3d gauge theory / quantum K-theory Correspondence

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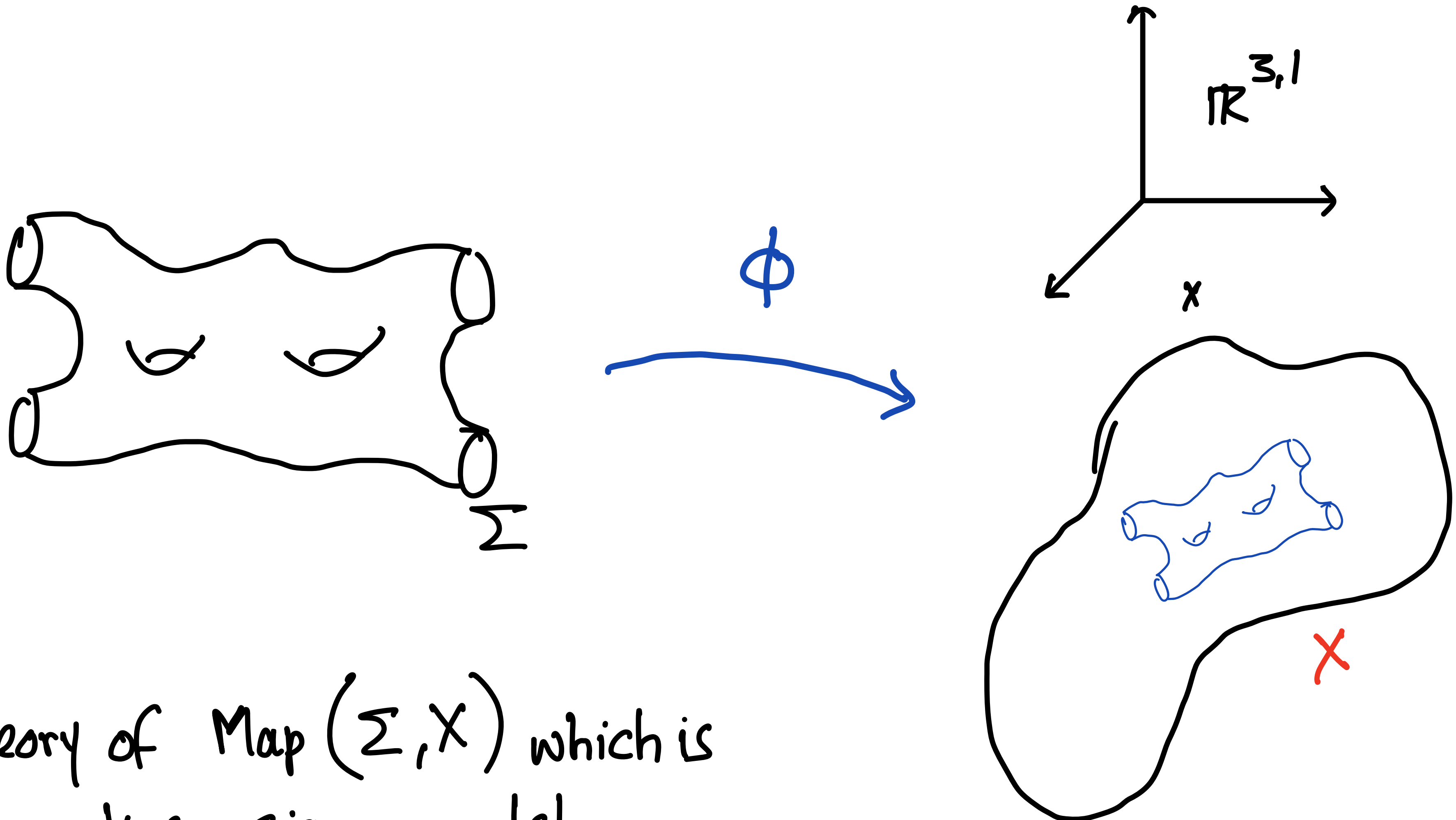
String theory



Theory of $Map(\Sigma, X)$ which is a non-linear sigma model

Complicated to work with

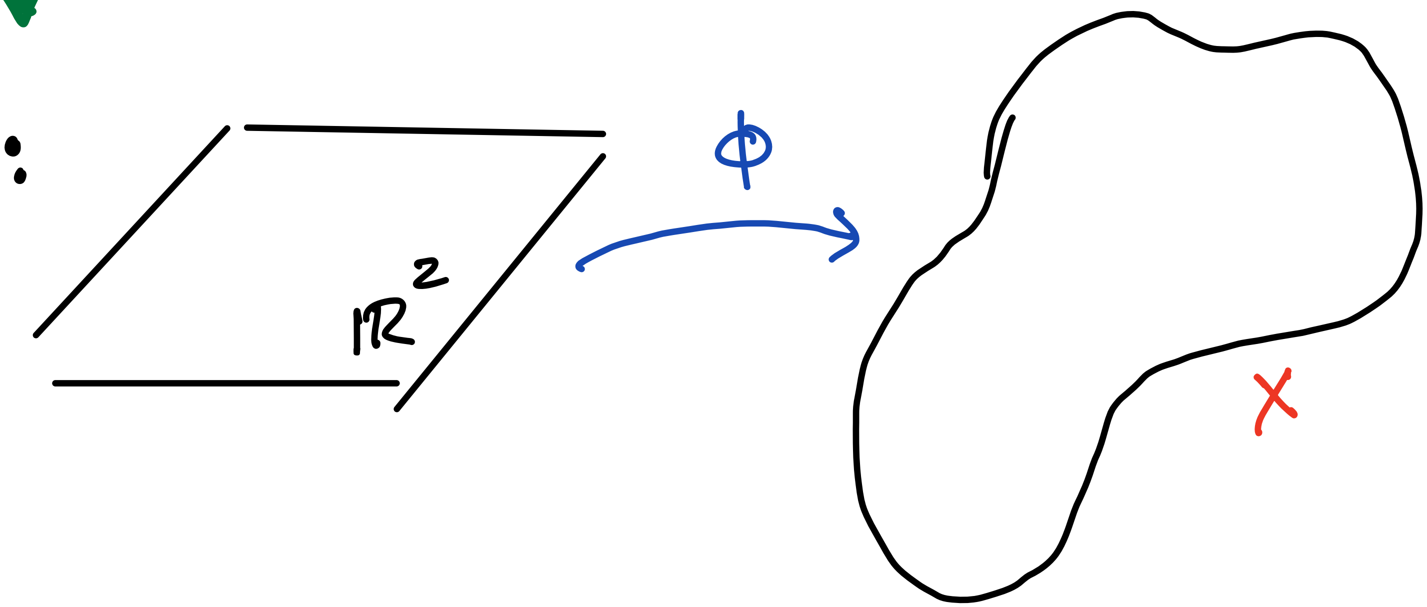
2d Gauged Linear Sigma Models [Witten '86]

(G, ϕ, W) on $\mathbb{R}^2$ with Higgs branch X

↑ Moduli space of vacua

$\Big\downarrow$ RG flow

2d NLSM :



In these 2d GLSMs, $\exists\, Q_A$: supercharge s.t. $Q_A^2 = 0$

- $\mathcal{R} \equiv Q_A$- Cohomology $= \{ \mathcal{O}(z) : Q_A \mathcal{O} = 0$ and $\mathcal{O} \neq \{Q_A, \Lambda\} \}$

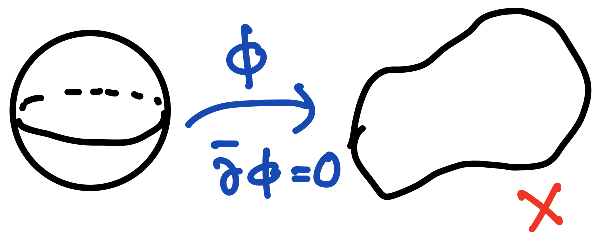
As sets:

$$\mathcal{R}_{2d} \cong H^\bullet(X)$$

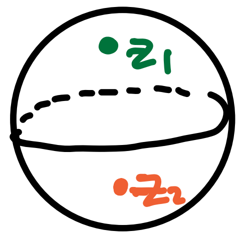
$$\mathcal{O}_i \leftrightarrow [\omega_i(z)]$$

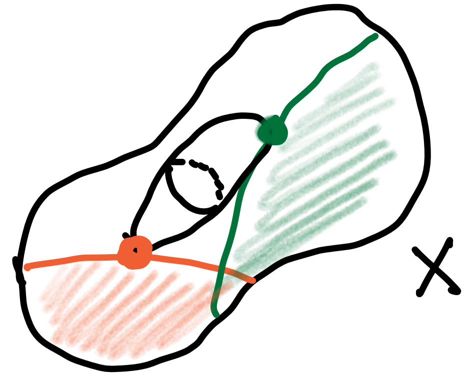
- Putting this Q_A-subsector on S^2:

$$\int [D\Phi] \exp\left(-S[\Phi]\right) = \int \left(\cdots \right)$$

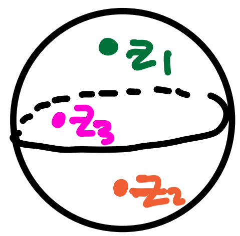
$\{$  $\} \equiv \overline{\mathcal{M}}_{0,\bullet}(X)$

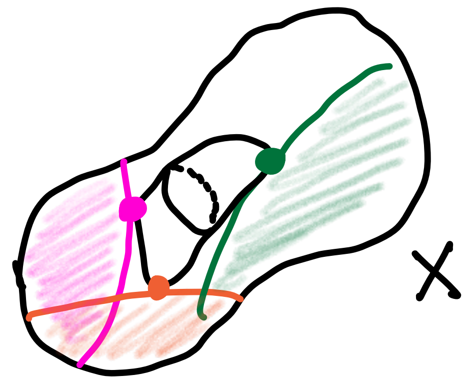
In particular:

$\eta_{ij} = \sum_{d \in H_2(X,\mathbb{Z})} q^d \, \# \Big\{$ 

$\xrightarrow[\bar\partial \phi = 0]{\phi}$ 

$\Big\}$

$C_{ijk} = \sum_{d \in H_2(X,\mathbb{Z})} q^d \, \# \Big\{$ 

$\xrightarrow[\bar\partial \phi = 0]{\phi}$ 

$\Big\}$

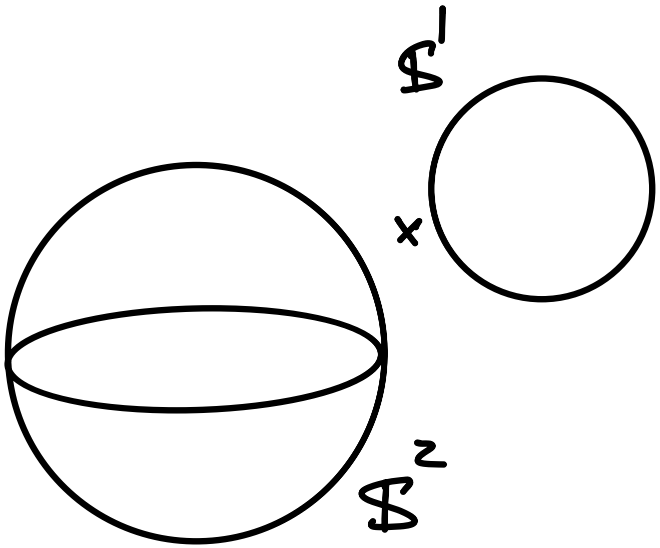
Fusion rules:

$$O_i * O_j = C_{ij}{}^k O_k \Rightarrow \omega_i \wedge_q \omega_j = C_{ij}{}^k \omega_k = \omega_i \wedge \omega_j + O(q)$$

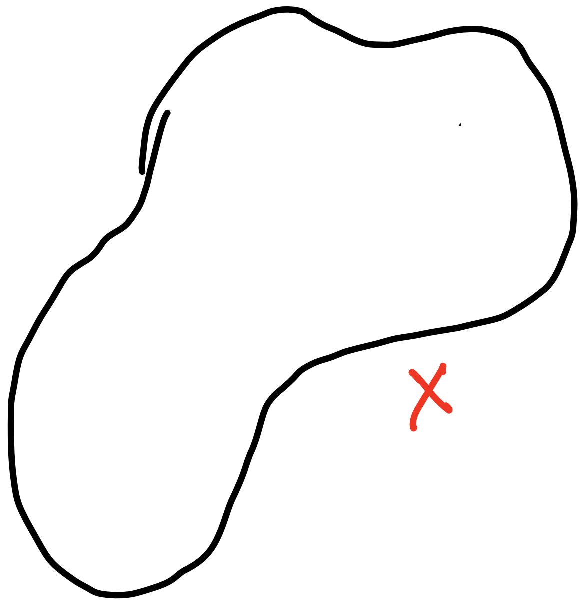
↳ quantum product

↳ $QH^\bullet(X)$.

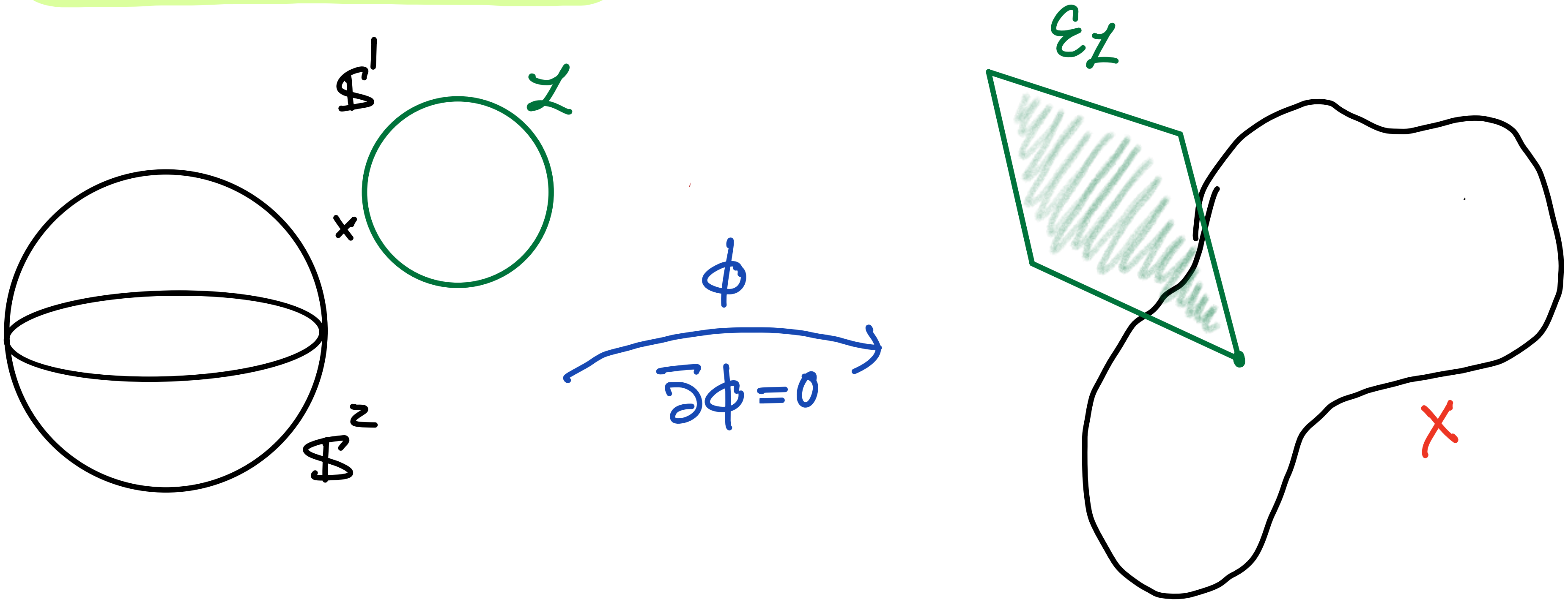
3d uplift of the story



$$\xrightarrow[\bar\partial \phi = 0]{\phi}$$



3d uplift of the story

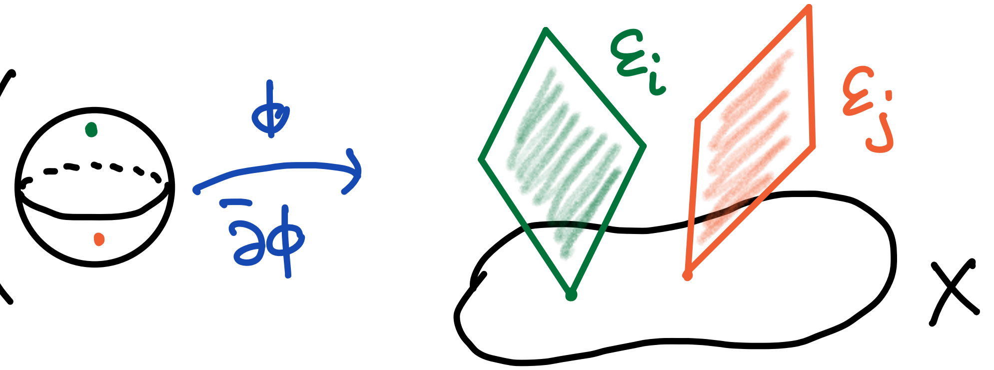


- $R = \left\{ \begin{array}{l} \text{BPS line operators} \\ \text{wrapping } S^1 \end{array} \right\} = K(X)$

- $\langle \mathcal{L} \dots \rangle_{S^2 \times S^1} = \sum_{d \in H_2(X,\mathbb{Z})} q^d \chi_T^{vir}\left(\overline{\mathcal{M}}_{0,d}(X), \mathcal{O}^{vir} \otimes \mathcal{E}_{\mathcal{L}} \otimes \dots\right)$

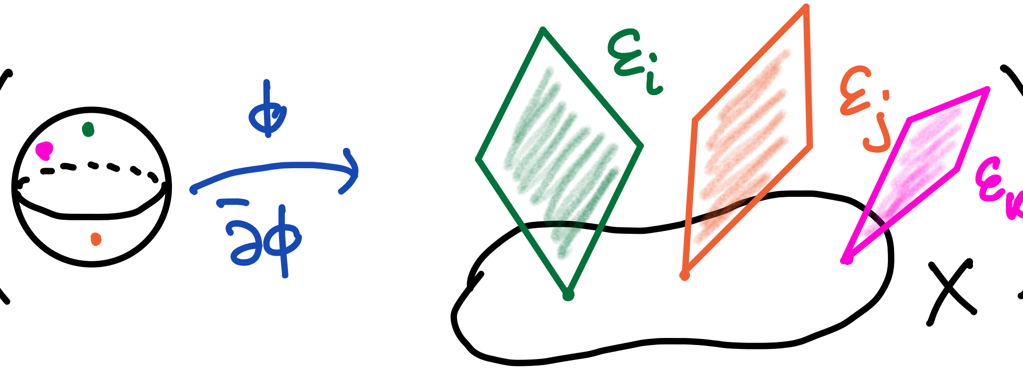
In particular

$$g_{ij} = \sum_{d \in H_2(X,\mathbb{Z})} q^d \, \chi\Big($$



$$\Big)$$

$$N_{ijk} = \sum_{d \in H_2(X,\mathbb{Z})} q^d \, \chi\Big($$



$$\Big)$$

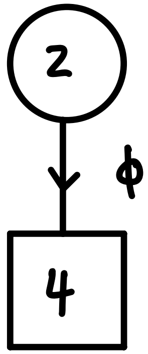
Fusion rules:

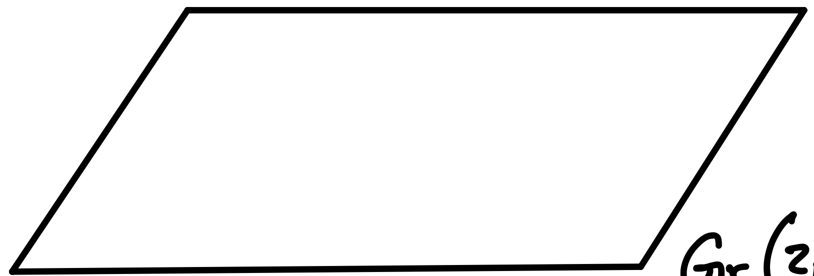
$$\mathcal{L}_i * \mathcal{L}_j = N_{ij}{}^{k} \mathcal{L}_k \Rightarrow \mathcal{E}_i \otimes_q \mathcal{E}_j = N_{ij}{}^{k} \mathcal{E}_k = \mathcal{E}_i \otimes \mathcal{E}_j + O(q)$$

↑ quantum tensor product

$\Rightarrow QK(X)$

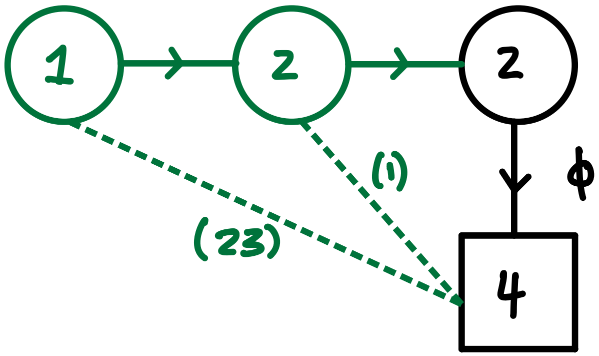
Example: Grassmannian manifold $Gr(k,n)$

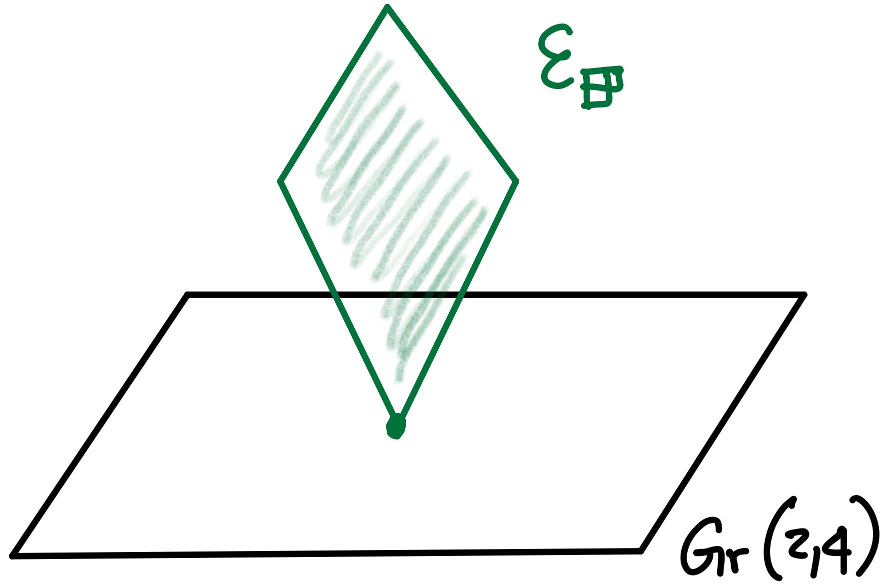




$Gr(2,4)$

Example: Grassmannian manifold $Gr(k,n)$





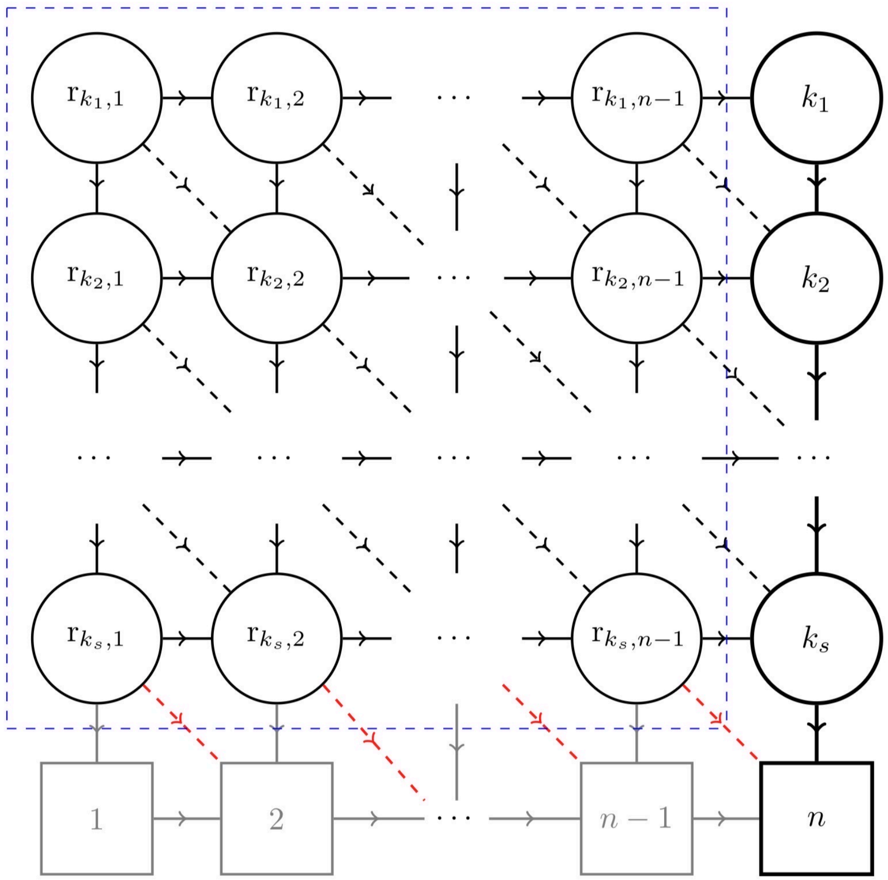
$$Z_{QM} = (1-x_1)(1-x_2)(1-x_1x_2) = \mathrm{ch}(\mathcal{E}_{⊞})$$

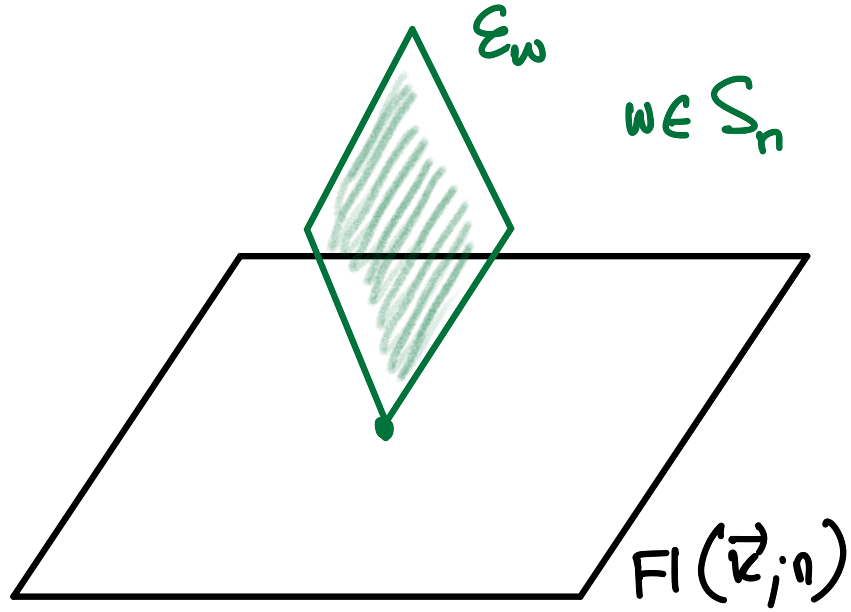
↳ Grothendieck polynomial
for partition ⊞ ⊂ ⊞

$$
\begin{aligned}
\mathcal{O}_{\ydiagram{1}}\,\mathcal{O}_{\ydiagram{1}} &= \left(1-\frac{y_3}{y_2}\right)\mathcal{O}_{\ydiagram{1}} + \frac{y_3}{y_2}\mathcal{O}_{\ydiagram{1,1}} + \frac{y_3}{y_2}\mathcal{O}_{\ydiagram{2}} - \frac{y_3}{y_2}\mathcal{O}_{\ydiagram{2,1}}\,,\\
\mathcal{O}_{\ydiagram{1}}\,\mathcal{O}_{\ydiagram{1,1}} &= \left(1-\frac{y_3}{y_1}\right)\mathcal{O}_{\ydiagram{1,1}} + \frac{y_3}{y_1}\mathcal{O}_{\ydiagram{2,1}}\,,\\
\mathcal{O}_{\ydiagram{1}}\,\mathcal{O}_{\ydiagram{2}} &= \left(1-\frac{y_4}{y_2}\right)\mathcal{O}_{\ydiagram{2}} + \frac{y_4}{y_2}\mathcal{O}_{\ydiagram{2,1}}\,,\\
\mathcal{O}_{\ydiagram{1}}\,\mathcal{O}_{\ydiagram{2,1}} &= q\frac{y_4}{y_1} - q\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_4}{y_1}\right)\mathcal{O}_{\ydiagram{2,1}} + \frac{y_4}{y_1}\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{1}}\,\mathcal{O}_{\ydiagram{2,2}} &= q\frac{y_3y_4}{y_1y_2}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_3y_4}{y_1y_2}\right)\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{1,1}}\,\mathcal{O}_{\ydiagram{1,1}} &= \left(1-\frac{y_2}{y_1}\right)\left(1-\frac{y_3}{y_1}\right)\mathcal{O}_{\ydiagram{1,1}} + \left(1-\frac{y_2}{y_1}\right)\frac{y_3}{y_1}\mathcal{O}_{\ydiagram{2,1}} + \frac{y_2}{y_1}\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{1,1}}\,\mathcal{O}_{\ydiagram{2}} &= q\frac{y_4}{y_1} + \left(1-\frac{y_4}{y_1}\right)\mathcal{O}_{\ydiagram{2,1}}\,,\\
\mathcal{O}_{\ydiagram{1,1}}\,\mathcal{O}_{\ydiagram{2,1}} &= q\left(1-\frac{y_2}{y_1}\right)\frac{y_4}{y_1} + q\frac{y_2y_4}{y_1^2}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_2}{y_1}-\frac{y_4}{y_1}+\frac{y_2y_4}{y_2^2}\right)\mathcal{O}_{\ydiagram{2,1}} + \left(\frac{y_2}{y_1}-\frac{y_4}{y_1}\right)\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{1,1}}\,\mathcal{O}_{\ydiagram{2,2}} &= q\left(1-\frac{y_3}{y_1}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + q\frac{y_3}{y_1}\mathcal{O}_{\ydiagram{2}} + \left(1-\frac{y_3}{y_1}-\frac{y_4}{y_1}+\frac{y_3y_4}{y_1^2}\right)\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{2}}\,\mathcal{O}_{\ydiagram{2}} &= \left(1-\frac{y_4}{y_3}\right)\left(1-\frac{y_4}{y_2}\right)\mathcal{O}_{\ydiagram{2}} + \left(1-\frac{y_4}{y_3}\right)\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{2,1}} + \frac{y_4}{y_3}\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{2}}\,\mathcal{O}_{\ydiagram{2,1}} &= q\left(1-\frac{y_4}{y_3}\right)\frac{y_4}{y_1} + q\frac{y_4^2}{y_1y_3}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_4}{y_3}-\frac{y_4}{y_1}+\frac{y_4^2}{y_1y_3}\right)\mathcal{O}_{\ydiagram{2,1}} + \left(1-\frac{y_4}{y_1}\right)\frac{y_4}{y_3}\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{2}}\,\mathcal{O}_{\ydiagram{2,2}} &= q\left(1-\frac{y_4}{y_2}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + q\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{1,1}} + \left(1-\frac{y_4}{y_2}-\frac{y_4}{y_1}+\frac{y_4^2}{y_1y_2}\right)\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{2,1}}\,\mathcal{O}_{\ydiagram{2,1}} &= q\left(1-\frac{y_2}{y_1}\right)\left(1-\frac{y_4}{y_3}\right)\frac{y_4}{y_1} + q\left(\frac{y_2}{y_1}-\frac{y_4}{y_1}+\frac{y_4}{y_3}-\frac{y_2y_4}{y_1y_3}\right)\mathcal{O}_{\ydiagram{1}} + q\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1,1}} + q\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{2}}\\
&\quad + \left[\left(1-\frac{y_2}{y_1}\right)\left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_3}\right) - q\frac{y_4}{y_1}\right]\mathcal{O}_{\ydiagram{2,1}} + \left(1-\frac{y_4}{y_1}\right)\left(\frac{y_2}{y_1}-\frac{y_4}{y_1}+\frac{y_4}{y_3}-\frac{y_2y_4}{y_1y_3}\right)\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{2,1}}\,\mathcal{O}_{\ydiagram{2,2}} &= q\left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + q\left(1-\frac{y_3}{y_1}\right)\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{1,1}} + q\left(\frac{y_3}{y_1}-\frac{y_3y_4}{y_1y_2}\right)\mathcal{O}_{\ydiagram{2}} + q\frac{y_3y_4}{y_1y_2}\mathcal{O}_{\ydiagram{2,1}}\\
&\quad + \left(1-\frac{y_4}{y_2}+\frac{y_4^2}{y_1y_2}-\frac{y_3}{y_1}-\frac{y_4}{y_1}+\frac{y_3y_4}{y_1y_2}-\frac{y_3y_4^2}{y_1^2y_2}+\frac{y_3y_4}{y_1^2}\right)\mathcal{O}_{\ydiagram{2,2}}\,,\\
\mathcal{O}_{\ydiagram{2,2}}\,\mathcal{O}_{\ydiagram{2,2}} &= q^2\frac{y_3y_4}{y_1y_2} + q\left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\left(1-\frac{y_4}{y_2}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}}\\
&\quad + q\left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{1,1}} + \left(\frac{y_3}{y_1}-\frac{y_3^2}{y_1y_2}-\frac{y_3y_4}{y_1y_2}+\frac{y_3^2y_4}{y_1y_2^2}\right)\mathcal{O}_{\ydiagram{2}}\\
&\quad + \left(\frac{y_3}{y_2}-\frac{y_3^2y_4}{y_1y_2^2}\right)\mathcal{O}_{\ydiagram{2,1}} + \left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)\mathcal{O}_{\ydiagram{2,2}}\,.
\end{aligned}
$$

Another example: Partial flag manifolds $Fl(k_1, k_2, \dots, k_s; n)$

$$\left\{ 0 \subset V_1 \subset V_2 \subset \cdots \subset V_s \subset \mathbb{C}^n \ ; \ \dim_{\mathbb{C}} V_\ell = k_\ell \right\}$$





1d index as a Chern character

$$\mathcal{I}_w^{(1\mathrm{d})}\begin{bmatrix} q \\ \mathbf{k} \end{bmatrix}(x,y) = \oint_{\mathrm{JK}} (\mathrm{dM})\, \mathrm{Z}^{\mathrm{ver}}_{\mathrm{chiral}}\, \mathrm{Z}^{\mathrm{hor}}_{\mathrm{chiral}}\, \mathrm{Z}^{\mathrm{black}}_{\mathrm{fermi}}\, \mathrm{Z}^{\mathrm{red}}_{\mathrm{fermi}}\,,$$

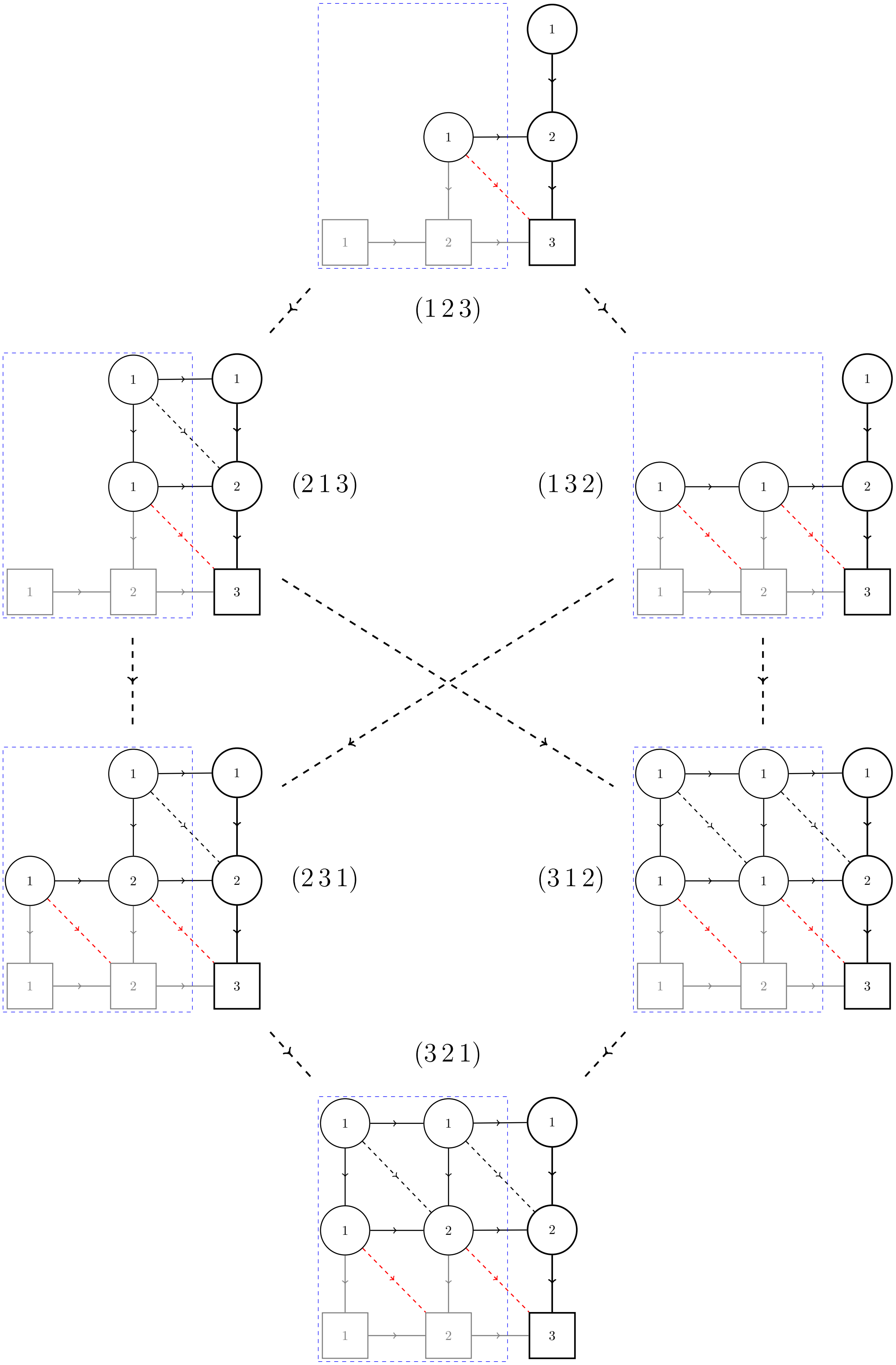
$$\mathrm{Z}^{\mathrm{ver}}_{\mathrm{chiral}} := \prod_{\ell=1}^{s-1}\prod_{i=1}^{n-1}\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,i}}\prod_{\beta=1}^{\mathrm{r}_{k_{\ell+1},i}} \left(1-\frac{z_\alpha^{(k_\ell,i)}}{z_\beta^{(k_{\ell+1},i)}}\right)^{-1},$$

$$\mathrm{Z}^{\mathrm{hor}}_{\mathrm{chiral}} := \prod_{\ell=1}^{s}\left[\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,n-1}}\prod_{a=1}^{k_\ell}\left(1-\frac{z_\alpha^{(k_\ell,n-1)}}{x_a^{(\ell)}}\right)^{-1}\prod_{i=1}^{n-2}\prod_{\beta=1}^{\mathrm{r}_{k_\ell,i}}\prod_{\gamma=1}^{\mathrm{r}_{k_\ell,i+1}}\left(1-\frac{z_\beta^{(k_\ell,i)}}{z_\gamma^{(k_\ell,i+1)}}\right)^{-1}\right],$$

$$\mathrm{Z}^{\mathrm{black}}_{\mathrm{fermi}} := \prod_{\ell=1}^{s-1}\left[\prod_{\gamma=1}^{\mathrm{r}_{k_\ell,n-1}}\prod_{a=1}^{k_{\ell+1}}\left(1-\frac{z_\gamma^{(k_\ell,n-1)}}{x_a^{(\ell+1)}}\right)\prod_{i=1}^{n-1}\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,i}}\prod_{\beta=1}^{\mathrm{r}_{k_{\ell+1},i+1}}\left(1-\frac{z_\alpha^{(k_\ell,i)}}{z_\beta^{(k_{\ell+1},i+1)}}\right)\right],$$

$$\mathrm{Z}^{\mathrm{red}}_{\mathrm{fermi}} := \prod_{i=1}^{n-1}\prod_{\alpha=1}^{\mathrm{r}_{k_s,i}}\left(1-\frac{z_\alpha^{(k_s,i)}}{y_{n-i}}\right).$$

$$(\mathrm{dM}) := \prod_{\ell=1}^{s}\prod_{i=1}^{n-1}\left[\Delta^{(k_\ell,i)}(z)\,\frac{1}{\mathrm{r}_{k_\ell,j}\,!}\prod_{\alpha=1}^{\mathrm{r}_{k_\ell,i}}\frac{dz_\alpha^{(k_\ell,i)}}{2\pi i\, z_\alpha^{(k_\ell,i)}}\right],\qquad \Delta^{(k_\ell,i)}(z) := \prod_{1\le\alpha\neq\beta\le \mathrm{r}_{k_\ell,i}}\left(1-\frac{z_\alpha^{(k_\ell,i)}}{z_\beta^{(k_\ell,i)}}\right).$$


(1 2 3)
(2 1 3)
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(2 3 1)
(3 1 2)
(3 2 1)

Thank You !