Lorentz transformation

GFT-1

L = 6

Lorentz transformation is a linear, homogeneous change of co-ordinates from x^μ to $\bar{x}^\mu$

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$$

[Set of linear transformation (i.e Λ) which preserves $t^2 - x^2$ form a group] → Lorentz group

interval $x^2 = c^2t^2 - \vec{x}^2$ is invariant

$= x^\mu x_\mu \rightarrow x^T g x = x'^T g x'$

$= (\Lambda x)^T g (\Lambda x)$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad (\Lambda^\mu{}_\nu \text{ must obey})$$

$= x^T \Lambda^T g \Lambda x$

$= x^T g x$

$\Rightarrow \boxed{\Lambda^T g \Lambda = g}$

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\left[\text{or} \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \right. \quad x^2 = x^2 - c^2t^2$$

$\det \Lambda^T \det g \det \Lambda = \det g$

$\Rightarrow \det \Lambda = \pm 1$

→ Set of all Lorentz transformation forms a group. product of any two L.T is a Lorentz transformation

($a, b \in G$, $ab \in G$ → associativity property.)

→ for every L.T there is an inverse

Λ^{-1}

→ there exist an identity transformation

$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}$

→ To find inverse $(\Lambda^{-1})^{\mu}{}_{\nu}$

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

$$\Rightarrow \Lambda_{\nu\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

$$\Rightarrow g^{\rho'\rho} \Lambda_{\nu\rho} \Lambda^{\nu}{}_{\sigma} = g^{\rho'\rho} g_{\rho\sigma}.$$

$$\Lambda_{\nu\rho} g^{\rho\rho'} = \Lambda_{\nu}{}^{\rho'}$$

$$\Rightarrow \boxed{\Lambda_{\nu}{}^{\rho'} \Lambda^{\nu}{}_{\sigma} = \delta^{\rho'}_{\sigma}}$$

By definition $(\Lambda^{-1})^{\rho'}{}_{\nu} \Lambda^{\nu}{}_{\sigma} = \delta^{\rho'}_{\sigma}$

$$\Rightarrow \boxed{(\Lambda^{-1})^{\rho'}{}_{\nu} = \Lambda_{\nu}{}^{\rho'}}$$

infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$$

Use $\quad g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$

$$\Rightarrow g_{\mu\nu} \left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right) = g_{\rho\sigma}$$

$$\Rightarrow g_{\mu\nu} \delta^{\mu}{}_{\rho} \delta^{\nu}{}_{\sigma} + g_{\mu\nu} \delta^{\nu}{}_{\sigma} \delta\omega^{\mu}{}_{\rho} + g_{\mu\nu} \delta^{\mu}{}_{\rho} \delta\omega^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

$$\Rightarrow \cancel{g_{\rho\sigma}} + g_{\mu\sigma} \delta\omega^{\mu}{}_{\rho} + g_{\rho\nu} \delta\omega^{\nu}{}_{\sigma} = \cancel{g_{\rho\sigma}}$$

$$\Rightarrow g_{\sigma\mu} \delta\omega^{\mu}{}_{\rho} + \delta\omega_{\rho\sigma} = 0$$

$$\Rightarrow \boxed{\delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} = 0}$$

$$\boxed{\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}}$$

Assignment-2

$\delta\omega$ is antisymmetric in both indices ρ & σ.

→ ρ & σ run $0, 1, 2, 3$. Because of antisymmetric $\delta\omega_{\rho\rho} = 0$. 6 independent component.

$\longrightarrow$ Six independent infinitesimal Lore[ntz]
transformation

$\quad \hookrightarrow$ 3 rotations $\quad \delta\omega_{ij} = -\epsilon_{ijk}\,\hat{n}_k$

$\quad \hookrightarrow$ 3 boosts $\quad \delta\omega_{i0} = \hat{n}_i\,\delta\eta$

for a boost along direction $\hat{n}$ & rapidity $\delta\eta$.

$\longrightarrow$ ~~from~~ $\cancel{(\Lambda^{-1})^{\nu}{}_{\sigma} = \Lambda_{\sigma}{}^{\nu}}$

det Λ

$\longrightarrow$ The other properties are $\det(\Lambda) = \pm 1$

(Ex: proof)

transformations with $\det(\Lambda) = +1$ proper L.T

$\det(\Lambda) = -1$ improper L.T

$\longrightarrow$ proper L.T are subgroup of Lorentz group.

Lec 2.

orthochronous

$$g_{\mu\nu} = g_{\rho\sigma} \Lambda^{\rho}_{\ \mu} \Lambda^{\sigma}_{\ \nu}$$

$$g_{00} = 1 = g_{\rho\sigma} \Lambda^{\rho}_{\ 0} \Lambda^{\sigma}_{\ 0}$$

$\Lambda^{0}_{\ 0} \geq +1$.

$|\Lambda^{0}_{\ 0}| \geq 1 \Leftarrow \boxed{1 = (\Lambda^{0}_{\ 0})^2 - (\Lambda^{i}_{\ 0})^2}$

(non-orthochronous) $\Lambda^{0}_{\ 0} \leq -1$

→ We are interested in proper & orthochronous Lorentz transformation.

→ We associate a ~~unitary operator~~ $U(\Lambda)$ to each of the proper & orthochronous L.T. ~~symmetries in quantum theory are represented by unitary operators~~.

SO(3,1) special orthogonal, 3 rotation, 1 time, non-compact, ~~semi~~simple group

→ finite dimensional unitary rep^n doesn't exist

→ $U(\Lambda'\Lambda) = U(\Lambda')\, U(\Lambda)$

→ $U(1 + \widetilde{\delta\omega}) = \mathbb{1} + \frac{i}{2} \widetilde{\delta\omega}_{\mu\nu} M^{\mu\nu}$

infinitesimal transformation of the ~~unitary~~ operator.

→ $M^{\mu\nu} = -M^{\nu\mu}$ are set of ~~Hermitian~~ operators, known as generators of L.G.

This equation tells, starting from

we have an infinitesimal group element

$$\mathbb{1} + \frac{i}{2}\,\delta\omega_{\mu\nu}\,M^{\mu\nu}.$$

Successive application of this operation

will give a finite element

$$\lim_{n\to\infty}\left(1+\frac{i}{2}\,\delta\omega_{\mu\nu}M^{\mu\nu}\right)^n$$

$$= \exp\left[\frac{i}{2}\,\delta\omega_{\mu\nu}\,M^{\mu\nu}\right]$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i g_{\nu\rho} M_{\mu\sigma} - i g_{\mu\rho} M_{\nu\sigma} - i g_{\nu\sigma} M_{\mu\rho} + i g_{\mu\sigma} M_{\nu\rho}$$

$\longrightarrow$ (2)

$\longrightarrow$ This is the Lie algebra of $SO(3,1)$

We introduce $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

$\longrightarrow$ Most general representation of $M_{\mu\nu}$ that satisfies Eq (2)

$$M_{\mu\nu} = \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{Hermitian generators}} + S_{\mu\nu} \longrightarrow \text{Hermitian}$$

P/P/S

Refer to group theory Before this.
discuss Lie group & Lie algebra

$[M_{ij}, M_{kl}]$

$= -i\delta_{jk} M_{il} + i\delta_{ik} M_{jl} + i\delta_{jl} M_{ik} - i\delta_{il} M_{jk}$

This is SU(2) algebra.

$-i\epsilon_{123} J_3$

$J_3 = \frac{1}{2}\epsilon_{312} M_{12} + \frac{1}{2}\epsilon_{321} M_{21}$

$= \frac{1}{2}\epsilon_{312} M_{12} + \frac{1}{2}\epsilon_{312} M_{12}$

{A more familiar expression}

$= \epsilon_{312} M_{12}$

$= M_{12}$

→ We define

$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$

$\epsilon_{123} = +1$

$[J_i, J_j] = i\epsilon_{ijk} J_k$

$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$

→ Boost generators

SU(2) & SO(3) are related. Rotation generators are Hermitian.

$K_i = M_{0i}$

$\cancel{\frac{1}{2}}$

$[M_{0i}, M_{0j}]$

$= i g_{i0} M_{0j}$

$- i g_{00} M_{ij}$

$- i g_{ij} M_{00}$

$+ i g_{0j} M_{i0}$

$= -i M_{ij}$

$[K_i, K_j]$

$= -i$

→ $[K_i, K_j] = -i\epsilon_{ijk} J_k$

→ $[J_i, K_j] = i\epsilon_{ijk} K_k$

→ ~~K_i~~ J_i are Hermitian generators.

→ K_i non-Hermitian

→ Introduce Linear combination

$N_i = \frac{1}{2}[J_i + iK_i]$

→ $N_i \neq N_i^\dagger$

$\boxed{[K_1, K_2] = -i M_{12} = -i J_3}$

We have

$$[N_i, N_j^\dagger] = 0$$

$$[N_i, N_j] = i\epsilon_{ijk} N_k$$

$$[N_i^\dagger, N_j^\dagger] = +i\epsilon_{ijk} N_k^\dagger$$

$\longrightarrow$ N_i & $N_i^\dagger$ independently obey the Lie algebra of $SU(2)$

L. T

$$[N_i, N_j]$$

$$= \frac{1}{2} \times \frac{1}{2} \left[(J_i + i K_i), (J_j + i K_j) \right]$$

$$= \frac{1}{4} \left\{ [J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] - [K_i, K_j] \right\}$$

$$= \frac{1}{4} \left[i\epsilon_{ijk} J_k + i\epsilon_{ijk} K_k \times i - i\epsilon_{jik} K_k \times i + i\epsilon_{ijk} J_k \right]$$

$$= \frac{i \times 2}{4} \epsilon_{ijk} \left[J_k + i K_k \right]$$

$$= \frac{i}{\cancel{2}} \epsilon_{ijk} N_k$$

$$\boxed{[N_i, N_j] = i\epsilon_{ijk} N_k}$$ ✓

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger$$

$$[(J_i - iK_i), (J_j - iK_j)] \times \frac{1}{4}$$

$$= \frac{1}{4} \left\{ [J_i, J_j] - i[J_i, K_j] - i[K_i, J_j] + [K_i, K_j] \right\}$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger$$

a)

N_i & $N_i^\dagger$ independently obey the Lie algebra of SU(2).

→ Casimir operator

✓ $N_i N_i = N^2$ eigenvalue $n(n+1)$

✓ $N_i^\dagger N_i^\dagger = (N^\dagger)^2$ " $m(m+1)$

✓ $m, n = 0, \frac{1}{2}, 1, 3/2 \ldots$ similar to SU(2) representation

→ label the representation as (n, m)

✓ → two SU(2) groups are not independent as related by Hermitian conjugate

→ In general, representations of the Lorentz group are not Hermitian ✓ conjugate eigenstates.

→ Since $J_i = N_i + N_i^\dagger$, we can identify representation by the spin of the $m+n$.

a) $(0,0)$ spin zero $\rightarrow$ scalar representation

b) $(\frac{1}{2}, 0) \rightarrow$ has spin $\frac{1}{2}$ & is a left-handed spinor

c) $(0, \frac{1}{2}) \rightarrow$ " spin $\frac{1}{2}$ & a right-handed spinor.

b & c are called weyl. spinor and has two component spin up & down.

d) a Dirac spinor is represented as
$(\frac{1}{2}, 0) \oplus (0 \frac{1}{2})$ ✓

$2j+1$
$2m+1$

$\frac{1}{2}+1$
$= 2$

ψ_L and ψ_R

e) ? $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ (4 vector)
$\hookrightarrow A_\mu$

f) $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0,0) \oplus (1, 0)$
$\hookrightarrow$ scalar