QCD Lecture 1

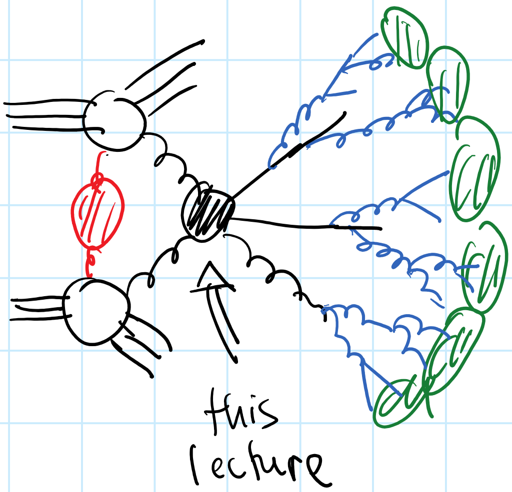
QCD → theory of strong interaction

↳ a remarkably rich and vast subject.

Describing different QCD regime require specialized theoretical tools

This lecture will focus specifically on perturbative QCD and its application to collider physics

I will mainly discuss fixed order computation within pQCD framework which describe hard scattering part of the theoretical simulation of particle collision at collider experiments



QCD ⊂ SM

closely connected to MC lectures

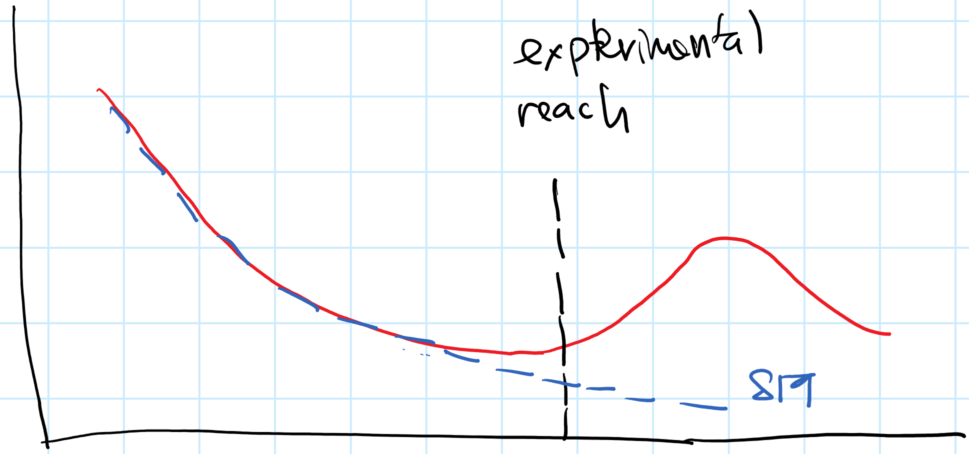
→ matching to fixed order

→ parton shower

→ hadronization

BSM / DM ?

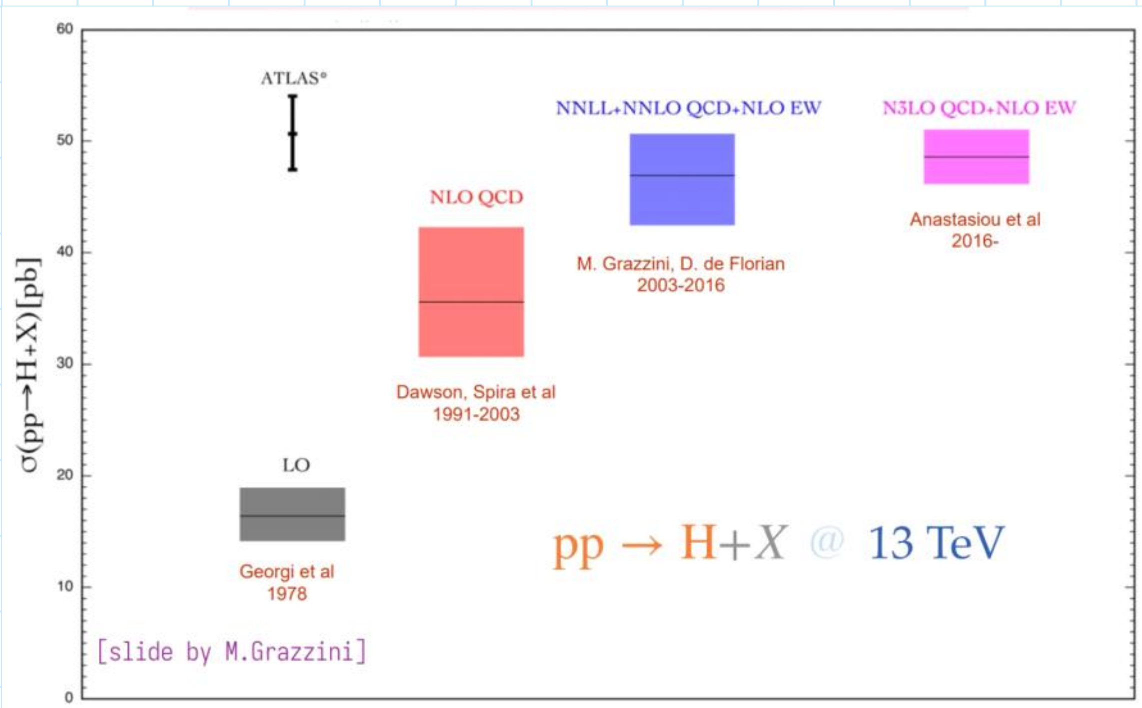
↳ as we will see in the course of this lecture higher-order corrections in pQCD framework is crucial to obtain reliable theory predictions which serve as background to BSM searches at collider.



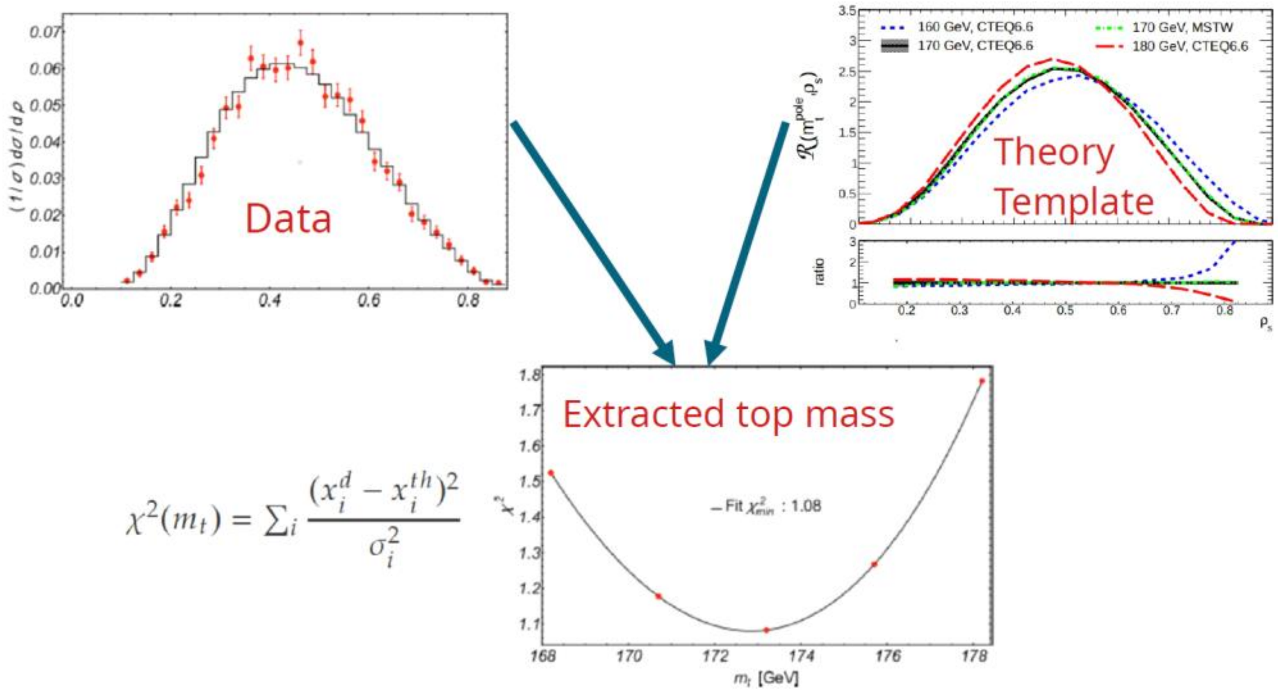
→ precision measurement → data
→ theory

reliable → normalisation
→ theory uncertainty

example : $pp \to H$ cross section



→ employ pQCD predictions to extract SM parameters.



→ need to have theory uncertainty under control

<u>GOAL</u> : Working knowledge of perturbative QCD, appreciate the impact of perturbative QCD calculation on physical observables.

<u>Disclaimer</u>: I will not be able to cover all pQCD topics in 4 lectures. Insted, topics selected in this lecture should represent the important features of QCD

- I will not be able to cover all the details of the calculation carried out in the lectures but they should be available in the lecture notes.

- Learning by doing : the lecturer will be structured around examples that illustrate important issues, computational technology commonly encountered in pQCD

PLAN

- Introduction
- QCD Lagrangian $\rightarrow$ Feynman rules
- Tree level techniques for QCD amplitude
- QCD corrections to $e^+e^- \rightarrow$ hadrons
- QCD corrections to DIS
- Aspects of QCD renormalisation
- Jet cross section

- Problem sets will be given.

  Discussions ~30 minutes in the beginning of lectures #3 & #4 on the problem sets.

  Students are invited to present their solutions

- Ideally students are familiar with:

  - Basics of QFT & Relativistic QM
  - Standard scattering amplitude computation with Feynman diagram (tree level)
  - One-loop computation
  - Basics of collider physics: cross sections, etc.

A bit of history

1950s - 1960s $\Rightarrow$ experiments had discovered a large number of strongly interacting particles (mesons & baryons)

$\hookrightarrow$ look for underlying structure

1964 $\Rightarrow$ Murray Gell-Mann and George Zweig independently proposed the "quark" model. Quarks come in three flavours: up, down, strange.

Properties $\Rightarrow$ spin $1/2$, $Q_u = 2/3$, $Q_d = -1/3$, $Q_s = -1/3$

example: $p = uud$

$n = udd$ ... etc

Δ^{++} puzzle: Δ^{++}: uuu with aligned spin (up)

For the ground state, the spatial, spin and flavor parts are symmetric, conflicting with Fermi-Dirac statistics. $\psi(1,2,3) = -\psi(2,1,3)$

For the spin $3/2$, $S_z = +\frac{3}{2}$, $|\Delta^{++}, \uparrow\uparrow\uparrow\rangle = |u^\uparrow u^\uparrow u^\uparrow\rangle$

New quantum number: color (Han, Nambu, Greenberg)

$$|\Delta^{++}, \uparrow\uparrow\uparrow\rangle = \frac{1}{\sqrt{6}} \varepsilon_{abc} |u^{a\uparrow} u^{b\uparrow} u^{c\uparrow}\rangle$$

with $a,b,c = r,g,b$.

Deep inelastic scattering (DIS) experiment at SLAC showed that the proton contains point-like scattering centers. Feynman described these as "partons"

These partons (quarks) seemed almost free when probed at a very short distances (high energy) in DIS, but no isolated quarks were observed.
↳ 1967-1968

1973 : Gross, Wilczek and Politzer

⇒ non-Abelian gauge theories can be asymptotically free : interaction becomes weaker at short distances

⇒ SU(3) non-Abelian gauge theory of color ⇒ QCD

QCD Lagrangian

QCD as an SU(3) gauge theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F_{a}^{\mu\nu} + \sum_i \bar{\psi}_i (i\not{\partial} - m)_{ij} \psi_j$$

$$+ \mathcal{L}_{GF} + \mathcal{L}_{ghost}$$

where $F_{\mu\nu}^{a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \underbrace{- g_s f^{abc} A_\mu^b A_\nu^c}$

gluon self interaction absent in QED

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s t_{ij}^c A_\mu^c$$

$i = 1, \ldots, 3$: quark in fundamental representation
$a = 1, \ldots, 8$: gluon is in adjoint representation

The QCD Lagrangian is invariant under $SU(3)_c$ local gauge transformation

$$\psi_i'(x) = U_{ij}(x) \psi_j(x)$$

where $U_{ij}(x) = e^{i g_s \alpha^a t_{ij}^a}$

$$A_\mu'(x) = U(x) A_\mu(x) U^{-1}(x) + \frac{i}{g_s} U(x) \partial_\mu U^{-1}(x)$$

$$A_\mu(x) = A_\mu^a(x) t^a$$

For an infinitesimal transformation

$$U(x) \simeq 1 + i\alpha^a(x) t^a$$

$$A_\mu^a \to A_\mu^a - \frac{1}{g_s} \partial_\mu \alpha^a - f^{abc} \alpha^b A_\mu^c$$

$\Rightarrow$ Gluon fields are related by gauge transformations

Path integral quantisation

$$Z = \int DA \, e^{iS[A]}$$

integrates over infinitely many gauge equivalent configurations

Gauge fixing term

$$\mathcal{L}_{GF} = \frac{1}{2\lambda} (\partial^\mu A_\mu^a)^2$$

$\Rightarrow$ inserting this gauge condition into the path integral will produce a Jacobian, known as Faddeev-Popov determinant.

$$\det M = \int D\bar{c} \, Dc \, e^{i\int d^4x \, \bar{c}^a M^{ab} c^b}$$

$$M^{ab} = \partial^\mu D_\mu^{ab}$$

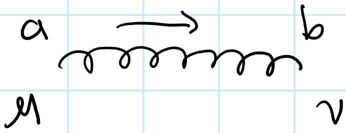
hence

$$\mathcal{L}_{ghost} = \bar{c}^a \partial^\mu D_\mu^{ab} c^b$$

$$= \partial_\mu \bar{c}^a \partial^\mu c^a + g_s f^{abc} (\partial^\mu \bar{c}^a) A_\mu^b c^c$$

ghost decouples in QED, but interacts with gluon in QCD

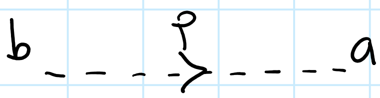
From the Lagrangian we can write down the QCD Feynman rules

gluon : 

$: \dfrac{i\delta^{ab}}{p^2+i\epsilon}\left[-g^{\mu\nu} + (1-\lambda)\dfrac{p^\mu p^\nu}{p^2+i\epsilon}\right]$

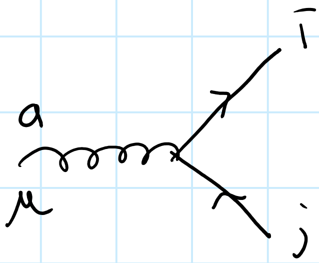
quark : 

$: \dfrac{i\delta_{ij}}{\not{p} - m + i\epsilon}$

ghost : 

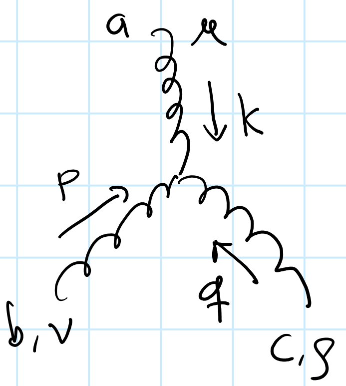
$: \dfrac{i\delta^{ab}}{p^2+i\epsilon}$

quark-gluon vertex :



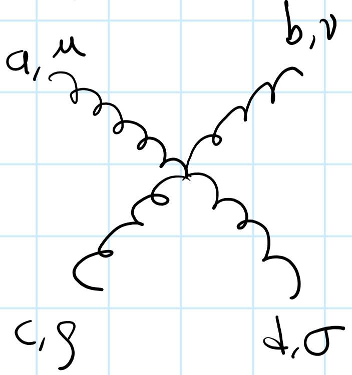
$$i g_s t^a_{ij} \gamma^a$$

three-gluon vertex



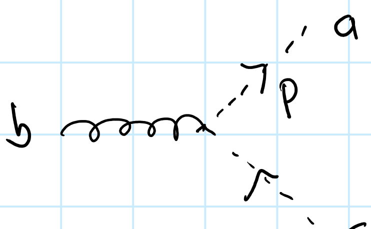
$$= g_s f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu \right]$$

four-gluon vertex



$$= -ig_s^2 \Big[f^{abe} f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace} f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade} f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right) \Big]$$

ghost-gluon vertex



$$= -g f^{abc} p^\mu$$

we will work in Feynman gauge $(\lambda = 1)$ in the remaining of this lecture unless stated otherwise

$i\epsilon \Rightarrow$ Feynman proscription $\rightarrow$ how to handle pole at $p^2 = 0$
$\rightsquigarrow$ $\epsilon \to 0^+ \Rightarrow$ ensure time ordering of Feynman prop.

Techniques for tree level QCD scattering amplitudes

Computation of scattering amplitudes in QCD often leads to explosion of algebraic expressions, even at tree level. This complexity grows with the number of external particles ("multiplicity"). In the Feynman diagram approach, in particular, the 3- and 4-gluon vertices are quite complicated. In many cases intermediate analytic expressions tend to be vastly more complicated than the final results.

In this lecture we will employ two approaches to efficiently compute scattering amplitude:

1. colour decomposition
2. spinor helicity formalism

I. Colour decomposition

Every amplitude, or Feynman diagram expression can be separated into colour and kinematic parts such that

$$M = \sum_i c_i A_i$$

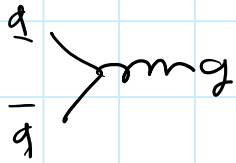
c_i is the basis of colour factor and A_i is pure kinematic part usually called "colour ordered partial amplitude"

The partial amplitude A_i might be related to each other, and exploiting the relations among A_i allows us to compute only the independent set of partial amplitudes. In addition in many cases A_i contains fewer Feynman diagrams than that of M.

To arrive at a colour factor basis, we need to perform colour algebra. Recall the colour factors appearing in the QCD Feynman rules

$i = 1, \ldots, N_c$

$a = 1, \ldots, N_c^2 - 1$

t^a_{ij} from 

f^{abc} from  

Colour algebra for the generators and structure constants of $SU(N_c)$:

$$\mathrm{tr}(t^a) = 0$$

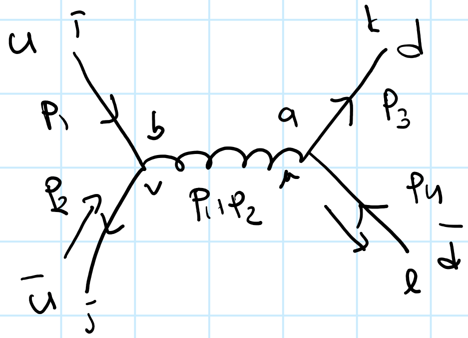
$$\mathrm{tr}(t^a t^b) = \frac{\delta^{ab}}{2}$$

$$[t^a, t^b] = i f^{abc} t^c$$

Fierz identity

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

Example: $u\bar{u} \to d\bar{d}$



$$\mathcal{M} = \bar{u}(p_3)\, i g_s\, t^a_{kl}\, \gamma^\mu\, v(p_4)\ \frac{-i g_{\mu\nu}\delta^{ab}}{(p_1+p_2)^2}\ \bar{v}(p_2)\, i g_s\, t^b_{ji}\, \gamma^\nu\, u(p_1)$$

$$= i g_s^2\ t^a_{ji} t^a_{kl}\ \frac{1}{2p_1\cdot p_2}\left[\bar{u}_3 \gamma^\mu v_4\right]\left[\bar{v}_2 \gamma_\mu u_1\right]$$

$$\hookrightarrow \frac{1}{2}\left(\delta_{jl}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{kl}\right)$$

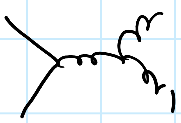
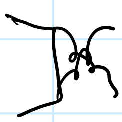
$$= \delta_{jl}\delta_{ki} A_1 + \frac{1}{N_c}\delta_{ji}\delta_{kl} A_2$$

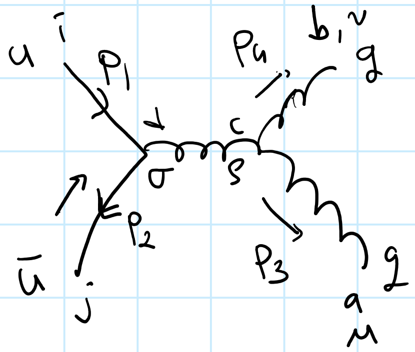
where $A_1 = \dfrac{i g_s^2}{2}\ \dfrac{1}{2p_1\cdot p_2}\left[\bar{u}_3 \gamma^\mu v_4\right]\left[\bar{v}_2 \gamma_\mu u_1\right]$

$$A_2 = -A_1$$

Example: $u\bar{u} \to gg$

There are 3 Feynman diagrams contributing to $u\bar{u} \to gg$

 (s-channel)  (t-channel)  (u-channel)



$$M_s = \bar{v}(p_2)\, i g_s\, t^d_{ji} \gamma^\sigma u(p_1) \times \frac{-i g_{\sigma\rho} \delta^{cd}}{(p_1+p_2)^2}$$

$$\times\, g_s f^{abc} \Big[g^{\mu\nu}(-p_3+p_4)^\rho$$

$$+\, g^{\nu\rho}(-p_4-p_3-p_4)^\mu$$

$$+\, g^{\rho\mu}(p_3+p_4+p_3)^\nu \Big]$$

$$\times\, \epsilon^{*\mu}(p_3)\, \epsilon^{*\nu}(p_4)$$

$$M_s = g_s^2\, \overbrace{t^c_{ji} f^{abc}}^{-i[t^a,t^b]_{ji}} \frac{1}{s_{12}} \epsilon^{*\mu}(p_3) \epsilon^{*\nu}(p_4) \times [\bar{v}_2 \gamma_\rho u_1]$$

$$\times \Big[g^{\mu\nu}(-p_3+p_4)^\rho + g^{\nu\rho}(-p_3-2p_4)^\mu + g^{\rho\mu}(2p_3+p_4)^\nu \Big]$$

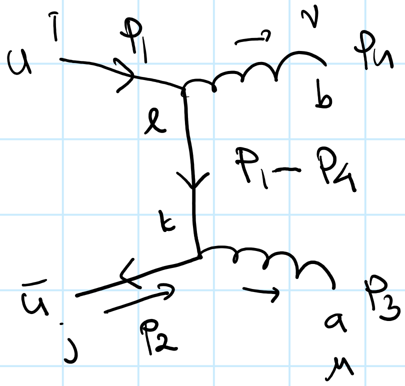
$$= t^a_{jk} t^b_{ki} A_{s,1} + t^b_{jk} t^a_{ki} A_{s,2}$$

where $A_{s,1} = -i g_s^2 \frac{1}{s_{12}} \epsilon^{*\mu}(p_3) \epsilon^{*\nu}(p_4) [\bar{v}_2 \gamma_\rho u_1]$

$$\times \Big[g^{\mu\nu}(-p_3+p_4)^\rho + g^{\nu\rho}(-p_3-2p_4)^\mu + g^{\rho\mu}(2p_3+p_4)^\nu \Big]$$

and $A_{s,2} = -A_{s,1} \big|_{p_3 \leftrightarrow p_4}$

Note that A_s is symmetric under $p_3 \leftrightarrow p_4$ exchange.



$$M_t = \bar{v}(p_2)\, i g_s t^a_{jk} \gamma^\mu \, i \frac{\cancel{p}_1 - \cancel{p}_4}{(p_1 - p_4)^2} \delta_{kl}$$

$$\times\, i g_s t^b_{li} \gamma^\nu u(p_1) \cdot \epsilon^{*\mu}(p_3)\, \epsilon^{*\nu}(p_4)$$

$$M_t = t^a_{jk} t^b_{ki} \, A_t$$

where $A_t = -i g_s^2 \, \epsilon^{*\mu}(p_3) \epsilon^{*\nu}(p_4) \times \bar{v}_2 \gamma^\mu \dfrac{(\cancel{p}_1 - \cancel{p}_4)}{(p_1 - p_4)^2} \gamma^\nu u(p_1)$

For the u channel diagram you can show that

$$M_u = t^b_{jk} t^a_{ki} \, A_u$$

where $A_u = A_t \, (p_3 \leftrightarrow p_4)$

The total contribution to $u\bar{u} \to gg$ scattering is

$$M = M_s + M_t + M_u$$

$$= t^a_{jk} t^b_{ki} (A_t + A_{s,1}) + t^b_{jk} t^a_{ki} (A_u - A_{s,1})$$

$$\equiv t^a_{jk} t^b_{ki} A_{ab} + t^b_{jk} t^a_{ki} A_{ba}$$

where $A_{ba} = A_{ab} \, (p_3 \leftrightarrow p_4)$

II. Spinor Helicity formalism

Let's recall the Dirac equation for massless particle in momentum space

$$\not{p}\, u(p) = 0$$

with $\not{p} = \gamma^\mu p_\mu$ and the Dirac matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

$$(\gamma^\mu)^* = \gamma^2\gamma^\mu\gamma^2$$

Weyl representation

2x2 matrix

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

such that

$$\not{p} = \begin{pmatrix} 0 & 0 & p^+ & \bar{p}_\perp \\ 0 & 0 & p_\perp & p^- \\ p^- & -\bar{p}_\perp & 0 & 0 \\ -p_\perp & p^+ & 0 & 0 \end{pmatrix}$$

where $p^\pm = p_0 \pm p_3$

$p_\perp = p_1 + ip_2$ and $\bar{p}_\perp = p_1 - ip_2$

The solution to massless Dirac equation $\not{p}\, u(p) = 0$ is

$$u(p) = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}\, e^{i\varphi_p} \\ \sqrt{p^-}\, e^{-i\varphi_p} \\ -\sqrt{p^+} \end{pmatrix}$$

φ_p is the phase factor given by

$$e^{\pm i\varphi_p} = \frac{p_1 \pm i p_2}{\sqrt{p_1^2 + p_2^2}} = \frac{p_1 \pm i p_2}{\sqrt{p^+ p_-}}$$

The spinor solution can be decomposed into different helicity states using projection operator:

$$\gamma_R = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_L = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

in the Weyl representation

We now have

$$u_+(p) = \gamma_R\, u(p) = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}\, e^{i\varphi_p} \\ 0 \\ 0 \end{pmatrix} ; \quad u_-(p) = \gamma_L\, u(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p^-}\, e^{-i\varphi_p} \\ -\sqrt{p^+} \end{pmatrix}$$

The conjugate spinors are then

$$\bar{u}_+(p) = u_+^\dagger(p)\gamma^0 = \left(0, 0, \sqrt{p^+}, \sqrt{p^-}\, e^{-i\varphi_p}\right)$$

$$\bar{u}_-(p) = \left(\sqrt{p^-}\, e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right)$$

where the solutions satisfy the following normalization

$$u_\pm^\dagger(p)\, u_\pm(p) = 2p^0$$

We will use the following notation for $u_\pm$ and $v_\pm$

$$|i\rangle = u_+(p_i) = v_-(p_i)$$

$$|i] = u_-(p_i) = v_+(p_i)$$

$$\langle i| = \bar{u}_-(p_i) = \bar{v}_+(p_i)$$

$$[i| = \bar{u}_+(p_i) = \bar{v}_-(p_i)$$

We can now define spinor products and spinor strings

$$\langle ij\rangle = \bar{u}_-(p_i)\, u_+(p_j)$$

$$\langle i\gamma^\mu j] = \bar{u}_-(p_i)\gamma^\mu u_-(p_j)$$

$$\langle i \not{p}_k j] = \bar{u}_-(p_i) \not{p}_k u_-(p_j)$$

... etc

Useful identities

$$\langle ij\rangle = -\langle ji\rangle, \quad [ij] = -[ji]$$

$$\langle ij\rangle^* = [ji] \quad (\text{for real momenta})$$

$$\langle ij] = [ij\rangle = 0$$

$$\not{p}_i = |i\rangle[i| + |i]\langle i|$$

Schouten identity $\quad \langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0$

Fierz identity $\quad \frac{1}{2}\langle i\gamma_\mu j]\gamma^\mu = |j]\langle i| + |i\rangle[j|$

$$\frac{1}{2}[i\gamma_\mu j\rangle\gamma^\mu = |j\rangle[i| + |i]\langle j|$$

$$\langle i\gamma^\mu j] = [j\gamma^\mu i\rangle$$

$$\langle i\gamma^\mu j]^* = \langle j\gamma^\mu i]$$

$$\langle i\gamma^\mu i] = 2p_i^\mu$$

$$\langle i\gamma^\mu j\rangle = [i\gamma^\mu j] = 0$$

Massless vector boson polarisation can be written in terms of spinor products as follows

$$\varepsilon_+^{*\mu}(p_i, r) = \frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu i]}{\langle ri\rangle}$$

$$\varepsilon_-^{*\mu}(p_i, r) = -\frac{1}{\sqrt{2}}\frac{[r\gamma^\mu i\rangle}{[ri]}$$

We can show that these correspond to the usual polarization vector by choosing

$$p_i = (E_p, 0, 0, E_p) \qquad p_i^+ = 2E_p \quad p_i^- = 0$$

$$r = (E_r, 0, 0, -E_r) \qquad r^+ = 0 \;,\; r^- = 2E_r$$

$$\langle r\gamma^\mu i] = \bar{u}_-(r)\gamma^\mu u_-(p_i)$$

$$u_-(p_i) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{2E_p} \end{pmatrix}$$

$$\bar{u}_-(r) = e^{i\varphi_r}\sqrt{2E_r}$$

$$\langle r\gamma^0 i] = 0$$

$$\langle r\gamma^1 i] = 2\sqrt{E_pE_r}\, e^{i\varphi_r}$$

$$\langle r\gamma^2 i] = -i\, 2\sqrt{E_pE_r}\, e^{i\varphi_r}$$

$$\langle r\gamma^3 i] = 0$$

$$\langle r i\rangle = 2\sqrt{E_pE_r}\, e^{i\varphi_r}$$

$$\varepsilon_+^{*\mu}(p_i, r) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

analogously we can show that

$$\varepsilon_-^{*\mu}(p_i, r) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$$

The spinor representation of massless polarization vector also satisfy the following properties

$$\left[\varepsilon_{\pm}^{*\mu}(p,r)\right]^{*} = \varepsilon_{\mp}^{*\mu}(p,r)$$

$$\varepsilon_{\pm}^{*\mu}(p,r)\cdot p_{\mu} = 0$$

$$\varepsilon_{\pm}^{*\mu}(p,r)\left[\varepsilon^{*}_{\pm,\mu}(p,r)\right]^{*} = -1$$

$$\varepsilon_{\pm}^{*\mu}(p,r)\left[\varepsilon^{*}_{\mp,\mu}(p,r)\right]^{*} = 0$$

Now back to the $u\bar{u}\to d\bar{d}$ scattering we considered in the colour decomposition section

$$A_1 = \frac{ig_s^2}{2}\,\frac{1}{2p_1\cdot p_2}\left[\bar{u}_3\gamma^{\mu}v_4\right]\left[\bar{v}_2\gamma_{\mu}u_1\right]$$

keeping track of the helicity states

$$u(p_1,h_1)+\bar{u}(p_2,h_2)\to d(p_3,h_3)+\bar{d}(p_4,h_4)$$

helicity label $\to$ $h_1h_2h_3h_4$

nonzero helicity configurations : $+-+-$

$+--+$

$-++-$

$-+-+$

Let us now compute $+-+-$ configuration

$$A_1^{+-+-} = \frac{i g_s^2}{2} \frac{1}{2 p_1 \cdot p_2} \bar{u}_+(p_3)\gamma^\mu v_-(p_4)\, \bar{v}_-(p_2)\gamma_\mu u_+(p_1)$$

$$= \frac{i g_s^2}{2} \frac{1}{2 p_1 \cdot p_2} [3\gamma^\mu 4\rangle [2\gamma_\mu 1\rangle$$

$$= \frac{i g_s^2}{2} \frac{2}{2 p_1 \cdot p_2} \langle 14\rangle [32]$$

$$= i g_s^2 \frac{\langle 14\rangle [32]}{s_{12}}$$ where $s_{12} = (p_1+p_2)^2$

The other helicity configurations can be obtained by

$$A_1^{-+-+} = A_1^{+-+-}\Big|_{\text{flip}} = i g_s^2 \frac{[41]\langle 23\rangle}{s_{12}}$$

$$\boxed{\begin{aligned} \text{flip}: \langle ij\rangle &\to [ji] \\ [ij] &\to \langle ji\rangle \end{aligned}}$$

$$A_1^{+--+} = A_1^{+-+-}(p_3 \leftrightarrow p_4)$$

$$= i g_s^2 \frac{\langle 13\rangle [42]}{s_{12}}$$

$$A_1^{-++-} = A_1^{+--+}\Big|_{\text{flip}}$$

$$= +i g_s^2 \frac{[31]\langle 24\rangle}{s_{12}}$$

Now we can compute squared amplitude

$$|A_1^{+-+-}|^2 = |A^{-+-+}|^2 = g_s^4 \frac{2p_1\cdot p_4\, 2p_2\cdot p_3}{s_{12}^2} = g_s^4 \frac{u^2}{s^2}$$

use $s_{12} = s$, $p_1\cdot p_4 = p_2\cdot p_3 = -\frac{u}{2}$, $2p_1\cdot p_3 = 2p_2\cdot p_4 = -\frac{t}{2}$

$$|A^{+--+}|^2 = |A^{-++-}|^2 = g_s^4 \frac{2p_1\cdot p_3\, 2p_2\cdot p_4}{s_{12}^2} = g_s^4 \frac{t^2}{s^2}$$

$$\sum_{\text{helicity}} |A_1|^2 = 2g_s^4 \frac{t^2+u^2}{s^2}$$

$$M = \underbrace{\left(\delta_{jl}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{kl}\right)}_{C} A_1$$

$$\boxed{\delta_{ij}\delta_{ji} = N_c}$$

$$\sum_{\text{color}} |M|^2 = \sum_{\text{color}} C^* C \; |A_1|^2$$

$$= \left(\delta_{lj}\delta_{ik} - \frac{1}{N_c}\delta_{ij}\delta_{lk}\right)\left(\delta_{jl}\delta_{ki} - \frac{1}{N_c}\delta_{ji}\delta_{kl}\right)|A|^2$$

$$= \left(N_c^2 - \frac{1}{N_c}N_c - \frac{1}{N_c}N_c + \frac{1}{N_c^2}N_c^2\right)|A|^2$$

$$= (N_c^2 - 1)|A|^2$$

$$\sum_{\text{color, helicity}} |M|^2 = (N_c^2-1)\cdot 2g_s^4 \frac{t^2+u^2}{s^2}$$

$|A_1|^2$ can also be computed using the standard trace techniques:

$$A_1 = \frac{i g_s^2}{2} \frac{1}{2p_1 \cdot p_2} [\bar{u}_3 \gamma^\mu v_4][\bar{v}_2 \gamma_\mu u_1]$$

$$\sum_{spin} |A_1|^2 = \frac{g_s^4}{4} \frac{1}{s_{12}^2} \mathrm{tr}[\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \mathrm{tr}[\not{p}_2 \gamma_\mu \not{p}_1 \gamma_\nu]$$

$$\mathrm{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

$$= \frac{g_s^4}{4} \frac{16}{s_{12}^2} \left(p_3^\mu p_4^\nu - p_3 \cdot p_4 \, g^{\mu\nu} + p_3^\nu p_4^\mu \right) \times \left(p_{2\mu} p_{1\nu} - p_1 \cdot p_2 \, g_{\mu\nu} + p_{2\nu} p_{1\mu} \right)$$

$$= \frac{4 g_s^4}{s_{12}^2} \Big[p_2 \cdot p_3 \, p_1 \cdot p_4 - \cancel{p_1 \cdot p_2 \, p_3 \cdot p_4} + p_3 \cdot p_1 \, p_2 \cdot p_4 - \cancel{p_3 \cdot p_4 \, p_1 \cdot p_2} + \cancel{p_1 \cdot p_2 \, p_3 \cdot p_4} \, 4 - \cancel{p_1 \cdot p_2 \, p_3 \cdot p_4} \; p_2 \cdot p_4 \, p_1 \cdot p_3 - \cancel{p_1 \cdot p_2 \, p_3 \cdot p_4} \quad p_2 \cdot p_3 \, p_1 \cdot p_4 \Big]$$

$$= \frac{4 g_s^4}{s_{12}^2} \left[2 p_2 \cdot p_3 \, p_1 \cdot p_4 + 2 p_2 \cdot p_4 \, p_1 \cdot p_3 \right]$$

$p_2 \cdot p_3 = p_1 \cdot p_4 = -\frac{u}{2}$, $p_1 \cdot p_3 = p_2 \cdot p_4 = -\frac{t}{2}$, $s_{12} = s$

$$\sum_{spin} |A_1|^2 = 2 g_s^4 \frac{t^2 + u^2}{s^2} \quad //$$