Stochastic Dynamics Towards Generalized Gibbs Ensembles

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Plan of the talk

Equilibrium vs non-equilibrium systems

Thermodynamic equilibrium:

- Many physical systems relax to a stationary state that forgets initial conditions.
- The steady state is usually described by Gibbs–Boltzmann statistics.

Non-equilibrium systems:

- Some systems do *not* forget initial conditions.
- They may display memory, non-thermal stationary states, or constrained dynamics.

In this talk: We focus on **integrable non-equilibrium systems**, where many conserved quantities, as many as the number of degrees of freedom, prevent thermalisation.

Stationary state of isolated systems

Degrees of freedom and conserved quantities:

- A system has N degrees of freedom (d.o.f.).
- Let the number of conserved quantities be n.

Non-integrable system:

$$n < N,$$

typically only the **energy** is conserved.
Microcanonical measure:

$$\rho \propto \delta(H(\{x_\mu, p_\mu\}) - E)\,.$$

Integrable system:

$$n = N, \qquad I_\mu(\{x_\mu, p_\mu\}) = \mathcal{I}_\mu,$$

$$\rho = \prod_{\mu=1}^{N} \delta(I_\mu(\{x_\mu, p_\mu\}) - \mathcal{I}_\mu)\,.$$

Each initial condition fixes all $\mathcal{I}_\mu$.

Open systems and canonical ensemble

For open systems, using a **microcanonical** description is impractical.

Instead, we use the **canonical ensemble:**

$$\rho = \frac{e^{-\beta H}}{Z(\beta)},$$

valid and well understood for **non-integrable** systems.

Question for integrable systems:

- Are local observables still described by a canonical measure?
- Or do we need a more general ensemble?

This leads to the **Generalized Gibbs Ensemble (GGE)**:

$$\rho_{\text{GGE}} = \frac{\exp\left[-\beta \sum_\mu \gamma_\mu I_\mu\right]}{Z(\{\beta\gamma_\mu\})}.$$

The parameters $\beta\gamma_\mu$ are fixed by the constraints:

$$I_\mu(0^+) = I_\mu(t) = \langle I_\mu \rangle_{\text{GGE}}.$$

Ergodicity and observables

For an observable $A(\{x_\mu, p_\mu\})$,
Ensemble average (GGE):

$$\langle A \rangle_{\text{GGE}} = \int \prod_\mu dp_\mu \, dx_\mu \; \rho_{\text{GGE}}(\{p_\mu, x_\mu\}) \, A(\{p_\mu, x_\mu\}).$$

Time average (Newton dynamics):

$$\bar{A} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\text{GGE}}}^{t_{\text{GGE}}+\tau} dt' \; A(\{p_\mu(t'), x_\mu(t')\}).$$

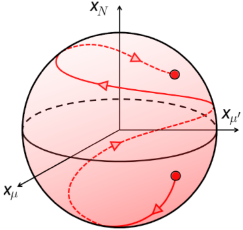
Ergodicity hypothesis

$$\langle A \rangle_{\text{GGE}} \stackrel{?}{=} \bar{A}$$

Why the Neumann model?

To test ergodicity in an integrable system, we need:

- A closed classical model with known integrals of motion I_μ.
- Explicit dynamics to compute $\bar{A}$.
- An analytic GGE expression for $\langle A \rangle_{\text{GGE}}$.



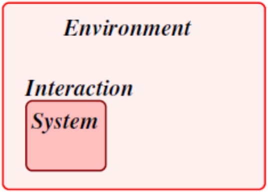
Neumann model: particle on an N-dimensional sphere in an anisotropic harmonic potential.

Ergodicity was achieved for this model.
Barbier, Cugliandolo, Lozano, Nessi. 2022

Goal: sample ρ_{GGE}

Objective of this work:

Extend this setup to an **open system**:



- Couple the integrable model to a **thermal bath** via a Langevin dynamics.

- **Sample the GGE**: check that

$$\langle A \rangle_{\mathrm{GGE}} = \lim_{t\to\infty} \langle A \rangle_{\xi},$$

where ξ labels different realizations of the thermal noise.

Neumann model

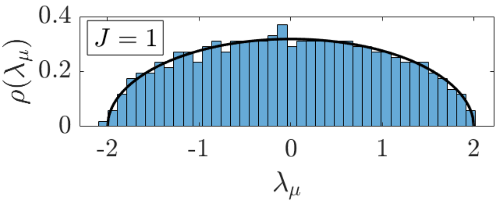
Potential energy

$$\mathcal{H}_{\text{pot}} = -\frac{1}{2}\sum_{\mu=1}^{N} \lambda_\mu x_\mu^2,$$

with ordered eigenvalues of a GOE matrix:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N,$$

distributed according to the Wigner semicircle law



$$\rho(\lambda_\mu) = \frac{1}{2\pi J^2}\sqrt{(2J)^2 - \lambda_\mu^2},$$

$$-2J \leq \lambda_\mu \leq 2J.$$

Neumann model

Potential energy

$$\mathcal{H}_{\text{pot}} = -\frac{1}{2}\sum_{\mu=1}^{N} \lambda_\mu x_\mu^2,$$

with ordered eigenvalues of a GOE matrix:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N,$$

distributed according to the Wigner semicircle law:

$$\rho(\lambda_\mu) = \frac{1}{2\pi J^2}\sqrt{(2J)^2 - \lambda_\mu^2}, \qquad |\lambda_\mu| \leq 2J.$$

Adding Newton dynamics: Include kinetic energy + spherical constraint

$$\mathcal{H} = \frac{1}{2m}\sum_\mu p_\mu^2 - \frac{1}{2}\sum_\mu \lambda_\mu x_\mu^2 + z\left(\sum_\mu x_\mu^2 - N\right).$$

Integrals of motion (Uhlenbeck, 1980s)

The Neumann model:

$$\mathcal{H} = \frac{1}{2m}\sum_{\mu} p_{\mu}^2 - \frac{1}{2}\sum_{\mu} \lambda_{\mu} x_{\mu}^2 + z\left(\sum_{\mu} x_{\mu}^2 - N\right).$$

is integrable with conserved quantities:

$$I_{\mu} = x_{\mu}^2 + \frac{1}{mN}\sum_{\nu(\neq\mu)} \frac{(x_{\mu}p_{\nu} - x_{\nu}p_{\mu})^2}{\lambda_{\nu} - \lambda_{\mu}}.$$

- Each I_{μ} is conserved: $\dot{I}_{\mu} = 0$.
- The dynamics is thus non-chaotic and exactly solvable in principle.

Types of solutions and phases

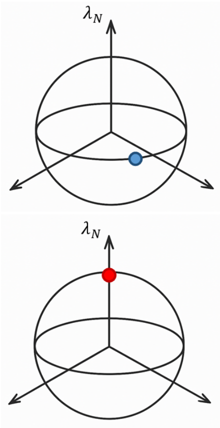
Drawn from canonical equilibrium with $\lambda_\mu^{(0)}$ at K:

- **Extended phase:** all modes share similar amplitude when $K < 1$,

$$\langle x_N^2 \rangle = \mathcal{O}(1).$$

- **Condensed phase:** localization along the last mode λ_N when $K > 1$,

$$\langle x_N^2 \rangle = qN.$$

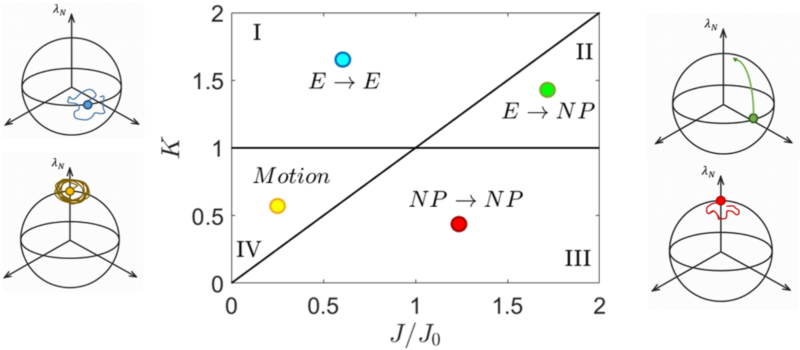


Instantaneous quench

Start from equilibrium at K and perform a sudden change in macroscopic energy $J \to \lambda_\mu = \frac{J}{J_0}\lambda_\mu^{(0)}$.

- $J/J_0 < 1$: energy injection.
- $J/J_0 > 1$: energy extraction.

Phase diagram



Generalized Gibbs Ensemble Hamiltonian

To describe steady states preserving all integrals of motion:

$$\mathcal{H}_{\mathrm{GGE}} = \sum_{\mu} \gamma_\mu I_\mu + z \left(\sum_{\mu} x_\mu^2 - N \right) .$$

Compare with the equilibrium Hamiltonian ($\gamma_\mu = -\lambda_\mu/2$):

$$\mathcal{H} = \frac{1}{2m} \sum_{\mu} p_\mu^2 - \frac{1}{2} \sum_{\mu} \lambda_\mu x_\mu^2 + z \left(\sum_{\mu} x_\mu^2 - N \right) .$$

Interpretation:

- The dynamics now preserves $\{I_\mu\}$ rather than just total energy.
- Long-time limit $\Rightarrow$ GGE.

Canonical measure:

$$\rho_{\mathrm{GGE}}(x, p) = \frac{1}{Z(\{\gamma_\mu\})} \exp \left[-\beta \sum_{\mu} \gamma_\mu I_\mu(x, p) \right] .$$

Newton Dynamics vs GGE

To describe steady states that preserve all integrals of motion:

$$\mathcal{H}_{\text{GGE}} = \sum_{\mu} \gamma_\mu I_\mu + z\left(\sum_{\mu} x_\mu^2 - N\right).$$

Solutions for x_μ and p_μ are obtained through:

- Different initial trajectories and the evolution under Newton dynamics.
- The GGE with a functional $\beta\gamma_\mu(\lambda_\mu, K, \frac{J}{J_0})$, imposing $I_\mu(0^+) = \langle I_\mu \rangle_{\text{GGE}}$.

It is verified that the observables satisfy:

$$\langle x_\mu^2 \rangle = \overline{\langle x_\mu^2 \rangle_{\text{i.c.}}}, \qquad \langle p_\mu^2 \rangle = \overline{\langle p_\mu^2 \rangle_{\text{i.c.}}}.$$

Barbier, Cugliandolo, Lozano, Nessi. 2022.

Stochastic Langevin dynamics towards GGE

We couple the integrable system to a thermal bath:

$$\dot{x}_\mu = \frac{\partial \mathcal{H}_{\mathrm{GGE}}}{\partial p_\mu},$$
$$\dot{p}_\mu + \eta \dot{x}_\mu = -\frac{\partial \mathcal{H}_{\mathrm{GGE}}}{\partial x_\mu} + \xi_\mu(t),$$

with white noise:

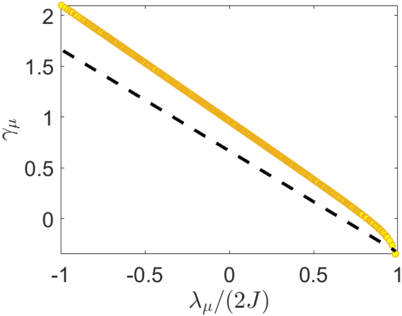
$$\langle \xi_\mu(t) \rangle = 0, \qquad \langle \xi_\mu(t) \xi_\nu(t') \rangle = 2\eta k_B T \, \delta_{\mu\nu} \delta(t - t').$$

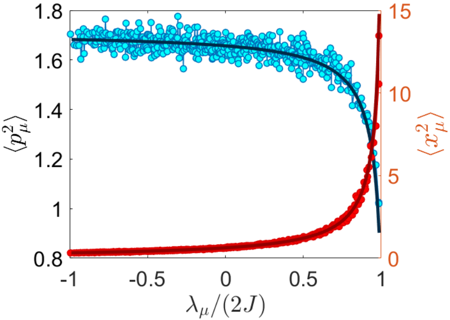
Goal: verify numerically and analytically that the stochastic dynamics reproduces the GGE averages:

$$\langle x_\mu^2 \rangle_{\mathrm{GGE}} = \lim_{t \to \infty} \langle x_\mu^2 \rangle_\xi, \quad \langle p_\mu^2 \rangle_{\mathrm{GGE}} = \lim_{t \to \infty} \langle p_\mu^2 \rangle_\xi.$$

Stochastic Langevin dynamics towards GGE

We use the GGE parameters $\beta\gamma_\mu$ corresponding to $K = 1.5$ and $J = 1.3$, $J_0 = 1$. We simulate the Langevin dynamics with $\eta = 5$, $N = 500$, $T = 1.5$ in phase I.



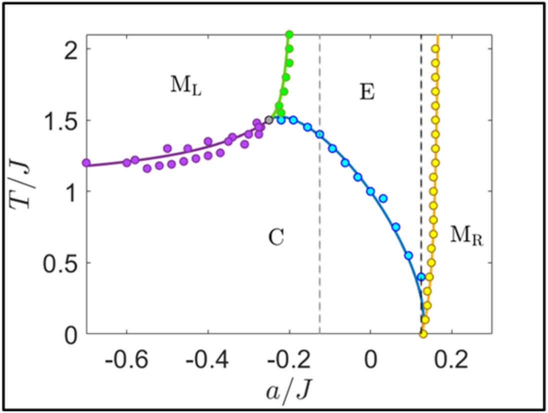


Similarly for the other phases except phase IV. We need a dynamic description.

Cugliandolo, RGA, Lozano, Stariolo. 2025.

Phase diagram for quadratic γ_μ

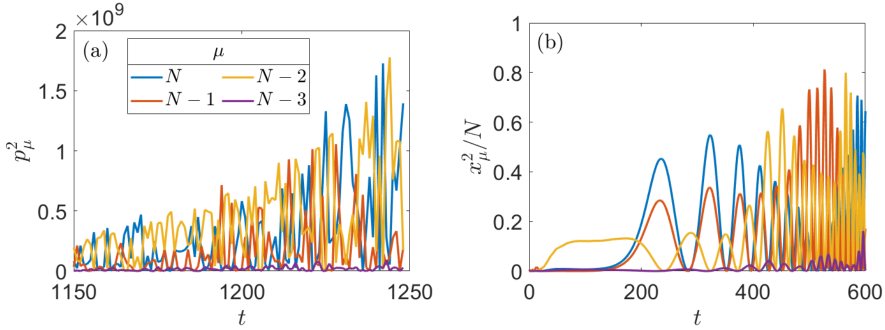
We use $\gamma_\mu = a\lambda_\mu^2 - \lambda_\mu/2$ because it allows us to directly compare analytical and numerical results.



Moving phases M_L and M_R

In the moving phase, momentum grows exponentially with time, and wave frequencies increase.

- When the fastest modes are the right ones ($\mu \approx N$): $\Rightarrow M_R$, occurs for $a > 1/8$ ($T = 0$).
- When the fastest modes are the left ones ($\mu \approx 1$): $\Rightarrow M_L$, occurs when $\gamma_1 < \gamma_N$ ($T = 0$).



Conclusions

- We constructed a modified Hamiltonian that includes all constants of motion of the Neumann model, ensuring convergence to the correct GGE.
- The deterministic dynamics were coupled to a thermal environment through a Langevin-type stochastic bath.
- We sampled the GGE using stochastic dynamics.
- Using a quadratic ansatz for γ_μ, we characterized the different dynamical phases.

Thank you for your attention!