Dirac field

Is there only one representation of Lorentz group?

All spin 1/2 fermions are represented by $\psi^{\alpha}(x^{\mu})$ where $\alpha = 1, 2, 3, 4$. α is spinor index.

spin

An intrinsic property of the particle.

$$x^{\mu} \rightarrow \Lambda^{\mu}_{\ \nu} x^{\nu} = x'^{\mu}$$

x^{μ} is old co-ordinate
x'^{μ} is new co-ordinate

how $\psi^{\alpha}(x^{\mu})$ changes?

Start with old co-ordinate $\psi^{\alpha}(x)$. This under $S[\Lambda]$ changes $\psi^{\alpha}(x) \longrightarrow S[\Lambda]\psi^{\alpha}(x)$
$= \psi'^{\alpha}(x')$

Using $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$

$$\psi'^{\alpha}(x') = S[\Lambda]^{\alpha}_{\ \beta}\, \psi^{\beta}(\Lambda^{-1} x')$$

Re-naming $x' \rightarrow x$

$$\boxed{\psi'(x) = S[\Lambda]\, \psi(\Lambda^{-1} x)}$$

*** In QM, spin is introduced as an intrinsic form of angular momentum. In relativistic QFT, spin has a deeper origin; it labels how a field transform under L.T. Different L.R correspond to particles with different spin.

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Lagrangian of a Dirac fermion

$\psi^{\alpha}(x)$

$$\mathcal{L} = \bar{\psi} i\gamma^{\mu} \partial_{\mu} \psi - m \bar{\psi} \psi$$

where $\bar{\psi} = \psi^{\dagger} \gamma^{0}$

What is this γ matrices ?

~~What~~ . Let us recap spinor representation.

Q: How does an electron field (i.e spin $\frac{1}{2}$ object) transforms under Lorentz transformation?

$\psi^{\alpha}(x)$ has non-trivial transformation property under Lorentz transformation $SO(3,1)$

Property of $SO(3,1)$

$x^\mu \Rightarrow \Lambda^\mu{}_\nu x^\nu = x'^\mu$

where Λ satisfies $x^\mu x_\mu = x'^\mu x'_\mu$

This gives $\Lambda^T \eta \Lambda = \eta$ (A)

Now expand Λ around $\mathbb{1}$

$$\boxed{\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu}$$

We get from (A) $\omega_{\mu\nu} = -\omega_{\nu\mu}$

i.e antisymmetric and hence it has 6 components.

→ One can impose successive infinitesimal transformation of $\Lambda^\mu{}_\nu$ to obtain

$$\boxed{\Lambda = \exp\left[\tfrac{1}{2}\tilde{\omega}_{\rho\sigma} M^{\rho\sigma}\right]} \quad (B)$$

→ $\omega^\mu{}_\nu \simeq \tilde{\omega}_{\rho\sigma} (M^{\rho\sigma})^\mu{}_\nu$ ($\rho, \sigma \to 0,1,2,3$; $\mu, \nu \to 0,1,2,3$)

→ 6 parameters 3 rotation + 3 boost
$\theta_1, \theta_2, \theta_3$ $\quad K_1, K_2, K_3$

→ 6 generators.

→ $(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$

where

$$[M^{\rho\sigma}, M^{\tau\nu}]$$
$$= \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu}$$
$$+ \eta^{\rho\nu} M^{\sigma\tau}$$
$$- \eta^{\sigma\nu} M^{\rho\tau}$$

$\longrightarrow$ Lie Algebra.

$\longrightarrow$ Λ in B is an element of Lie group $SO(3,1)$

$\longrightarrow$ $\underbrace{M^{01}, M^{02}, M^{03}}_{\text{rotation}}, \underbrace{M^{12}, M^{13}, M^{23}}_{\text{boost}}$.

Q $\to$ Is there any other matrices that satisfy Lorentz Algebra?

$\longrightarrow$ Clifford Algebra

A set of 4 matrices

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$
$$= 2\eta^{\mu\nu}$$
$$\mu = 0, 1, 2, 3$$

$\gamma^{0^2} = \mathbb{1}$ & $\gamma^{i^2} = -\mathbb{1}$

We define $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]$

$= \frac{1}{2}\gamma^\rho\gamma^\sigma \quad \rho \neq \sigma$

$= 0 \quad \text{for } \rho = \sigma$

One of the representation of γ^μ matrix is

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$\longrightarrow$ <u>Chiral repn</u>

$\longrightarrow$ We construct $S[\Lambda] = \exp\left[i/2\, \omega_{\rho\sigma} S^{\rho\sigma}\right]$

The Dirac field transforms as

$$\psi_\alpha(x) \longrightarrow S[\Lambda]_\alpha^{\ \beta}\, \psi_\beta[\Lambda^{-1}x]$$

$\longrightarrow$ parameters are same, but generators are different $S^{\rho\sigma}$.

$\longrightarrow$ Using chiral repn

$$S^{0i} = i/4\,[\gamma^0, \gamma^i] = -i/2 \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$S^{ij} = i/4\,[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

S^{0i} & S^{ij} are in block diagonal form.

$S[\Lambda]$ under boost   Ex : 3

$$= \begin{pmatrix} e^{-i/2\, \epsilon_i \sigma_i} & 0 \\ 0 & e^{+i/2\, \epsilon_i \sigma_i} \end{pmatrix}$$

$S[\Lambda] =$ under rotation

$$\begin{pmatrix} e^{-i/2\, \epsilon_{0K}\, \sigma^K} & \\ & e^{-i/2\, \epsilon_{0K}\, \sigma^K} \end{pmatrix}$$

Because S^{0i} & S^{ij} have these block diagonal form therefore ψ as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

$$= \Psi_L + \Psi_R$$

under infinitesimal transformation

under boost different

$$\begin{cases} \psi_L \to \left(\mathbb{1} - i\frac{\sigma \cdot \theta}{2} - \frac{\beta \cdot \sigma}{2}\right)\psi_L \\ \psi_R \to \left(\mathbb{1} - i\frac{\theta \cdot \sigma}{2} + \frac{\beta \cdot \sigma}{2}\right)\psi_R \end{cases}$$

More fundamental way to define Left & right chiral states

→ Problem is for a different choice of γ^μ, $S[\Lambda]$ might not have block diagonal form.

→ rather let us define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \& \quad \gamma^{5\dagger} = \gamma^5$$

$$\gamma^{5^2} = \mathbb{1}$$

$[\gamma^5, S_{\mu\nu}] = 0 \rightarrow \gamma^5$ is Lorentz invariant

→ $$\left.\begin{aligned} P_L &= \tfrac{1}{2}(\mathbb{1} - \gamma_5) \\ P_R &= \tfrac{1}{2}(\mathbb{1} + \gamma_5) \end{aligned}\right\} \text{ Lorentz invariant}$$

→ $$\left.\begin{aligned} P_L\psi &= \psi_L \\ P_R\psi &= \psi_R \end{aligned}\right\} \begin{aligned} \gamma_5\psi_L &= -\psi_L \\ \gamma_5\psi_R &= +\psi_R \end{aligned}$$

ψ_L & ψ_R are eigenstates of γ_5 matrix with eigenvalue -1 & $+1$.

Now that left chiral & right chiral field decomposition is clear, let us look back to SM.

e field $\quad e = e_L + e_R$

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$$

$$e^-_R \qquad \mu^-_R \qquad \tau^-_R$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L$$

$$u_R, d_R \qquad c_R, s_R \qquad t_R, b_R$$

each of this fields are 2 component Weyl spinor.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Eqn of motion ~~in~~ is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$\longrightarrow$ In massless limit $\quad \left.\begin{array}{l} i(\partial_0 + \sigma\cdot\vec{\nabla})\psi_R = 0 \\ i(\partial_0 - \sigma\cdot\vec{\nabla})\psi_L = 0. \end{array}\right]$

Quantization of fermion field

Consider $\psi(\vec{x},t)$ and $\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}$

$$= \bar{\psi} i\gamma^0$$
$$= \psi^\dagger i$$

as operators.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

In Schrödinger's picture

$$\psi(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^s u^s(p) e^{+i\vec{p}\cdot\vec{x}} + c_p^{s\dagger} v^s(p) e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\psi^\dagger(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^{s\dagger} u^{s\dagger}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + c_p^s v^{s\dagger}(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} \right]$$

Using $e^{iHt} b_p^s e^{-iHt} = b_p^s e^{-iE_{\vec{p}}t}$

and $e^{iHt} c_p^{s\dagger} e^{-iHt} = c_p^{s\dagger} e^{+iE_p t}$

In Heisenberg's picture

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_p^s u^s(p) e^{-ip_\mu x^\mu} + c_p^{s\dagger} v^s(p) e^{+ip_\mu x^\mu} \right]$$

$$\psi^\dagger(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big[b_p^{s\dagger} u_p^{s\dagger} e^{+ip_\mu x^\mu} + c_p^s v^s(p)^\dagger e^{-ip_\mu x^\mu} \Big]$$

Instead of commutation, we follow anticommutation [spin 1/2 particles are fermion, they satisfy Fermi-Dirac ~~properties~~ statistics]

$$\longrightarrow \{ b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger} \} = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\,\delta^{rs}$$

$$\{ c_{\vec{p}}^r, c_{\vec{q}}^{s\dagger} \} = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\,\delta^{rs}$$

all other anti-commutators are 0.

$\longrightarrow$ Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \Big[b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s + c_{\vec{p}}^{s\dagger} c_{\vec{p}}^s - (2\pi)^3\delta^3(0) \Big]$$

$\longrightarrow$ Ground/vacuum state is $b_{\vec{p}}^s |0\rangle = c_{\vec{p}}^s |0\rangle = 0$ for all $\vec{p}$

→ Create one particle state

$|\vec{p}, r\rangle = b_p^{r\dagger}|0\rangle$

→ Anti-particle $|\vec{q}, s\rangle = c_q^{s\dagger}|0\rangle$

→ $|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle = -|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle$

Two similar / identical spin $1/2$ fermion can not exist.

→ The u & v spinor satisfy the following properties

$$\sum_s u^s(p)\,\bar{u}^s(p) = \not{p} + m$$

$$\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m$$

→ Eq^n is $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$\psi(x) = u(p)e^{-ip.x}$ / $v(p)\,e^{+ip.x}$ | satisfies this eq^n.

→ in $u^s(p)\,e^{-ip.x}$ or $v^s(p)\,e^{+ip.x}$

$p^2 = m^2$ & $p^0 > 0$

$\longrightarrow$ The spinor $u^s(p)$ and $v^s(p)$ have the following form

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi^s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi^s \end{pmatrix} \qquad \left\{ \begin{array}{l} \sigma = (\mathbb{1}, \vec{\sigma}) \\ \bar{\sigma} = (\mathbb{1}, -\vec{\sigma}) \end{array} \right.$$

$\hookrightarrow p_\mu \sigma^\mu$

and $$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta^s \\ -\sqrt{p \cdot \bar{\sigma}}\, \eta^s \end{pmatrix}$$

$\hookrightarrow p_\mu \sigma^\mu$

$\longrightarrow$ $\bar{u}^r(p)\, u^s(p) = 2m\, \delta^{rs}$

$u^{r\dagger}(p)\, u^s(p) = 2E_p\, \delta^{rs}$

$\longrightarrow$ $\bar{v}^r(p)\, v^s(p) = -2m\, \delta^{rs}$

$v^{r\dagger}(p)\, v^s(p) = 2E_p\, \delta^{rs}$

$\longrightarrow$ $\xi^{r\dagger} \xi^s = \delta^{rs}$ & $\eta^{r\dagger} \eta^s = \delta^{rs}$

$r, s = 1, 2$

$\longrightarrow$ $\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$

$\hookrightarrow$ orthogonal

$\longrightarrow$ ξ or $\eta \rightarrow$ $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or η^1 & η^2

Interpretation is b_p^{rt} acting on vacuum create a particle (electron) with momentum $\vec{p}$ & spin s $|\vec{p}, s\rangle$ and $c_p^{s\dagger}$ acting on vacuum create an antiparticle with same mass as electron, but all conserved additive quantum charges opposite. [example, electromagnetic charge of e^- is $-ve$ and e^+ is $+ve$]

The antiparticle state is denoted as

$|\vec{p}, \vec{s}\rangle$

→ The propagator, similar to complex scalar field can be obtained as

$$S_{F\alpha\beta}(x - x') = \langle 0|T(\psi_\alpha(x)\bar{\psi}_\beta(x'))|0\rangle$$

$$= \psi_\alpha(x)\bar{\psi}_\beta(x') \text{ if } x^0 > x'^0$$

$$= -\bar{\psi}_\beta(x')\psi_\alpha(x) \text{ if } x'^0 > x^0$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} S_F(p)$$

$$\rightarrow \quad S_F(p) = \frac{i}{\not{p} - m + i\epsilon}$$

where $\not{p} = p_\mu \gamma^\mu$

can be obtained from

$$\boxed{(i\gamma^\mu \partial_\mu - m)\, S(x - x') = i\delta^4(x - x')}$$

$\rightarrow$ Pole of the propagator at
$p^2 = m^2$ i.e $p_0^2 = E_p^2$

In a nutshell

free scalar theory or free Dirac Lagrangian is accompanied ~~by~~ with a Lagrangian

$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$ or $(\partial_\mu \phi)(\partial^\mu \phi)^* - m^2 |\phi|^2$
for scalar

and fermion $\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

$\rightarrow$ We quantise the fields and field excitations are particles.

$\rightarrow$ Particles have definite mass and one can define propagator (amplitude of propagation)