Non-Abelian Gauge Theory

Manimala Mitra

Contents

1 Non-Abelian Gauge Theory **2**

1.1 Introduction 2
1.2 Quantum Electrodynamics: The Prototype Gauge Theory 2
1.2.1 The Electromagnetic Field Strength Tensor and QED Lagrangian 4
1.2.2 Maxwell's Equations from the QED Lagrangian 5
1.3 Historical Motivation for Non-Abelian Symmetry 9
1.4 From Isospin to SU(N) 10
1.5 Global and Local $SU(N)$ Symmetry 10
1.5.1 N-component fermion multiplet 10
1.5.2 Global $U(N)$ and $SU(N)$ 10
1.5.3 The Groups $SU(2)$ and $SU(3)$ 13
1.5.4 Local $SU(N)$ Symmetry 16
1.5.5 The Yang–Mills Lagrangian 17
1.5.6 The Yang–Mills Lagrangian 19
1.5.7 Feynman Rules 22
1.5.8 General Yang–Mills Theory with Product Gauge Groups 26
1.6 The Standard Model as a Yang–Mills Gauge Theory 28
1.6.1 Gauge Fields 28
1.6.2 Field Strength Tensors 28
1.6.3 Left- and Right-Handed Fermions 29
1.6.4 The Higgs Doublet 29
1.6.5 Covariant Derivative 30
1.6.6 The Standard Model Lagrangian 30

Chapter 1

Non-Abelian Gauge Theory

1.1 Introduction

Gauge symmetry is the fundamental principle underlying the modern description of elementary particle interactions. Rather than introducing forces by hand, the gauge principle shows that interactions arise naturally when a global symmetry is promoted to a local symmetry. Quantum electrodynamics (QED) is based on the Abelian group $U(1)$ and successfully explains electromagnetic interactions. However, the weak and strong interactions possess a much richer internal structure that cannot be described by a single phase transformation. This motivates the study of non-Abelian gauge theories, first formulated by Yang and Mills in 1954, and ultimately leads to the gauge structure of the Standard Model,

$$SU(3)_C \times SU(2)_L \times U(1)_Y.$$

Throughout this chapter we develop the mathematical framework of non-Abelian gauge theories beginning with their historical origin, before introducing the general Yang–Mills formalism.

Before discussing non-Abelian gauge symmetry it is however useful to recall the simplest gauge theory, Quantum Electrodynamics (QED). QED illustrates the gauge principle in its simplest form and provides the conceptual bridge to Yang–Mills theory.

1.2 Quantum Electrodynamics: The Prototype Gauge Theory

Before discussing non-Abelian gauge theories, it is instructive to examine the simplest example of a gauge theory, namely Quantum Electrodynamics (QED). Although QED is based on the Abelian gauge group $U(1)$, it illustrates all the essential ingredients of the gauge principle. The construction of Yang–Mills theories follows exactly the same logic, the only difference being that the gauge transformations are generated by a non-Abelian Lie group.

Consider the free Dirac Lagrangian describing a spin-$\frac{1}{2}$ particle of mass m,

$$\mathcal{L}_0 = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi. \tag{1.1}$$

This Lagrangian is invariant under the global phase transformation

$$\psi(x) \to e^{-i\alpha}\psi(x), \tag{1.2}$$

where the parameter α is independent of the space-time coordinates. Such a transformation changes only the overall phase of the field, leaving all measurable physical quantities unchanged. Consequently, the theory possesses a continuous global $U(1)$ symmetry.

According to Noether's theorem, every continuous global symmetry gives rise to a conserved current. Applying Noether's theorem to the global $U(1)$ symmetry yields the conserved current associated with global $U(1)$ transformation

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{1.3}$$

which satisfies the continuity equation

$$\partial_\mu j^\mu = 0. \tag{1.4}$$

Integrating the time component over all space gives

$$Q = \int d^3x\, j^0(x), \tag{1.5}$$

which represents the conserved charge.

At this stage one may ask whether Nature requires the phase transformation to be the same everywhere in space-time. Since the absolute phase of a field is not observable, there is no compelling physical argument requiring the phase parameter to be constant. It is therefore natural to consider the more general transformation

$$\alpha \to \alpha(x), \tag{1.6}$$

where the phase may vary from one space-time point to another. Such a transformation is called a *local gauge transformation*.

Unfortunately, the free Dirac Lagrangian is no longer invariant under this transformation. Indeed,

$$\psi(x) \to e^{-i\alpha(x)}\psi(x), \tag{1.7}$$

implies

$$\partial_\mu\psi \to e^{-i\alpha(x)}\left(\partial_\mu - i\partial_\mu\alpha\right)\psi, \tag{1.8}$$

where the additional term involving $\partial_\mu\alpha(x)$ has no counterpart in the original Lagrangian. Consequently,

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi \tag{1.9}$$

is no longer invariant, and local gauge symmetry is lost.

Rather than abandoning local symmetry, we seek a modified derivative that transforms in the same manner as the fermion field itself. This requirement leads naturally to the definition of the covariant derivative,

$$D_\mu = \partial_\mu + ieA_\mu, \tag{1.10}$$

where the newly introduced vector field $A_\mu(x)$ is called the gauge field. Demanding that

$$D_\mu\psi \to e^{-i\alpha(x)}D_\mu\psi \tag{1.11}$$

uniquely determines the transformation law of the gauge field,

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha. \tag{1.12}$$

The remarkable feature of this construction is that the interaction between the charged fermion and the electromagnetic field is not introduced by hand. Instead, it appears inevitably as a consequence of insisting that the theory remain invariant under local $U(1)$ gauge transformations. In this sense, gauge symmetry does not merely constrain the form of the interaction—it determines it. The current,

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{1.13}$$

satisfies the continuity equation

$$\partial_\mu j^\mu = 0. \tag{1.14}$$

Integrating the time component over all space gives

$$Q = \int d^3x\, j^0(x), \tag{1.15}$$

which in this scenario represents the conserved electromagnetic charge.

1.2.1 The Electromagnetic Field Strength Tensor and QED Lagrangian

The gauge field must possess its own dynamics. This is described by the field strength tensor, which can be obtained from the commutator of two covariant derivatives,

$$[D_\mu, D_\nu] = [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu] \tag{1.16}$$
$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{1.17}$$

Hence,

$$F_{\mu\nu} = \frac{1}{ie}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.18}$$

The antisymmetry of $F_{\mu\nu}$ follows immediately,

$$F_{\mu\nu} = -F_{\nu\mu}.$$

Its components are related to the electric and magnetic fields by

$$F_{0i} = -E_i, \qquad F_{ij} = -\epsilon_{ijk} B_k.$$

The kinetic term for the photon field is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.19}$$

Combining the gauge-field and fermion sectors gives the complete QED Lagrangian,

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.} \tag{1.20}$$

Substituting

$$D_\mu = \partial_\mu + ieA_\mu,$$

one obtains

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu. \tag{1.21}$$

The first term describes the free propagation of the photon, the second term represents the free fermion, while the third term describes the interaction between the fermion and the electromagnetic field.

Since

$$j^\mu = \bar{\psi}\gamma^\mu \psi,$$

the interaction Lagrangian may be written as

$$\boxed{\mathcal{L}_{\text{int}} = -ej^\mu A_\mu.} \tag{1.22}$$

Thus the electromagnetic interaction arises naturally from the gauge principle. No interaction term has been introduced by hand; it is completely determined by requiring local $U(1)$ gauge invariance. This elegant idea forms the foundation of all modern gauge theories and serves as the prototype for the construction of non-Abelian Yang–Mills theories.

1.2.2 Maxwell's Equations from the QED Lagrangian

An important consequence of the QED Lagrangian is that Maxwell's equations arise naturally as the Euler–Lagrange equations for the gauge field. The complete QED Lagrangian is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{1.23}$$

where

$$D_\mu = \partial_\mu + ieA_\mu, \tag{1.24}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.25}$$

Expanding the covariant derivative, the QED Lagrangian becomes

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{1.26}$$

To obtain the equation of motion for the gauge field, we apply the Euler–Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0. \tag{1.27}$$

The interaction term gives

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -e\bar{\psi}\gamma^\nu\psi = -ej^\nu, \tag{1.28}$$

where

$$j^\nu = \bar{\psi}\gamma^\nu\psi \tag{1.29}$$

is the electromagnetic current.
Next, consider the gauge-field kinetic term. Since

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.30}$$

one finds

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}. \tag{1.31}$$

Substituting these results into the Euler–Lagrange equation gives

$$-ej^\nu + \partial_\mu F^{\mu\nu} = 0. \tag{1.32}$$

Hence the equation of motion for the electromagnetic field is

$$\boxed{\partial_\mu F^{\mu\nu} = ej^\nu.} \tag{1.33}$$

This is precisely Maxwell's equation in covariant form with the four-current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{1.34}$$

acting as the source of the electromagnetic field.

The remaining two Maxwell equations,

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0,} \tag{1.35}$$

follow identically from the definition of the field-strength tensor, where

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{1.36}$$

is the dual electromagnetic field tensor.

Thus all four of Maxwell's equations are contained within the QED Lagrangian. The inhomogeneous equations arise from the Euler–Lagrange equation, while the homogeneous equations are geometric identities resulting from the antisymmetry of the field-strength tensor.

Evaluation of $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$. Ex: 1

The kinetic term of the electromagnetic field is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}, \tag{1.37}$$

where

$$F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho. \tag{1.38}$$

To evaluate $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$, we first apply the chain rule,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_\mu A_\nu)}. \tag{1.39}$$

Using the product rule,

$$\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_\mu A_\nu)} = F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} + F_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)}. \tag{1.40}$$

Since

$$F^{\rho\sigma} = g^{\rho\alpha}g^{\sigma\beta}F_{\alpha\beta},$$

the metric tensor is independent of the fields, so

$$\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = g^{\rho\alpha}g^{\sigma\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}.$$

Hence the two terms are identical, giving

$$\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_\mu A_\nu)} = 2F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}. \tag{1.41}$$

Therefore,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2} F^{\rho\sigma} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}. \tag{1.42}$$

Now differentiate the field-strength tensor,

$$\begin{aligned} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \qquad (1.43) \\ &= \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho. \qquad (1.44) \end{aligned}$$

Substituting into the previous equation,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{2} F^{\rho\sigma} \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \right) \qquad (1.45) \\ &= -\frac{1}{2} \left(F^{\mu\nu} - F^{\nu\mu} \right). \qquad (1.46) \end{aligned}$$

Since the field-strength tensor is antisymmetric,

$$F^{\nu\mu} = -F^{\mu\nu},$$

we finally obtain

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}.} \tag{1.47}$$

Recovery of Maxwell's Equations, Ex: 2

The covariant equation

$$\partial_\mu F^{\mu\nu} = e j^\nu \tag{1.48}$$

contains two of Maxwell's equations. To demonstrate this, we express the field-strength tensor in terms of the electric and magnetic fields,

$$F^{0i} = E^i, \qquad F^{ij} = -\epsilon^{ijk} B^k, \tag{1.49}$$

and write the four-current as

$$j^\mu = (\rho, \mathbf{J}), \tag{1.50}$$

where ρ is the charge density and $\mathbf{J}$ is the current density.

Gauss's Law Choosing $\nu = 0$ in Eq. (1.48) gives

$$\partial_\mu F^{\mu 0} = e\rho. \tag{1.51}$$

Since

$$F^{00} = 0,$$

this reduces to

$$\partial_i F^{i0} = e\rho. \tag{1.52}$$

Using

$$F^{i0} = E^i,$$

we obtain

$$\boxed{\nabla \cdot \mathbf{E} = e\rho,} \tag{1.53}$$

which is Gauss's law.

Ampère–Maxwell Law For the spatial components, $\nu = i$,

$$\partial_\mu F^{\mu i} = eJ^i. \tag{1.54}$$

Separating the time and space derivatives,

$$\partial_0 F^{0i} + \partial_j F^{ji} = eJ^i. \tag{1.55}$$

Since

$$F^{0i} = E^i,$$

and

$$F^{ji} = -\epsilon^{jik} B^k,$$

we obtain

$$\boxed{\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = e\mathbf{J},} \tag{1.56}$$

which is the Ampère–Maxwell equation.

The remaining two Maxwell equations do not arise from the Euler–Lagrange equation. Instead, they follow identically from the definition of the field-strength tensor.

Define the dual tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{1.57}$$

The antisymmetry of $F_{\mu\nu}$ immediately implies the Bianchi identity,

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0.} \tag{1.58}$$

This single equation contains the remaining two Maxwell equations.

Gauss's Law for Magnetism Taking $\nu = 0$,

$$\partial_i \tilde{F}^{i0} = 0. \tag{1.59}$$

Since

$$\tilde{F}^{i0} = B^i,$$

we obtain

$$\boxed{\nabla \cdot \mathbf{B} = 0,} \tag{1.60}$$

showing that isolated magnetic monopoles do not exist in classical electrodynamics.

Faraday's Law For $\nu = i$,

$$\partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} = 0. \tag{1.61}$$

Using

$$\tilde{F}^{0i} = B^i, \qquad \tilde{F}^{ji} = \epsilon^{jik} E^k,$$

one obtains

$$\boxed{\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.} \tag{1.62}$$

Thus the four familiar Maxwell equations,

$$\nabla \cdot \mathbf{E} = e\rho, \tag{1.63}$$
$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = e\mathbf{J}, \tag{1.64}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{1.65}$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.66}$$

are compactly encoded in the two Lorentz-covariant equations

$$\partial_\mu F^{\mu\nu} = e j^\nu, \tag{1.67}$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{1.68}$$

1.3 Historical Motivation for Non-Abelian Symmetry

The origin of non-Abelian symmetry lies in nuclear physics. During the 1930s experiments showed that the strong interaction treats protons and neutrons almost identically. After subtracting electromagnetic effects, the nuclear force between two protons, two neutrons, or a proton and a neutron is nearly the same. This remarkable property is known as the charge independence of the strong interaction.

The masses of the proton and neutron are

$$m_p = 938.272 \text{ MeV}, \qquad m_n = 939.565 \text{ MeV},$$

differing by only about 1.293 MeV. Such a tiny difference compared with their average mass suggests that these particles are closely related. Werner Heisenberg proposed that the proton and neutron should be viewed as two internal states of a single particle, the nucleon. This idea led to the introduction of *isospin*.

Isospin is an internal quantum number analogous to ordinary spin, but it acts in an abstract internal space rather than physical space. The proton and neutron form an isospin doublet,

$$\Psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix},$$

whose components correspond to the two eigenstates of isospin,

$$I = \frac{1}{2}, \qquad I_3 = \pm\frac{1}{2}.$$

The free nucleon Lagrangian,

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi,$$

is invariant under global rotations in isospin space,

$$\Psi(x) \to U\Psi(x), \qquad U \in SU(2).$$

Physically this means that, in the limit of exact isospin symmetry, the strong interaction cannot distinguish whether it is acting on a proton or on a neutron.

The success of isospin suggested that internal symmetries might play a fundamental role in particle physics. Yang and Mills asked whether this internal symmetry could be promoted from a global symmetry to a local one, allowing the transformation matrix to vary from one space-time point to another. This simple question led to the birth of non-Abelian gauge theory and ultimately to the gauge structure of the Standard Model.

1.4 From Isospin to SU(N)

The nucleon doublet provides the simplest example of an internal multiplet. More generally, one may consider an N-component field

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix},$$

whose components represent different internal states of the same physical system. The most general transformation preserving probability is represented by a unitary matrix acting in this internal space. Restricting the determinant to unity leads to the special unitary group $SU(N)$, which forms the mathematical foundation of Yang–Mills theories. In the following section we study the structure of the group $SU(N)$ and show how its properties determine the dynamics of the corresponding gauge fields.

1.5 Global and Local $SU(N)$ Symmetry

1.5.1 N-component fermion multiplet

Consider

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix},$$

where each ψ_i is a Dirac spinor. The free Lagrangian is

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = \sum_{i=1}^{N} \bar{\psi}_i (i\gamma^\mu \partial_\mu - m)\psi_i.$$

1.5.2 Global $U(N)$ and $SU(N)$

Under a global transformation $\Psi \to U\Psi$, with $U^\dagger U = I$,

$$\mathcal{L}_0' = \bar{\Psi}U^\dagger(i\gamma^\mu \partial_\mu - m)U\Psi = \mathcal{L}_0.$$

Imposing $\det U = 1$ defines

$$SU(N) = \{U \,|\, U^\dagger U = I,\ \det U = 1\}.$$

Writing

$$U = e^{i\theta^a T^a},$$

the generators satisfy

$$(T^a)^\dagger = T^a, \qquad \mathrm{Tr}(T^a) = 0,$$
$$[T^a, T^b] = i f^{abc} T^c, \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}.$$

Global $U(N)$ and $SU(N)$ details

The free Lagrangian introduced in the previous section possesses a much larger internal symmetry than the proton–neutron example. Consider the transformation

$$\Psi(x) \to U\Psi(x), \tag{1.69}$$

where U is an $N \times N$ matrix acting on the internal degrees of freedom of the multiplet. The multiplet $\Psi(x)$ is an N-component vector in an internal complex vector space. A transformation acting on this multiplet should preserve the natural inner product defined on this space. This requires

$$\Psi^\dagger\Psi = \Psi^\dagger U^\dagger U\Psi = \Psi^\dagger\Psi, \tag{1.70}$$

which immediately implies

$$U^\dagger U = I. \tag{1.71}$$

Matrices satisfying this condition are called *unitary matrices*. The set of all $N \times N$ unitary matrices forms the group $U(N)$.

Since the transformation matrix is independent of space and time, the derivative transforms simply as

$$\partial_\mu(U\Psi) = U\,\partial_\mu\Psi. \tag{1.72}$$

Consequently,

$$\begin{aligned}\mathcal{L}_0' &= \bar{\Psi}U^\dagger\,(i\gamma^\mu\partial_\mu - m)\,U\Psi\\ &= \bar{\Psi}\,(i\gamma^\mu\partial_\mu - m)\,\Psi = \mathcal{L}_0,\end{aligned} \tag{1.73}$$

where we have used the unitarity condition $U^\dagger U = I$. Thus the free Lagrangian is invariant under global $U(N)$ transformations.

A general unitary matrix may have an arbitrary phase,

$$\det U = e^{i\phi}, \tag{1.74}$$

where ϕ is a real parameter. This overall phase corresponds to an independent $U(1)$ symmetry and does not describe rotations within the internal multiplet itself. To isolate the purely non-Abelian part of the symmetry, we impose the additional condition

$$\det U = 1. \tag{1.75}$$

The resulting group is called the *Special Unitary Group*,

$$SU(N) = \left\{ U \,\middle|\, U^\dagger U = I,\ \det U = 1 \right\}. \tag{1.76}$$

The group $SU(N)$ is a continuous Lie group. Any element sufficiently close to the identity can therefore be written as

$$U = e^{i\theta^a T^a}, \tag{1.77}$$

where θ^a are real parameters and T^a are the generators of the group. The index a labels the independent generators.

Expanding the exponential to first order,

$$U = I + i\theta^a T^a + \mathcal{O}(\theta^2), \tag{1.78}$$

and substituting into the unitarity condition,

$$U^\dagger U = I, \tag{1.79}$$

gives

$$(I - i\theta^a T^{a\dagger})(I + i\theta^b T^b) = I. \tag{1.80}$$

Keeping only terms linear in the infinitesimal parameters,

$$I + i\theta^a \left(T^a - T^{a\dagger} \right) = I. \tag{1.81}$$

Since the parameters θ^a are arbitrary, the coefficient of every parameter must vanish independently. Hence,

$$\boxed{(T^a)^\dagger = T^a.} \tag{1.82}$$

Thus every generator of $SU(N)$ is Hermitian. The condition $\det U = 1$ imposes another important restriction on the generators. Using the identity

$$\det\left(e^M\right) = e^{\mathrm{Tr}(M)}, \tag{1.83}$$

we obtain

$$1 = \det(U) = \det\left(e^{i\theta^a T^a}\right) = e^{\,i\theta^a \mathrm{Tr}(T^a)}. \tag{1.84}$$

Since this equation must hold for arbitrary values of the parameters θ^a, it follows that

$$\boxed{\mathrm{Tr}(T^a) = 0.} \tag{1.85}$$

Therefore the generators of $SU(N)$ are traceless. Physically, this condition removes the overall $U(1)$ phase transformation, leaving only genuine rotations in the internal N-dimensional space.

An important question is how many independent generators the group possesses. A general complex $N \times N$ matrix contains N^2 complex entries, corresponding to $2N^2$ real parameters. The unitarity condition $U^\dagger U = I$ imposes N^2 independent constraints, leaving N^2 real parameters. Finally, the condition $\det U = 1$ removes one additional parameter. Hence the dimension of the group is

$$\boxed{N^2 - 1.} \tag{1.86}$$

Thus the group $SU(N)$ possesses exactly $N^2 - 1$ independent generators. For example, $SU(2)$ has three generators, corresponding to the three Pauli matrices, while $SU(3)$ has eight generators, represented by the Gell–Mann matrices.

The generators do not, in general, commute with one another. Instead, they satisfy the commutation relation

$$\boxed{[T^a, T^b] = if^{abc}T^c,} \tag{1.87}$$

where the coefficients f^{abc} are called the *structure constants*. These constants completely characterize the Lie algebra associated with the group. Unlike the group itself, which describes finite transformations, the Lie algebra determines the behaviour of infinitesimal transformations and therefore forms the mathematical foundation of gauge theories.

The closure of the commutator is one of the defining properties of a Lie algebra: the commutator of any two generators is again a linear combination of the generators. For Abelian groups such as $U(1)$, all generators commute and the structure constants vanish. In contrast, for non-Abelian groups such as $SU(2)$ and $SU(3)$, the generators do not commute, giving rise to non-zero structure constants. It is precisely this non-commutativity that leads to the self-interaction of gauge bosons in Yang–Mills theories, a feature absent in Quantum Electrodynamics.

Finally, the generators are conventionally normalized according to

$$\boxed{\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab},} \tag{1.88}$$

which fixes the overall normalization of the generators and greatly simplifies the construction of gauge-invariant Lagrangians.

1.5.3 The Groups $SU(2)$ and $SU(3)$

Having discussed the general mathematical properties of the group $SU(N)$, it is useful to consider the two most important examples in particle physics, namely $SU(2)$ and $SU(3)$. These groups form the basis of the electroweak and strong interactions, respectively.

The Group $SU(2)$

The group $SU(2)$ consists of all 2×2 unitary matrices with unit determinant,

$$SU(2) = \left\{ U \mid U^\dagger U = I, \ \det U = 1 \right\}.$$

Since the dimension of the group is

$$N^2 - 1 = 2^2 - 1 = 3,$$

the group possesses three independent generators. In the fundamental representation, these generators are chosen to be one-half of the Pauli matrices,

$$T^a = \frac{\sigma^a}{2}, \qquad a = 1, 2, 3,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices are Hermitian,

$$(\sigma^a)^\dagger = \sigma^a,$$

traceless,

$$\mathrm{Tr}(\sigma^a) = 0,$$

and satisfy the commutation relation

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

Consequently, the generators satisfy the Lie algebra

$$[T^a, T^b] = i\epsilon^{abc}T^c,$$

where ϵ^{abc} is the completely antisymmetric Levi–Civita tensor. The corresponding structure constants are therefore

$$f^{abc} = \epsilon^{abc}.$$

The generators are normalized according to

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}.$$

In particle physics, the fundamental representation of $SU(2)$ acts on two-component multiplets, such as the proton–neutron isospin doublet or the left-handed fermion doublets of the electroweak theory.

The Group $SU(3)$

The group $SU(3)$ consists of all 3×3 unitary matrices with unit determinant,

$$SU(3) = \left\{ U \mid U^\dagger U = I,\ \det U = 1 \right\}.$$

Its dimension is

$$N^2 - 1 = 3^2 - 1 = 8,$$

and therefore $SU(3)$ possesses eight independent generators.
In the fundamental representation, the generators are chosen to be

$$T^a = \frac{\lambda^a}{2}, \qquad a = 1, \cdots, 8,$$

where λ^a are the eight Gell–Mann matrices,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Like the Pauli matrices, the Gell–Mann matrices are Hermitian and traceless,

$$(\lambda^a)^\dagger = \lambda^a, \qquad \mathrm{Tr}(\lambda^a) = 0,$$

and satisfy the Lie algebra

$$[T^a, T^b] = i f^{abc} T^c,$$

where f^{abc} are the structure constants of $SU(3)$. The normalization of the generators is chosen to be

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}.$$

In Quantum Chromodynamics (QCD), the fundamental representation of $SU(3)$ acts on the three colour states of a quark,

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix},$$

where q_r, q_g, and q_b denote the red, green, and blue colour components, respectively. Since the group has eight generators, the corresponding gauge theory contains eight gauge bosons, namely the eight gluons.

Fundamental and Adjoint Representations A representation of a Lie group specifies how the elements of the group act on a vector space. The simplest non-trivial representation is the *fundamental representation*, in which the group acts directly on an N-component vector. For $SU(N)$, a field in the fundamental representation transforms as

$$\Psi \to U\Psi,$$

where U is an $N \times N$ unitary matrix with $\det U = 1$. The generators in this representation are $N \times N$ Hermitian traceless matrices. Examples include the proton–neutron isospin doublet in $SU(2)$ and the colour triplet of quarks in $SU(3)$.

Another important representation is the *adjoint representation*, whose dimension is equal to the number of generators of the group, namely $N^2 - 1$. In this representation, the generators are represented by $(N^2 - 1) \times (N^2 - 1)$ matrices. Using the Lie algebra

$$[T^a, T^b] = i f^{abc} T^c,$$

the matrix elements of the generators in the adjoint representation are given by

$$(T^a_{\mathrm{adj}})_{bc} = -i f^{abc},$$

where the overall sign depends on the convention adopted for the Lie algebra. Thus, the structure constants determine the generators of the adjoint representation. Example is the Gluons transform as adjoint representation.

1.5.4 Local $SU(N)$ Symmetry

In the previous section, we showed that the free Dirac Lagrangian

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{1.89}$$

is invariant under the global $SU(N)$ transformation

$$\Psi(x) \to U\Psi(x), \qquad U = e^{i\theta^a T^a}, \tag{1.90}$$

where the transformation parameters θ^a are constants, independent of space and time. The invariance follows because the derivative transforms in the same manner as the field,

$$\partial_\mu(U\Psi) = U\partial_\mu\Psi, \tag{1.91}$$

since U is constant.

The principle of local gauge invariance demands a stronger symmetry. Instead of requiring the transformation to be identical at every point in space-time, we allow the group parameters to depend on the space-time coordinates,

$$\theta^a \to \theta^a(x). \tag{1.92}$$

The transformation matrix therefore becomes

$$U(x) = e^{i\theta^a(x)T^a}, \tag{1.93}$$

and the fermion multiplet transforms as

$$\boxed{\Psi(x) \to \Psi'(x) = U(x)\Psi(x).} \tag{1.94}$$

Unlike the global case, the derivative of the transformed field now becomes

$$\begin{aligned}\partial_\mu\Psi' &= \partial_\mu(U\Psi) \\ &= (\partial_\mu U)\Psi + U\partial_\mu\Psi.\end{aligned} \tag{1.95}$$

The additional term $(\partial_\mu U)\Psi$ is a consequence of the space-time dependence of the transformation matrix. It is absent for global transformations but survives for local transformations. Consequently,

$$\partial_\mu\Psi \not\to U\partial_\mu\Psi, \tag{1.96}$$

and therefore the kinetic term of the free Lagrangian is no longer invariant.

Rather than abandoning local symmetry, we modify the derivative operator itself. We introduce a new derivative, called the *covariant derivative*,

$$\boxed{D_\mu = \partial_\mu - igA_\mu,} \tag{1.97}$$

where g denotes the gauge coupling constant and

$$A_\mu = A_\mu^a T^a \tag{1.98}$$

with A_μ^a are the gauge fields. Since the generators T^a are matrices acting on the internal space, the gauge field is also matrix-valued.

The purpose of the covariant derivative is to compensate for the extra term $(\partial_\mu U)\Psi$ generated by the ordinary derivative. We therefore require the covariant derivative to transform in exactly the same way as the fermion field,

$$\boxed{D_\mu \Psi \to D'_\mu \Psi' = U(D_\mu \Psi).} \tag{1.99}$$

This condition is known as the *covariance condition.* It ensures that if the field transforms according to the fundamental representation of the gauge group, then its covariant derivative transforms in precisely the same manner. The locally invariant fermion Lagrangian is

$$\mathcal{L}_f = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi.$$

Substituting

$$D_\mu = \partial_\mu - igA_\mu,$$

one obtains

$$(\partial_\mu - igA'_\mu)(U\Psi) = U(\partial_\mu - igA_\mu)\Psi. \tag{1.100}$$

Expanding the left-hand side,

$$(\partial_\mu U)\Psi + U\partial_\mu \Psi - igA'_\mu U\Psi = U\partial_\mu \Psi - igUA_\mu \Psi. \tag{1.101}$$

The terms involving $U\partial_\mu \Psi$ cancel from both sides, leaving

$$(\partial_\mu U)\Psi - igA'_\mu U\Psi = -igUA_\mu \Psi. \tag{1.102}$$

Since this equation must hold for an arbitrary fermion field Ψ, we obtain

$$(\partial_\mu U) - igA'_\mu U = -igUA_\mu. \tag{1.103}$$

Multiplying from the right by U^{-1} gives

$$igA'_\mu = igUA_\mu U^{-1} - (\partial_\mu U)U^{-1}, \tag{1.104}$$

or equivalently,

$$\boxed{A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}.} \tag{1.105}$$

This is the transformation law of a non-Abelian gauge field. It differs from the transformation law of the electromagnetic field because

1.5.5 The Yang–Mills Lagrangian

Having introduced the covariant derivative and the field strength tensor, we are now in a position to construct the Lagrangian of a non-Abelian gauge theory. As in Quantum Electrodynamics, the complete Lagrangian should contain two parts. The first describes the interaction of the fermion multiplet with the gauge fields, while the second describes the dynamics of the gauge fields themselves.

The fermionic part of the Lagrangian is obtained by replacing the ordinary derivative by the covariant derivative,

$$\boxed{\mathcal{L}_f = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi,} \tag{1.106}$$

where

$$D_\mu = \partial_\mu - igA_\mu.$$

Since the covariant derivative transforms as

$$D'_\mu = UD_\mu U^{-1},$$

and the fermion fields transform as

$$\Psi' = U\Psi, \qquad \bar{\Psi}' = \bar{\Psi}U^{-1},$$

it follows immediately that

$$\begin{aligned}\mathcal{L}'_f &= \bar{\Psi}'(i\gamma^\mu D'_\mu - m)\Psi' \\ &= \bar{\Psi}U^{-1}(i\gamma^\mu UD_\mu U^{-1} - m)U\Psi \\ &= \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \\ &= \mathcal{L}_f. \end{aligned} \tag{1.107}$$

Hence the fermionic part of the Lagrangian is invariant under local $SU(N)$ gauge transformations.

The next step is to construct a kinetic term for the gauge fields. Since the gauge field is matrix-valued,

$$A_\mu = A^a_\mu T^a,$$

the field strength tensor is also matrix-valued,

$$F_{\mu\nu} = F^a_{\mu\nu} T^a.$$

Define

$$[D_\mu, D_\nu] = -igF_{\mu\nu},$$

giving

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

Hence

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu.$$

Under a local gauge transformation,

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1}.$$

Unlike the Abelian case, the quantity

$$F_{\mu\nu}F^{\mu\nu}$$

is itself a matrix. Therefore it is not gauge invariant, since

$$F'_{\mu\nu}F'^{\mu\nu} = UF_{\mu\nu}U^{-1}UF^{\mu\nu}U^{-1} = UF_{\mu\nu}F^{\mu\nu}U^{-1},$$

which is related to the original quantity by a similarity transformation.

To construct a gauge-invariant scalar, we take the trace over the internal group indices. Using the cyclic property of the trace,

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB),$$

we obtain

$$
\begin{aligned}
\mathrm{Tr}(F'_{\mu\nu}F'^{\mu\nu}) &= \mathrm{Tr}\left(UF_{\mu\nu}U^{-1}UF^{\mu\nu}U^{-1}\right) \\
&= \mathrm{Tr}\left(UF_{\mu\nu}F^{\mu\nu}U^{-1}\right) \\
&= \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}U^{-1}U\right) \\
&= \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}).
\end{aligned}
\tag{1.108}
$$

Hence,

$$
\boxed{\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})}
$$

is gauge invariant and is therefore the appropriate kinetic term for the gauge fields.

1.5.6 The Yang–Mills Lagrangian

Having established the transformation properties of the fermion fields, the gauge fields, and the field strength tensor, we are now in a position to write down the complete Lagrangian describing a non-Abelian gauge theory. The Yang–Mills Lagrangian consists of two parts. The first describes the dynamics of the gauge fields, while the second describes the interaction of the gauge fields with the fermion multiplet.

The complete Yang–Mills Lagrangian is

$$
\boxed{\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi.}
\tag{1.109}
$$

The first term is the kinetic energy of the gauge fields, whereas the second term is the Dirac Lagrangian in which the ordinary derivative has been replaced by the covariant derivative,

$$
D_\mu = \partial_\mu - igA_\mu.
$$

Consequently, the interaction between the fermion multiplet and the gauge fields arises naturally from the requirement of local gauge invariance and does not need to be introduced by hand.

The gauge field strength tensor is given by

$$
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu.
$$

Substituting this expression into the kinetic term,

$$
-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},
$$

one immediately notices an important difference from Quantum Electrodynamics. Unlike the electromagnetic field tensor,

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
$$

the non-Abelian field strength tensor contains an additional term proportional to

$$
gf^{abc}A^b_\mu A^c_\nu.
$$

This term originates from the non-commutativity of the generators,

$$
[T^a, T^b] = if^{abc}T^c,
$$

and therefore vanishes only for Abelian groups, for which the structure constants are zero.

To understand the physical consequences, let us expand the gauge kinetic term,

$$F^a_{\mu\nu} F^{a\mu\nu}.$$

Using

$$F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + g f^{abc} A^b_\mu A^c_\nu,$$

one obtains three different kinds of contributions,

$$F^a_{\mu\nu} F^{a\mu\nu} = (\partial A)^2 + 2g(\partial A)AA + g^2 A^4. \tag{1.110}$$

Although this notation is schematic, it clearly illustrates the structure of the Lagrangian. The first term contains two gauge fields and represents the propagation of the gauge bosons. The second term contains three gauge fields and gives rise to three-gauge-boson interaction vertices. The last term contains four gauge fields and produces four-gauge-boson interaction vertices.

Thus, the Yang–Mills Lagrangian predicts that gauge bosons can interact directly with one another. This remarkable feature has no analogue in Quantum Electrodynamics.

To see this more clearly, recall that the electromagnetic field tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which is linear in the gauge field. Consequently,

$$F_{\mu\nu} F^{\mu\nu}$$

contains only quadratic terms in the photon field. There are no cubic or quartic terms, and therefore photons do not interact directly with other photons at tree level.

In contrast, the field strength tensor of a non-Abelian gauge theory is nonlinear in the gauge fields because of the commutator term,

$$ig[A_\mu, A_\nu].$$

As a consequence, the Yang–Mills Lagrangian automatically contains both cubic and quartic interaction terms. The existence of these self-interactions is therefore a direct consequence of the non-Abelian nature of the gauge group.

This distinction between Abelian and non-Abelian gauge theories has profound physical consequences. In Quantum Chromodynamics, the eight gluons carry colour charge and interact strongly with one another through these self-interaction vertices. These interactions are responsible for many characteristic features of the strong interaction, including asymptotic freedom and colour confinement. On the other hand, the photon is electrically neutral and does not possess such self-interactions. Consequently, Quantum Electrodynamics remains a linear gauge theory to an excellent approximation.

The Yang–Mills Lagrangian therefore provides a unified mathematical framework for all non-Abelian gauge theories. By choosing the appropriate gauge group, it forms the basis of both Quantum Chromodynamics, based on the symmetry group $SU(3)_C$, and the electroweak theory, based on the symmetry group $SU(2)_L \times U(1)_Y$. Together these gauge symmetries constitute the Standard Model of particle physics.

Inclusion of a Complex Scalar Multiplet

So far, we have considered only fermionic matter fields transforming under the fundamental representation of the gauge group. Suppose that, in addition to the fermion multiplet $\Psi(x)$, the theory also contains a complex scalar multiplet

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_N(x) \end{pmatrix},$$

which transforms under the fundamental representation of $SU(N)$ as

$$\boxed{\Phi(x) \to \Phi'(x) = U(x)\Phi(x),}$$

where

$$U(x) = e^{i\theta^a(x)T^a}.$$

The free Lagrangian for the scalar multiplet is

$$\mathcal{L}_{\text{scalar}} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - V(\Phi),$$

where $V(\Phi)$ denotes the scalar potential. Since the ordinary derivative does not transform covariantly under a local gauge transformation, the kinetic term is not gauge invariant.

To restore local gauge invariance, the ordinary derivative is replaced by the covariant derivative,

$$\boxed{D_\mu \Phi = (\partial_\mu - igA_\mu)\Phi.}$$

The covariant derivative satisfies

$$D_\mu \Phi \to D'_\mu \Phi' = U(D_\mu \Phi),$$

so that

$$(D_\mu \Phi)^\dagger (D^\mu \Phi)$$

is gauge invariant.

The gauge-invariant Lagrangian for the scalar multiplet is therefore

$$\boxed{\mathcal{L}_{\text{scalar}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi).}$$

The scalar potential must also be invariant under the gauge symmetry. The simplest renormalizable potential is

$$\boxed{V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2,}$$

where μ^2 and λ are real constants. Since

$$\Phi^\dagger \Phi \to \Phi^\dagger U^\dagger U \Phi = \Phi^\dagger \Phi,$$

the potential is manifestly invariant under $SU(N)$ gauge transformations.

Including both fermionic and scalar matter fields, the complete Yang–Mills Lagrangian takes the form

$$\boxed{\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi\\ &+ (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi).\end{aligned}}$$

This expression forms the basis of a general non-Abelian gauge theory containing both fermions and scalar fields. In the Standard Model, the scalar multiplet is identified with the Higgs doublet, which transforms as a fundamental representation of the electroweak group $SU(2)_L$.

Yukawa Interactions Besides gauge interactions, scalar fields may also couple directly to fermions through Yukawa interactions. Let Ψ_1 and Ψ_2 denote two fermion multiplets transforming under the representations R_1 and R_2 of the gauge group, while the scalar multiplet Φ transforms under the representation R_Φ. Under a gauge transformation,

$$\Psi_1 \to U_{R_1}\Psi_1, \qquad \Psi_2 \to U_{R_2}\Psi_2, \qquad \Phi \to U_{R_\Phi}\Phi,$$

where the representation matrices U_{R_i} depend on the representation carried by the corresponding field.

The most general renormalizable Yukawa interaction can be written schematically as

$$\boxed{\mathcal{L}_Y = -y\,\bar{\Psi}_1\,\Phi\,\Psi_2 + \text{h.c.},}$$

where y is the Yukawa coupling constant. Strictly speaking, the indices associated with the gauge group are contracted with appropriate invariant tensors, which have been suppressed for simplicity.

The Yukawa interaction is gauge invariant only if the product of representations contains a singlet, i.e.,

$$R_1^* \otimes R_\Phi \otimes R_2 \supset \mathbf{1}.$$

If this condition is not satisfied, no gauge-invariant Yukawa interaction can be written down. Thus, the allowed Yukawa couplings are completely determined by the representation content of the theory.

In the Standard Model, for example, the left-handed fermions transform as $SU(2)_L$ doublets, the right-handed fermions transform as singlets, while the Higgs field is an $SU(2)_L$ doublet. Their tensor product contains an $SU(2)_L$ singlet, allowing gauge-invariant Yukawa interactions that generate fermion masses after spontaneous symmetry breaking.

1.5.7 Feynman Rules

The principal objective of constructing the Yang–Mills Lagrangian is to calculate scattering amplitudes involving fermions, scalar fields and gauge bosons. These amplitudes are evaluated using perturbation theory, in which every term of the interaction Lagrangian corresponds to a Feynman diagram. The Feynman rules specify the propagators and interaction vertices associated with each field. They are obtained by expanding the Lagrangian in powers of the fields and identifying the coefficients of the interaction terms.

In general, the Feynman rules are derived by following three simple steps.

1. Identify the quadratic terms in the Lagrangian. These determine the propagators.

2. Identify the interaction terms involving three or more fields. These determine the interaction vertices.

3. Replace every derivative acting on a field by the corresponding momentum carried by that field according to

$$\partial_\mu \to -ip_\mu,$$

and multiply each interaction term by an overall factor of i.

The most commonly used Feynman rules are summarized below.

Fermion propagator The quadratic fermion Lagrangian

$$\bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

gives the propagator

$$\boxed{\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}.}$$

Scalar propagator The scalar kinetic term

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) - m_\phi^2 \Phi^\dagger \Phi$$

gives the propagator

$$\boxed{\frac{i}{p^2 - m_\phi^2 + i\epsilon}.}$$

Gauge Boson Propagator The propagator of a field is obtained by inverting the quadratic part of its Lagrangian. For the Yang–Mills field, the free Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu.$$

Retaining only the quadratic terms in the gauge fields and integrating by parts, the Lagrangian may be written as

$$\boxed{\mathcal{L} = \frac{1}{2} A^a_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu\right) A^a_\nu.}$$

The corresponding kinetic operator is therefore

$$K^{\mu\nu} = g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu.$$

In momentum space,

$$K^{\mu\nu}(p) = -p^2 g^{\mu\nu} + p^\mu p^\nu.$$

Notice that

$$K^{\mu\nu} p_\nu = -p^2 p^\mu + p^\mu p^2 = 0.$$

Thus the kinetic operator possesses a zero eigenvalue corresponding to the gauge degree of freedom. Consequently, its inverse does not exist. Since the propagator is defined as the inverse of the kinetic operator,

$$D_{\mu\nu} = i(K^{-1})_{\mu\nu},$$

the gauge boson propagator cannot be obtained directly from the Yang–Mills Lagrangian. This is a direct consequence of gauge invariance, which implies that the gauge field contains redundant, unphysical degrees of freedom.

To obtain an invertible kinetic operator, one introduces a gauge-fixing term into the Lagrangian,

$$\boxed{\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial_\mu A^{a\mu})^2,}$$

where ξ is an arbitrary gauge parameter. The quadratic part of the Lagrangian then becomes

$$\boxed{\mathcal{L} = \frac{1}{2}A^a_\mu \left[g^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A^a_\nu,}$$

whose inverse now exists. The gauge boson propagator is therefore

$$\boxed{D^{ab}_{\mu\nu}(p) = \frac{i\delta^{ab}}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right].}$$

This expression is known as the covariant gauge propagator. Different choices of the gauge parameter ξ correspond to different gauge choices. For $\xi = 1$, one obtains the **Feynman gauge**,

$$\boxed{D^{ab}_{\mu\nu}(p) = \frac{ig_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}.}$$

This is the simplest form of the propagator and is widely used in perturbative calculations. For $\xi = 0$, one obtains the **Landau gauge**,

$$\boxed{D^{ab}_{\mu\nu}(p) = \frac{i\delta^{ab}}{p^2 + i\epsilon}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right).}$$

The Landau gauge is a transverse gauge since

$$p^\mu D_{\mu\nu}(p) = 0.$$

Although the explicit form of the propagator depends on the choice of gauge, all physical observables, such as scattering cross

Fermion–Gauge Boson Vertex Expanding the covariant derivative,

$$D_\mu = \partial_\mu - igA_\mu,$$

the interaction term becomes

$$-g\,\bar{\Psi}\gamma^\mu A^a_\mu T^a \Psi.$$

Hence the corresponding vertex factor is

$$\boxed{-ig\gamma^\mu T^a.}$$

This is the direct generalization of the QED vertex $-ie\gamma^\mu$.

Scalar–Gauge Boson Vertex The scalar kinetic term

$$(D_\mu\Phi)^\dagger(D^\mu\Phi)$$

contains the interaction

$$igA^a_\mu\left[(\partial^\mu\Phi^\dagger)T^a\Phi-\Phi^\dagger T^a(\partial^\mu\Phi)\right].$$

The corresponding vertex factor is

$$\boxed{ig(p+p')^\mu T^a,}$$

where p and p' denote the incoming and outgoing scalar momenta.

Two Scalar–Two Gauge Boson Vertex The same kinetic term also contains the quartic interaction

$$g^2\Phi^\dagger A^a_\mu T^a A^{b\mu}T^b\Phi,$$

leading to a four-point interaction between two scalars and two gauge bosons.

Yukawa Vertex The Yukawa interaction

$$\mathcal{L}_Y=-y\,\bar\Psi_1\Phi\Psi_2+\text{h.c.}$$

gives the vertex

$$\boxed{-i\,y.}$$

Three-Gauge Boson Vertex Unlike Quantum Electrodynamics, the Yang–Mills Lagrangian contains cubic gauge boson interactions originating from the non-Abelian field strength tensor,

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu.$$

The corresponding three-gauge-boson vertex is

$$\boxed{gf^{abc}\left[g^{\mu\nu}(p-q)^\rho+g^{\nu\rho}(q-r)^\mu+g^{\rho\mu}(r-p)^\nu\right],}$$

where all momenta are taken to be incoming.

Four-Gauge Boson Vertex The quartic interaction in the Yang–Mills Lagrangian produces a four-gauge-boson vertex. Its expression is proportional to

$$g^2f^{abe}f^{cde},$$

together with permutations of the Lorentz indices. Although lengthy, this vertex is a direct consequence of the non-Abelian nature of the gauge symmetry.

The most striking feature of a Yang–Mills theory is the existence of the three-gauge-boson and four-gauge-boson vertices. These vertices arise because the gauge bosons themselves carry the gauge charge associated with the non-Abelian symmetry. In contrast, the photon does not carry electric charge and therefore has no tree-level self-interaction. This distinction is one of the defining differences between Abelian and non-Abelian gauge theories.

1.5.8 General Yang–Mills Theory with Product Gauge Groups

So far we have considered a Yang–Mills theory based on a single gauge group $SU(N)$. Many physically relevant theories, however, are based on a direct product of gauge groups. Suppose that the gauge symmetry of the theory is

$$G = SU(N) \times SU(M) \times U(1).$$

Each factor of the gauge group possesses its own gauge field and coupling constant. The corresponding gauge fields are

$$A_\mu = A_\mu^a T_N^a,$$

$$B_\mu = B_\mu^\alpha T_M^\alpha,$$

and

$$C_\mu,$$

associated with the groups $SU(N)$, $SU(M)$ and $U(1)$, respectively. Their gauge couplings are denoted by

$$g_N, \qquad g_M, \qquad g'.$$

The generators of the two non-Abelian groups satisfy

$$[T_N^a, T_N^b] = i f_N^{abc} T_N^c,$$

and

$$[T_M^\alpha, T_M^\beta] = i f_M^{\alpha\beta\gamma} T_M^\gamma,$$

while the $U(1)$ generator is Abelian and therefore satisfies

$$[Y, Y] = 0.$$

The field strengths corresponding to the three gauge groups are

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_N f_N^{abc} A_\mu^b A_\nu^c,$$

$$G_{\mu\nu}^\alpha = \partial_\mu B_\nu^\alpha - \partial_\nu B_\mu^\alpha + g_M f_M^{\alpha\beta\gamma} B_\mu^\beta B_\nu^\gamma,$$

and

$$C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu.$$

Notice that the $U(1)$ field strength contains no self-interaction term because the generator of an Abelian group commutes with itself.

The covariant derivative acting on a matter multiplet carrying representations under all three groups is

$$\boxed{D_\mu = \partial_\mu - i g_N A_\mu^a T_N^a - i g_M B_\mu^\alpha T_M^\alpha - i g' Y C_\mu.}$$

Here Y denotes the $U(1)$ charge (or hypercharge in the Standard Model).

The kinetic terms of the gauge fields are

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},$$

$$-\frac{1}{4}G^\alpha_{\mu\nu}G^{\alpha\mu\nu},$$

and

$$-\frac{1}{4}C_{\mu\nu}C^{\mu\nu}.$$

Suppose that the theory contains a fermion multiplet Ψ and a complex scalar multiplet Φ, both transforming under appropriate representations of the gauge group. Their gauge-invariant kinetic terms are

$$\bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi,$$

and

$$(D_\mu\Phi)^\dagger(D^\mu\Phi).$$

The scalar self-interactions are described by the potential

$$V(\Phi),$$

which must be invariant under the full gauge group.

The theory may also contain Yukawa interactions between two fermion multiplets and the scalar multiplet,

$$\mathcal{L}_Y = -y_{ij}\bar{\Psi}_{1i}\Phi\Psi_{2j} + \text{h.c.},$$

provided that the product of representations forms an overall gauge singlet.

The complete renormalizable Lagrangian for a gauge theory based on

$$SU(N) \times SU(M) \times U(1)$$

is therefore

$$\boxed{\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G^\alpha_{\mu\nu}G^{\alpha\mu\nu} - \frac{1}{4}C_{\mu\nu}C^{\mu\nu} \\ &+ \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \\ &+ (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi) \\ &- y_{ij}\bar{\Psi}_{1i}\Phi\Psi_{2j} + \text{h.c.}\end{aligned}}$$

This Lagrangian provides the general framework for gauge theories based on product gauge groups. By choosing suitable values of N and M, together with appropriate representations for the matter fields, one obtains many physically important theories. The Standard Model of particle physics is obtained by choosing the gauge group

$$SU(3)_C \times SU(2)_L \times U(1)_Y,$$

which will be discussed in the next chapter.

1.6 The Standard Model as a Yang–Mills Gauge Theory

The formalism developed in the previous sections forms the mathematical foundation of the Standard Model of particle physics. The Standard Model is a renormalizable Yang–Mills gauge theory based on the direct product gauge group

$$\boxed{G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y,}$$

where $SU(3)_C$ describes the strong interaction, $SU(2)_L$ describes the weak interaction, and $U(1)_Y$ corresponds to the weak hypercharge symmetry. Every elementary particle transforms under a particular representation of this gauge group, and all interactions arise from the requirement of local gauge invariance.

1.6.1 Gauge Fields

Each factor of the gauge group possesses its own gauge field and gauge coupling,

$$SU(3)_C \quad \longrightarrow \quad G^a_\mu, \qquad a = 1, \cdots, 8,$$

$$SU(2)_L \quad \longrightarrow \quad W^i_\mu, \qquad i = 1, 2, 3,$$

$$U(1)_Y \quad \longrightarrow \quad B_\mu.$$

The corresponding gauge couplings are

$$g_s, \qquad g, \qquad g'.$$

The eight gluons are associated with the generators of $SU(3)$, the three weak gauge bosons correspond to the generators of $SU(2)$, while the Abelian group $U(1)_Y$ possesses a single gauge field.

1.6.2 Field Strength Tensors

The field strength tensors are

$$\boxed{G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu,}$$

$$\boxed{W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk} W^j_\mu W^k_\nu,}$$

and

$$\boxed{B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.}$$

Notice that only the non-Abelian field strength tensors contain self-interaction terms. Consequently, gluons and weak gauge bosons possess three- and four-gauge-boson vertices, whereas the Abelian gauge field B_μ has no self-interaction.

1.6.3 Left- and Right-Handed Fermions

Unlike Quantum Chromodynamics and Quantum Electrodynamics, the weak interaction is chiral. The left-handed and right-handed components of a fermion transform differently under $SU(2)_L$.

The projection operators

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2},$$

allow the decomposition

$$\psi_L = P_L\psi, \qquad \psi_R = P_R\psi.$$

Only the left-handed fermions transform as doublets of $SU(2)_L$, whereas the right-handed fermions are singlets.

For one generation of leptons,

$$L_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \sim (1, 2, -1),$$

$$e_R \sim (1, 1, -2).$$

Similarly, the quark multiplets are

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \sim \left(3, 2, \frac{1}{3}\right),$$

$$u_R \sim \left(3, 1, \frac{4}{3}\right),$$

$$d_R \sim \left(3, 1, -\frac{2}{3}\right).$$

Here the three entries denote the representations under

$$SU(3)_C, \qquad SU(2)_L, \qquad U(1)_Y,$$

respectively.

1.6.4 The Higgs Doublet

The Standard Model contains a single complex scalar doublet,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix},$$

which transforms as

$$\boxed{\Phi \sim (1, 2, 1).}$$

The Higgs field is responsible for spontaneous symmetry breaking and generates the masses of the gauge bosons and fermions through the Higgs mechanism.

1.6.5 Covariant Derivative

The covariant derivative acting on a field carrying colour, weak isospin and hypercharge is

$$\boxed{D_\mu = \partial_\mu - ig_s T^a G^a_\mu - ig\frac{\sigma^i}{2}W^i_\mu - ig'\frac{Y}{2}B_\mu,}$$

where

$$T^a = \frac{\lambda^a}{2}$$

are the generators of $SU(3)$ in the fundamental representation,

$$\frac{\sigma^i}{2}$$

are the generators of $SU(2)$, and Y denotes the weak hypercharge.

1.6.6 The Standard Model Lagrangian

The complete Standard Model Lagrangian can be written as

$$\boxed{\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{Gauge}} + \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}.}$$

The gauge sector is

$$\boxed{\mathcal{L}_{\text{Gauge}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}.}$$

The fermion kinetic term is

$$\boxed{\mathcal{L}_{\text{Fermion}} = \sum_f \bar{\psi}_f i\gamma^\mu D_\mu \psi_f.}$$

The Higgs sector is

$$\boxed{\mathcal{L}_{\text{Higgs}} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi),}$$

where

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2.$$

Finally, the Yukawa interactions are

$$\boxed{\begin{aligned}\mathcal{L}_{\text{Yukawa}} = &- y_d\,\bar{Q}_L\Phi d_R \\ &- y_u\,\bar{Q}_L\widetilde{\Phi}u_R \\ &- y_e\,\bar{L}_L\Phi e_R + \text{h.c.},\end{aligned}}$$

where

$$\widetilde{\Phi} = i\sigma_2\Phi^*.$$

After spontaneous symmetry breaking, these Yukawa interactions generate the masses of the charged leptons and quarks. In addition to the gauge invariant Lagrangian there are also two other type of terms we add, which are $\mathcal{L}_{GF}$ and $\mathcal{L}_{ghost}$, which are included to overcome the complexity of gauge field quantization.

Summary

The Standard Model is therefore a renormalizable Yang–Mills gauge theory based on the symmetry group

$$SU(3)_C \times SU(2)_L \times U(1)_Y.$$

Its interactions are completely determined by the principle of local gauge invariance. The gauge fields mediate the strong and electroweak interactions, the Higgs doublet induces spontaneous symmetry breaking, and the Yukawa interactions generate fermion masses. In the following chapter, we shall study spontaneous symmetry breaking in detail and show how the physical gauge bosons $W^{\pm}$, Z and γ emerge from the electroweak gauge fields.