theta_{d-1}$$

θ_{d-1} → azimuthal angle

The d-dimensional measure:

$$d^dk = k^{d-1}\,dk\,d\Omega_{d-1}$$

$$d\Omega_d = \sin^{d-2}\theta_1\sin^{d-3}\theta_2\cdots\sin\theta_{d-2}\,d\theta_1\ldots d\theta_{d-1}$$

$$\int d\Omega_d = \int_0^{\pi} d\theta_1\sin^{d-2}\theta_1\cdots\int_0^{\pi} d\theta_{d-2}\sin\theta_{d-2}\int_0^{2\pi} d\theta_{d-1}$$

$$= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

→ Yellow note (at the end of this document) ☆2

So
$$\int d\Omega_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}$$

$$\int d\Phi_2 = \int d\Omega_{d-1}\int dE_3\,E_3^{d-2}\,\delta(2E_3-\sqrt{s})\,\frac{1}{(2\pi)^{d-2}\,4E_3^2}$$

The $\gamma^* \to q\bar{q}$ production rate is

$$\Gamma^{(0)}_{\gamma^* \to q\bar{q}} = \frac{1}{2\sqrt{s}} \int d\Phi_2 \, \overline{\sum} |M^{(0)}|^2$$

$$= \frac{1}{2\sqrt{s}} \frac{1}{(2\pi)^{d-2}} \frac{2\pi^{(d-1)/2}}{4\,\Gamma\left(\frac{d-1}{2}\right)} \, 4 e^2 Q_q^2 \underbrace{\frac{(1-\epsilon)}{d-1}}_{\gamma^* \text{ polarization in } d\text{-dim}} N_c$$

$$\int dE_3 \frac{E_3^{d-2}}{E_3^2} \, \underbrace{\delta\left(2E_3 - \sqrt{s}\right)}_{\frac{1}{2}\delta\left(E_3 - \frac{\sqrt{s}}{2}\right)} \, s$$

$$= \frac{1}{2\sqrt{s}} \frac{1}{(2\pi)^{d-2}} \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} e^2 Q_q^2 (1-\epsilon) \frac{1}{2}$$

$$\times \frac{\left(\frac{\sqrt{s}}{2}\right)^{d-2}}{\left(\frac{\sqrt{s}}{2}\right)^2} \cdot s$$

$$= \alpha Q_q^2 N_c \sqrt{s} \, \frac{1-\epsilon}{3-2\epsilon} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

Now, let's do the 3-body phase space in d-dimensions

$$\int d\Phi_3 = \int \frac{d^{d-1}p_3}{(2\pi)^{d-1} 2E_3} \frac{d^{d-1}p_4}{(2\pi)^{d-1} 2E_4} \frac{d^{d-1}p_5}{(2\pi)^{d-1} 2E_5} (2\pi)^d \delta^{(d)}(p_3+p_4+p_5-Q)$$

integrate out p_5 $\qquad$ CM frame $Q = (\sqrt{s}, \vec{0})$

$$= \frac{1}{8(2\pi)^{2d-3}} \int \frac{d^{d-1}p_3 \, d^{d-1}p_4}{E_3 E_4 E_5} \delta(E_3+E_4+E_5-\sqrt{s})$$

$$= \frac{1}{8(2\pi)^{2d-3}} \int dE_3 \, E_3^{d-3} \, d\Omega_{d-1}^{(3)}$$

$$\times \, dE_4 \, E_4^{d-3} \, d\Omega_{d-1}^{(4)}$$

$$\times \, \frac{1}{E_5} \delta(E_3+E_4+E_5-\sqrt{s})$$

We will integrate $d\Omega_{d-1}^{(1)}$ (all angles of p_3)

For angles of p_4, we will integrate all angles

but angle w.r.t $p_3 \to \theta_{34}$

$$\int d\Omega_{d-1}^{(4)} = \int d\theta_{34} \sin^{d-3}\theta_{34} \int d\Omega_{d-2}^{(4)}$$

$$= \int_0^{\pi} \sin\theta_{34} \, d\theta_{34} \, \sin^{d-4}\theta_{34} \int d\Omega_{d-2}^{(4)}$$

$$= \int_{-1}^{1} d(\cos\theta_{34}) \sin^{d-4}\theta_{34} \int d\Omega_{d-2}^{(4)}$$

$$\int d\Phi_3 = \frac{1}{8(2\pi)^{2d-3}} \int dE_3\, E_3^{d-3}\, dE_4\, E_4^{d-3}\, d\Omega^{(3)}_{d-1}\, d\Omega^{(4)}_{d-2}$$

$$\int_{-1}^{1} d(\cos\theta_{34}) \sin^{d-4}\theta_{34}\, \frac{\delta(E_3+E_4+E_5-\sqrt{s})}{E_5}$$

we have that $\vec{p}_5 = -\vec{p}_3 - \vec{p}_4$

$$\vec{p}_5^{\,2} = \vec{p}_3^{\,2} + \vec{p}_4^{\,2} + 2\vec{p}_3\cdot\vec{p}_4$$

$$E_5 = |\vec{p}_5| = \sqrt{E_3^2 + E_4^2 + 2E_3E_4\cos\theta_{34}}$$

$$\int d(\cos\theta_{34})\, \delta\left(E_3+E_4+\sqrt{E_3^2+E_4^2+2E_3E_4\cos\theta_{34}} - \sqrt{s}\right)$$

consider $\int dx\, \delta[f(x)] = \sum_i \frac{\delta(x-\bar{x}_i)}{|f'(\bar{x}_i)|}$

where $x = \cos\theta_{34}$

$$f(x) = E_3+E_4+\sqrt{E_3^2+E_4^2+2E_3E_4x} - \sqrt{s}$$

$\bar{x}_i$ is obtained for $f(x)=0$

$$f(\bar{x}) = 0 \rightarrow \sqrt{s}-E_3-E_4 = \sqrt{E_3^2+E_4^2+2E_3E_4\bar{x}}$$

$$(\sqrt{s}-E_3-E_4)^2 = E_3^2+E_4^2+2E_3E_4\bar{x}$$

$$s+\cancel{E_3^2+E_4^2} - 2\sqrt{s}(E_3+E_4) + 2E_3E_4 = \cancel{E_3^2+E_4^2} + 2E_3E_4\bar{x}$$

$$\bar{x} = \frac{s-2\sqrt{s}(E_3+E_4)+2E_3E_4}{2E_3E_4} \equiv \cos\theta_{34}$$

$$f'(x) = \frac{1}{2}\left(E_3^2 + E_4^2 + 2E_3E_4 x\right)^{-1/2} \cdot 2E_3E_4$$

$$f'(\bar{x}) = \frac{1}{2}\left[E_3^2 + E_4^2 + s + 2E_3E_4 - 2\sqrt{s}\,(E_3+E_4)\right]^{-1/2} 2E_3E_4$$

$$= \frac{1}{2}\left[\left(\sqrt{s} - E_3 - E_4\right)^2\right]^{-1/2} 2E_3E_4$$

$$= \frac{E_3E_4}{\sqrt{s} - E_3 - E_4} = \frac{E_3E_4}{E_5}$$

$$\int dx\, \delta(f(x)) = \frac{\delta(x-\bar{x})}{\frac{E_3E_4}{E_5}}$$

So

$$\int d\Phi_3 = \frac{1}{8(2\pi)^{2d-3}} \int dE_3\, E_3^{d-4}\, dE_4\, E_4^{d-4}\, \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \frac{2\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}$$

$$\times \Big(1 - \underbrace{\cos^2\theta_{34}}_{\bar{x}}\Big)^{\frac{d-4}{2}}$$

$$\bar{x} = \frac{s - 2\sqrt{s}\,(E_3+E_4) + 2E_3E_4}{2E_3E_4}$$

also substituting $E_3 = \frac{x_3}{2}\sqrt{s}$, $E_4 = \frac{x_4}{2}\sqrt{s}$

$$\int d\Phi_3 = \int \frac{s}{128\pi^3}\left(\frac{4\pi}{s}\right)^{2\epsilon} \frac{dx_3\, dx_4}{\left[(1-x_3)(1-x_4)(-1+x_3+x_4)\right]^{\epsilon}} \frac{1}{\Gamma(2-2\epsilon)}$$

The real cross section is now

$$\Gamma^{(0)}_{\gamma^* \to q\bar{q}g} = \frac{1}{2\sqrt{s}} \int d\Phi_3 \, \overline{\sum} |M^{(0)}_{real}|^2$$

$$= \frac{1}{2\sqrt{s}} e^2 Q_q^2 g_s^2 N_c C_F \frac{8(1-\epsilon)}{3-2\epsilon} \frac{s}{128\pi^3} \left(\frac{4\pi}{s}\right)^{2\epsilon} \frac{1}{\Gamma(2-2\epsilon)}$$

$dx_3 dx_4$ integration

Yellow note

#3

$$\times \int_0^1 dx_3 \int_{1-x_3}^1 dx_4 \frac{x_3^2 + x_4^2 - \epsilon(2-x_3-x_4)^2}{(1-x_3)(1-x_4)}$$

$$\cdot (1-x_3)^{-\epsilon} (1-x_4)^{-\epsilon} (-1+x_3+x_4)^{-\epsilon}$$

$$= \alpha Q_q^2 N_c \sqrt{s} \frac{1-\epsilon}{3-2\epsilon} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$\times \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon)\right]$$

recall $\displaystyle r_\Gamma = \frac{\Gamma^2(1-\epsilon)\,\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$

$$\frac{1}{\Gamma(1-\epsilon)} = r_\Gamma + \mathcal{O}(\epsilon^3)$$

make g_s dimensionless

$$\Gamma^{(0)}_{\gamma^* \to q\bar{q}g} = \Gamma^{(0)}_{\gamma^* \to q\bar{q}g} \cdot \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \cdot r_\Gamma$$

$$\times \left[\frac{2}{\epsilon} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon)\right]$$

The virtual cross section is

$$\Gamma^{(1)}_{\gamma^* \to q\bar{q}} = \frac{1}{2\sqrt{s}} \int d\Phi_2 \overline{\sum} 2\,\mathrm{Re}\,(M^{(0)}_\mu)^* M^{(1)\mu}$$

$$= \frac{1}{\sqrt{s}} \int d\Phi_2 \overline{\sum} (M^{(0)}_\mu)^* M^{(0)\mu} \times C_F \frac{\alpha_s}{4\pi}$$
$$\times \left(\frac{4\pi\mu^2}{-s}\right)^\epsilon r_\Gamma \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + O(\epsilon) \right]$$

$$= \Gamma^{(0)}_{\gamma^* \to q\bar{q}} \; C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\epsilon r_\Gamma \left(-\frac{2}{\epsilon} - \frac{3}{\epsilon} - 8 + \pi^2 + O(\epsilon) \right)$$

use $\quad (-s)^{-\epsilon} = s^{-\epsilon} \left[1 + i\pi\epsilon - \frac{\epsilon^2\pi^2}{2} + \cdots \right]$

to analitically continue $(-s)^{-\epsilon}$

Summing up tree level, virtual and real corrections

$$\Gamma^{NLO}_{\gamma^* \to q\bar{q}} = \Gamma^{(0)}_{\gamma^* \to q\bar{q}} \left[1 + \overset{4/3}{C_F} \frac{\alpha_s}{2\pi} \frac{3}{2} + \cdots \right]$$

$$= \Gamma_{\gamma^* \to q\bar{q}} \left[1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right]$$

$$R^{NLO} = \frac{\Gamma^{NLO}_{\gamma^* \to q\bar{q}}}{\Gamma_{\gamma^* \to \mu^+\mu^-}} = N_c \sum_q Q_q^2 \left(1 + \frac{\alpha_s}{\pi} \right)$$

Few comments are in order

1.) We see that the $\frac{\#}{\epsilon^2}$ and $\frac{\#}{\epsilon}$ poles in the virtual corrections are completely cancelled by real corrections. Real corrections have only IR divergencies, hence, the ϵ poles appearing in the 1-loop computation are of IR nature

2.) In perturbation theory, α_s depends on μ^2 (we will discuss this later)

so

$$R^{NLO}(\mu^2) = N_c \sum_q Q_q^2 \left(1 + \frac{\alpha_s(\mu^2)}{\pi}\right)$$

3.) Cancellation of IR singularities

$\Rightarrow$ Kinoshita-Lee-Nauenberg (KLN) theorem

IR singularities in virtual & real corrections cancel out for $\underbrace{\text{inclusive quantity}}_{\substack{\text{insensitive to IR physics} \\ \text{(soft emissions and} \\ \text{collinear splittings)}}}$

IR-safe observables:

$$\mathcal{O}_{n+1}(p_1, \dots, p_n, k) \longrightarrow \mathcal{O}_n(p_1, \dots, p_n) \quad \text{as } k \to 0$$

$$\mathcal{O}_{n+1}(p_1, \dots, p_i, p_j, \dots) \to \mathcal{O}(p_1, \dots, p_{ij}, \dots)$$

$$\text{as } p_i \parallel p_j$$

★1 Scalar Feynman integral computation

$$I = \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^m} = i(-1)^m \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{[l_E^2+\Delta]^m}$$

go to Euclidean space $\quad l^0 = i l_E^0 \quad , \quad l_E^i = l_E^i$

$$l^2 = (l^0)^2 - \vec{l}^2$$
$$= -(l_E^0)^2 - \vec{l}_E^2$$
$$= l_E^2$$

$$d^d l = i\, d^d l_E \quad , \quad (l^2-\Delta)^m = (-1)^m (l_E^2+\Delta)$$

use Schwinger parametrisation

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty ds\, s^{n-1} e^{-sA}$$

$$\frac{1}{(l_E^2+\Delta)^n} = \frac{1}{\Gamma(n)} \int_0^\infty ds\, s^{n-1} e^{-s[l_E^2+\Delta]}$$

$$I = \frac{i(-1)^m}{\Gamma(m)} \int_0^\infty ds\, s^{n-1} e^{-s\Delta} \underbrace{\int \frac{d^d l_E}{(2\pi)^d} e^{-s l_E^2}}_{}$$

↳ Gaussian integral

$$\int d^d l_E\, e^{-s l_E^2} = \left(\frac{\pi}{s}\right)^{d/2}$$

$$I = \frac{i(-1)^m}{(4\pi)^{d/2}\Gamma(m)} \int_0^\infty ds\, s^{n-1-\frac{d}{2}} e^{-s\Delta}$$

Gamma function $\quad \int_0^\infty ds\, s^{a-1} e^{-\Delta s} = \frac{\Gamma(a)}{\Delta^a}$

$$I = \frac{i(-1)^m}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

☆2 D-dimensional solid angle

$$(\sqrt{\pi})^d = \left(\int dx\, e^{-x^2}\right)^d = \int d^d x\, e^{-\sum_{i=1}^{d} x_i^2}$$

$$= \int d\Omega_d \int_0^\infty dx\, x^{d-1} e^{-x^2} = \underbrace{\left(\int d\Omega_d\right) \cdot \frac{1}{2}\int_0^\infty d(x^2)\,(x^2)^{\frac{d}{2}-1} e^{-(x^2)}}_{\text{change of variable}}$$

$$= \left(\int d\Omega_d\right) \frac{1}{2}\Gamma\left(\frac{d}{2}\right)$$

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

☆3 Integration over x_3 and x_4 in $\gamma^* \to q\bar{q}g$ cross section

$$I = \int_0^1 dx_3 \int_{1-x_3}^1 dx_4\, \frac{x_3^2 + x_4^2 - \epsilon(2-x_3-x_4)^2}{(1-x_3)(1-x_4)} \times (1-x_3)^{-\epsilon}(1-x_4)^{-\epsilon}(-1+x_3+x_4)^{-\epsilon}$$

change of variables $\quad x_3 = 1-u$

$\quad x_4 = 1-v+uv = 1-v(1-u)$

Jacobian $\quad \frac{\partial(x_3,x_4)}{\partial(u,v)} = \begin{vmatrix} -1 & 0 \\ v & -(1-u) \end{vmatrix} = 1-u$

$$dx_3\, dx_4 = (1-u)\, du\, dv$$

$$I = \int_0^1 du \int_0^1 dv\, u^{-1-\epsilon} v^{-1-\epsilon} (1-u)^{-2\epsilon} (1-v)^{-\epsilon} P(u,v)$$

$$\boxed{P(u,v) = \left[(1-u)^2 + (1-v+uv)^2 - \epsilon(u+v-uv)^2\right]}$$

$$P(u,v) = 2 - 2u - 2v + (1-\epsilon)u^2 + (1-\epsilon)v^2$$
$$+ 2(1-\epsilon)uv + 2\epsilon u^2 v - 2(1-\epsilon)uv^2 + (1-\epsilon)u^2v^2$$

Define
$$J_{mn} = \int_0^1 du \int_0^1 dv \; u^{m-1-\epsilon}(1-u)^{-2\epsilon} v^{n-1-\epsilon}(1-v)^{-\epsilon}$$

u and v integration factorize

$$J_{mn} = B(m-\epsilon, 1-2\epsilon)\, B(n-\epsilon, 1-\epsilon)$$

in terms of Gamma function

$$J_{mn} = \frac{\Gamma(m-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(m+1-3\epsilon)} \frac{\Gamma(n-\epsilon)\Gamma(1-\epsilon)}{\Gamma(n+1-2\epsilon)}$$

So now

$$I = 2J_{00} - 2J_{10} - 2J_{01} + (1-\epsilon)J_{20} + (1-\epsilon)J_{02}$$
$$+ 2(1-\epsilon)J_{11} + 2\epsilon J_{21} - 2(1-\epsilon)J_{12} + (1-\epsilon)J_{22}$$

$$= \frac{2(1-2\epsilon)(\epsilon^2 - 2\epsilon + 2)}{\epsilon^2(1-3\epsilon)(2-3\epsilon)} \frac{\Gamma(1-\epsilon)^3}{\Gamma(1-3\epsilon)}$$

$$= \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2$$