L-2

① $\rightarrow$ Standard Model is based on the principle that — Any theory of fundamental nature, which explains physics at High Energy must be consistent with relativity & quantam theory.

$\rightarrow$ The problem of single particle relativistic wave equation is

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}\, \psi$$

in the above equation L.H.S $\frac{\partial \psi}{\partial t}$ & r.H.S $\sqrt{\nabla^2 \psi}$. upon expansion it will give infinite terms containing ∇^2 on ψ.

one can instead use

$$E = i\hbar \frac{\partial}{\partial t} \quad \& \quad \vec{p} = -i\hbar \vec{\nabla} \quad \text{in}$$

$$E^2 = p^2 + m^2$$

which will give

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right)\psi + m^2 \psi = 0$$

$$\Rightarrow \boxed{\Box\psi + m^2 \psi = 0} \Rightarrow \text{Klein-Gordon equation}$$

Relativistic invariant where ψ is a wave-function.

→ However the problem is this kind of equation can not describe a process where new particles are produced or particle number not conserved.

Slide on LHC/RHIC events

Other problems

→ probability density associated with wave-function $\rho = \psi^* \psi$ in non relativistic approach.

$$\& \quad j = \frac{-i\hbar}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \nabla \psi^* \right)$$

Together they satisfy continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} . \vec{j} = 0$$

With relativistic approach, define

$$j^\mu = (\rho, \vec{j}\,) = \frac{i\hbar}{m} \psi^* \overleftrightarrow{\partial^\mu} \psi$$

definition $\quad A \overleftrightarrow{\partial^\mu} B = \frac{1}{2} \left[A \partial^\mu B - (\partial^\mu A) B \right]$

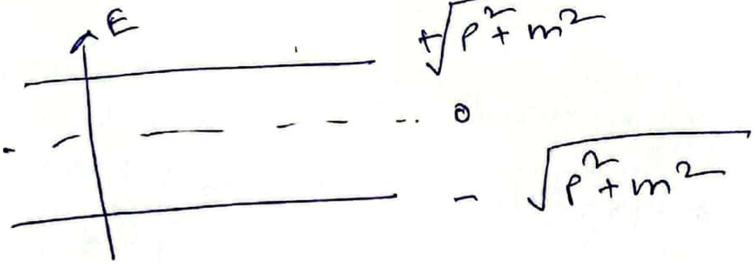
$$\text{So,} \quad j^0 = \frac{i\hbar}{2m} \left[\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right]$$

$$j^i = \frac{-i\hbar}{2m} \left[\psi^* \nabla \psi - (\nabla \psi^*) \psi \right]$$

$$\boxed{\partial_\mu j^\mu = 0}$$

However ρ now can be negative.
So ρ as probability density not viable.

→ Another problem with Klein-Gordon equation is solution of $p^2 = p_\mu p^\mu = m^2$ gives both $E = \pm\sqrt{p^2+m^2}$, i.e along with positive energy also negative energy solutions.



For a free particle, energy is constant and once positive solution is chosen, it will be at $+\sqrt{p^2+m^2}$. But for an interacting particle, there can be exchange of energy with other interacting states. So particle can cascade down to negative energy states with ground state energy infinitely negative.

$\longrightarrow$ The problem with causality.

free particle to propagate from x_0 to x

$$U(t) = \langle x | e^{-iHt} | x_0 \rangle$$

$$= \langle x | e^{-i\hat{p}^2/2m\, t} | x_0 \rangle = \int \frac{d^3p}{(2\pi)^3} \langle x | e^{-i(\,)} | p \rangle \langle p | x_0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{-ip^2/2m\, t} e^{ip.(x - x_0)}$$

$$= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(x-x_0)^2/2t}$$

The amplitude is non-zero for all x & t.

This indicates particle can propagate between x & x_0 in $t \to 0$ time.

with $E = p^2 + m^2$

$$U(t) \sim e^{-m\sqrt{x^2 - t^2}}$$ still non-zero, although can be small.

Causality is violated.

We adhere to the field description, rather than following single particle relativistic wave equation.

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4 Dynamics of fields

A field is a dynamical variable, $\phi(x^\mu)$.

→ Classical mechanics accompanied with finite number of generalised co-ordinates $q_a(t)$, a is index & conjugate momentum $\dot{q}_a(t)$

→ In QM, we consider q_a & $\dot{q}_a$ as operators.

→ In field theory field is labelled as $\phi_a(\vec{x},t)$, both a & $(\vec{x},t)$ are labels.

→ We are working with infinite number of degrees of freedom, at least one in each point x^μ.

→ Common example is Electric & Magnetic field $\vec{E}(\vec{x},t)$ & $\vec{B}(\vec{x},t)$ $\{A^\mu(\vec{x},t)\}$
$= (\phi, \vec{A})$
$\mu = 0, 1, 2, 3$

The field evolves (~~at any point x^μ~~) respecting Eqs of motion.

→ $\mathcal{L}$ (Lagrangian) is a function of $\varphi(\vec{x},t)$ & $\dot{\varphi}(\vec{x},t)$ & $\vec{\nabla}\varphi(\vec{x},t)$

[Analogous to classical mechanics $L(q, \dot{q})$]

→ Define Action $S = \int d^4x \, \mathcal{L}$

$\mathcal{L}$ is Lagrangian density.

→ Remember in classical mechanics

$$L = T - V = \frac{1}{2} m \dot{q}^2 - V$$

→ In field theory $\mathcal{L} = f(\varphi, \partial_\mu \varphi)$

(a) $\mathcal{L}$ doesn't depend explicitly on x^μ, rather the dependency appears via $\varphi(x^\mu)$ & $\partial_\mu \varphi(x^\mu)$

(b) We are not considering system $\mathcal{L} \to f(\varphi, \nabla\varphi, \nabla^2\varphi)$ etc, rather only $f(\varphi, \partial_\mu \varphi)$

Principle of Least Action

The field ϕ at ~~any~~ x^μ evolves in accordance to eqns of motion. The eqn of motion is derived from principle of least action.

$$\mathcal{L} \Rightarrow \mathcal{L}(\phi_\mu, \partial_\mu \phi)$$

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta \phi_a$$

$$+ \underbrace{\int d^4x \; \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right]}_{\text{A total derivative term}}$$

By Gauss theorem $\int_\Omega d^4x \; \partial_\mu F^\mu$

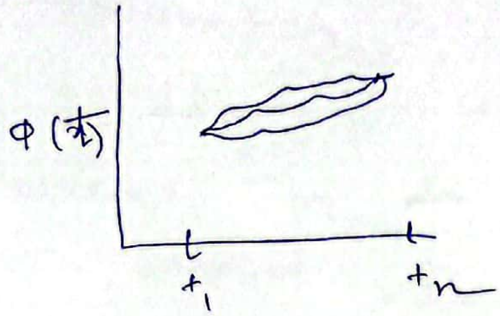
$$= \int_{\partial\Omega} ds_\mu F^\mu$$

$\rightarrow$ The total derivative term is $\left. \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right|_{x^\mu_{in}}^{x^\mu\, final}$ [Limits of surface integral]

$\rightarrow$ We consider $\delta \phi_a(\vec{x}, t_1) = \delta \phi_a(\vec{x}, t_2) = 0$

& as $|\vec{x}| \rightarrow \infty$, $\delta \phi_a(\vec{x}, t) \rightarrow 0$

CamScanner



φ will evolve along the path for which action is an extremum. i.e $\delta S = 0$

With this Euler-Lagrange Eqn

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi_a} = 0}$$

$\Rightarrow$ This is relativistically co-variant

[$\mathcal{L}$ is local, i.e $\int d^3x\, d^3y\, \varphi(x)\varphi(y)$ not present $\to$ violates causality

$\to$ no $\partial^2\varphi$ term or $\ddot{\varphi}$ term $\to$ Hamiltonian unbounded from below]

(A) Scalar field (Real)

$\cdot x^\mu \to \cdot x'^\mu$

$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$

$\varphi(x') = \varphi(x)$
$= \varphi(\Lambda^{-1}x')$
$x' \to x$
$\varphi(x) = \varphi(\Lambda^{-1}x)$

$\to$ $\varphi(\vec{x}, t)$ is a scalar field.

$\to$ A Lorentz invariant field.

i.e if $x^\mu \to \Lambda^\mu_{\ \nu} x^\nu$

$$\varphi(x^\mu) \to \varphi'(x^\mu) = \varphi(\Lambda^{-1}x)$$

*

$$\boxed{\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) - \frac{1}{2} m^2 \varphi^2}$$

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \frac{1}{2}(\partial^\nu \varphi)(\partial_\nu \varphi) - \frac{1}{2} m^2 \varphi^2$$

$\to$ flat Metric

Eq^n of motion

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{1}{2}(\partial_\rho \varphi)\eta^{\rho\sigma}\delta_{\sigma\mu} + \frac{1}{2}(\partial_\sigma \varphi)\eta^{\rho\sigma}\delta_{\rho\mu}$$

$$= (\partial^\mu \varphi)$$

Eq^n of motion becomes

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \varphi + m^2\varphi = 0$$

$$\Rightarrow \quad \boxed{(\Box + m^2)\varphi = 0}$$

The difference w.r.t relativistic q.m is φ is not a wave function.

Rather $\varphi(x^\mu)$ is a field whose value at every x^μ can be different.

Infinite number of degrees of freedom.

→ In general $\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi)}$

→ for complex scalar field $\varphi = \dfrac{\varphi^1 + i\varphi^2}{\sqrt{2}}$

$$\mathcal{L} = (\partial_\mu \varphi)(\partial^\mu \varphi^*) - m^2|\varphi|^2$$

& .S

Ex: 2

Prove that $\mathcal{L}$ with real Klein-Gordon field is Lorentz invariant.

Meaning

$x^\mu \to \Lambda^\mu_{\ \nu} x^\nu$ (or $y^\mu = \Lambda^{-1\mu}_{\ \ \ \nu} x^\nu$

for this transformation

$\mathcal{L}$ remains the same.

$\longrightarrow$ The Hamiltonian

$$H = \pi \dot{\varphi} - \mathcal{L}$$

with $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}}$ (conjugate momentum to φ)

$$\boxed{H = \frac{\pi^2}{2} + \frac{(\vec{\nabla}\varphi)^2}{2} + \frac{1}{2} m^2 \varphi^2}$$

has the structure $T + V$

Hamilton's eqn of motion

$$\left\{ \dot{q} = \frac{\partial H}{\partial p} \ \& \ \dot{p} = -\frac{\partial H}{\partial q} \right\}$$

in classical mechanics

$$\dot{\varphi}(\vec{x},t) = \frac{\partial H}{\partial \pi(\vec{x},t)} \quad \& \quad \dot{\pi}(\vec{x},t) = -\frac{\partial H}{\partial \varphi(\vec{x},t)}$$

A physical scalar particle with mass m is obtained by quantization of the scalar field.

{ A quick Guide to quantisation procedure [Canonical quantization] }

→ For real scalar field, we re-interpret the dynamical variables (i.e $\varphi(x^\mu)$ & $\pi(x^\mu)$) as operators that obey certain mathematical relation. ✓

→ Solve the theory by finding Eigenvalues & eigenstates of Hamiltonian
[Analogy is Harmonic Oscillator system]

→ Recall in QM, $[\hat{q}^i, \hat{p}^j] = i\delta^{ij}$

$$[\hat{q}^i, \hat{q}^j] = 0$$ ✓

$$[\hat{p}^i, \hat{p}^j] = 0$$

→ Generalise this

$$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y})$$ ✓

$$[\varphi(\vec{x}), \varphi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$$

equal time commutation relation.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\, \tilde{\phi}(\vec p)\, e^{i\vec p\cdot \vec x}$$

$$\phi(\vec x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec p}}} \left[a_{\vec p}\, e^{i\vec p\cdot\vec x} + a^\dagger_{\vec p}\, e^{-i\vec p\cdot\vec x} \right]$$

$$\frac{\partial^2 \tilde\phi(\vec p)}{\partial t^2} + \omega^2 \tilde\phi(\vec p) = 0$$

$$\pi(\vec x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec p}}{2}} \left[a_{\vec p}\, e^{i\vec p\cdot\vec x} - a^\dagger_{\vec p}\, e^{-i\vec p\cdot\vec x} \right]$$

The $a^\dagger_{\vec p}$ & $a_{\vec p}$ are creation and annihilation operators satisfying

$$[a_{\vec p}, a^\dagger_{\vec p'}] = (2\pi)^3 \delta^3(\vec p - \vec p')$$

$$\& \quad [a_{\vec p}, a_{\vec p'}] = [a^\dagger_{\vec p}, a^\dagger_{\vec p'}] = 0$$

The action of $a_{\vec p}$ is ground state $|0\rangle$ satisfies $a_{\vec p}|0\rangle = 0$ for all $\vec p$

and $a^\dagger_{\vec p}|0\rangle = |\vec p\rangle$

→ The Hamiltonian $H = \pi\dot\phi - \mathcal{L}$

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec p} \left(a^\dagger_{\vec p} a_{\vec p} \right)$$

& $H|\vec p\rangle = \omega_{\vec p} |\vec p\rangle$

with $\omega_{\vec p} = \sqrt{|\vec p|^2 + m^2}$

→ $|\vec p\rangle$ is an energy eigenstate of Hamiltonian with definite energy $\omega_{\vec p}$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\, \tilde{\phi}(p)\, e^{ip\cdot x}$$ → Harmonic oscillator system &

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\frac{\partial^2 \tilde{\phi}(p)}{\partial t^2} + \omega^2 \tilde{\phi}(p) = 0$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

The $a_{\vec{p}}^{\dagger}$ & $a_{\vec{p}}$ are creation and annihilation operator. satisfying

$$[a_{\vec{p}}, a_{\vec{p}'}^{\dagger}] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$$

$$\& \quad [a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^{\dagger}, a_{\vec{q}}^{\dagger}] = 0$$

The action of $a_{\vec{p}}$ is ground state $|0\rangle$ satisfies $a_{\vec{p}}|0\rangle = 0$ for all $\vec{p}$

and $a_{\vec{p}}^{\dagger}|0\rangle = |\vec{p}\rangle$

→ The Hamiltonian $H = \pi\dot{\phi} - \mathcal{L}$

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}} \left(a_{\vec{p}}^{\dagger} a_{\vec{p}} \right)$$

$$\& \quad H|\vec{p}\rangle = \omega_{\vec{p}} |\vec{p}\rangle$$

with $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$

→ $|\vec{p}\rangle$ is an energy eigenstate of Hamiltonian with definite energy $\omega_{\vec{p}}$

a Multiparticle state

$|\vec{p}_1, \ldots \vec{p}_n\rangle$ can be obtained as

$$|\vec{p}_1, \ldots \vec{p}_n\rangle = a^{\dagger}_{\vec{p}_1} \ldots a^{\dagger}_{\vec{p}_n} |0\rangle$$

& for scalar particle $|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle$

Propagator

The framework of QFT can successfully describe particle interaction (cross-section, decay rates etc)

one of the main ingredient to understand how the interactions occur is via the concept of propagator.

Meaning :

→ $\phi(x)|0\rangle$ implies $a^{\dagger}_p|0\rangle$, i.e a particle is created at x.

→ $\langle 0|\phi(x')$ implies particle is annihilated at x' as only annihilation part $\langle 0|a_p$ non zero.

→ $\boxed{\langle 0|\phi(x')\phi(x)|0\rangle}$

↳ Amplitude of propagation of a particle from x to x'.

Some Mathematics

We define $G(x-x')$, a green's function as

$$(\Box_x + m^2)\, G(x-x') = -i\delta^4(x-x')$$

$$G(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.(x-x')} \frac{i}{p^2-m^2+i\epsilon'}$$

The Feynman propagator is defined as $\dfrac{i}{p^2-m^2+i\epsilon}$

$\longrightarrow$ $$i\Delta_F(x-x') = \langle 0|T(\phi(x)\phi(x'))|0\rangle$$

$\hookrightarrow$ $\phi(x)\phi(x')$ if $x^0 > x'^0$

$\phi(x')\phi(x)$ if $x'^0 > x^0$

$\longrightarrow$ Connection with causality? Assignment - I

$$\boxed{\phi(x) = -\int dx\, G(x-x')\, J(x') + \phi_0}$$

(7) for complex scalar field

$$\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[b_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

i.e $$\phi^{\dagger} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[b_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} \right]$$

$$[b_{\vec{p}},\ b_{\vec{q}}^{\dagger}] = (2\pi)^3\, \delta^3(\vec{p}-\vec{q})$$

$$[c_{\vec{p}},\ c_{\vec{q}}^{\dagger}] = (2\pi)^3\, \delta^3(\vec{p}-\vec{q})$$

All other combination as 0

$$[\phi(\vec{x}),\ \pi(\vec{y})] = [\phi^{\dagger}(\vec{x}),\ \pi^{\dagger}(\vec{y})]$$
$$= i\,\delta^3(\vec{x}-\vec{y})$$

→ Two creation operator
$b_{\vec{p}}^{\dagger}$ & $c_{\vec{p}}^{\dagger}$

→ one of them for particle → $b_p^{\dagger}$ for particle

$c_p^{\dagger}$ for Anti-particle

→ And another for antiparticle

→ for a real scalar particle
particle & anti-particle same

→ H is $(b_p^{\dagger} b_{\vec{p}} + c_{\vec{p}}^{\dagger} c_{\vec{p}})$

Propagator for complex scalar field

$\langle 0 | \phi^{\dagger}(x) \, \phi^{\dagger}(y) | 0 \rangle$ for $x^0 > y^0$

$\longmapsto$ particle is created at y

This will give the combination

$\langle 0 | b_p \, b_p^{\dagger} | 0 \rangle \longrightarrow$ This implies particle is created at y by the operation of $\phi^{\dagger}(y)|0\rangle$

$\longrightarrow$ propagates to x and annihilates at x by the operation $\langle 0 | \phi(y)$

$\longrightarrow$ Antiparticle travels from x to y.

$\longrightarrow$ complex scalar field propagator for particle or antiparticle

$$\langle 0 | T(\phi(x) \, \phi^{\dagger}(y) | 0 \rangle$$

$$= \underbrace{\theta(x^0 - y^0) \langle 0 | \phi(x) \, \phi^{\dagger}(y) | 0 \rangle}_{\text{particle propagation from } y \text{ to } x} + \underbrace{\theta(y^0 - x^0) \langle 0 | \phi^{\dagger}(y) \, \phi(x) | 0 \rangle}_{\text{Antiparticle propagation from } x \text{ to } y.}$$

propagator $\dfrac{i}{p^2 - m^2 + i\epsilon}$

So far, we were working with Schrödinger picture, where $\varphi(x) \neq \varphi(\vec{x}, t)$

Given $\varphi(\vec{x})$, one can find out how $\varphi(x)$ changes in Heisenberg's picture, where $\varphi(x)$ also depend on t.

$$\varphi_H(x) = e^{iHt}\,\varphi(\vec{x})\,e^{-iHt}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-i p_\mu x^\mu} + a_{\vec{p}}^\dagger\, e^{+i p_\mu x^\mu} \right)$$