Positivity properties of scattering amplitudes

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+

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and

work in progress with Sara Ditsch and Johannes Henn

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Introduction

In this talk we shall discuss several examples of quantities that satisfy positivity properties across different orders in perturbation theory

$$A \sim \sum_{L=0}^{\infty} g^L A^{(L)}$$

Motivation

- **Positivity of Integrated objects:** Positivity properties of integrands have been investigated extensively, but at the level of integrated objects are far less studied.

- **Uncover patterns in perturbative data:** see if these hint at some deeper underlying structure.

- **Input to Bootstraps:** see if these properties can be used in numerical/ analytic bootstrap programs.

Completely monotone functions

- A $\mathbf{C}^\infty$-function $f(x)$, is called completely monotone (CM) in a region $R \subset \mathbb{R}$ if it satisfies

$$(-\partial_x)^n f(x) \geq 0\,, \quad \text{for} \quad n \geq 0\,, \quad \forall x \in R\,.$$

- In particular,

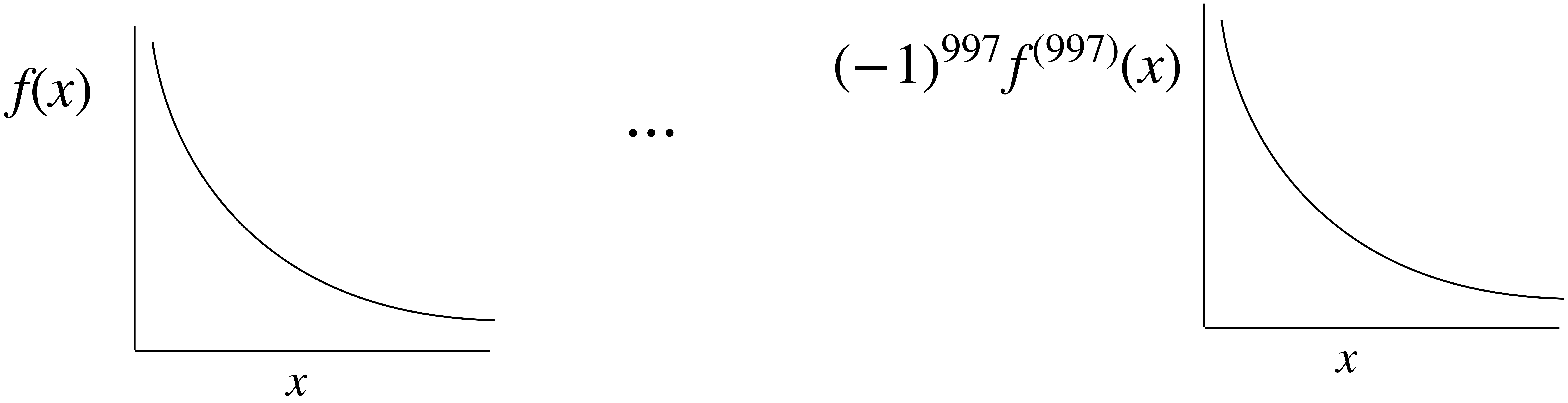
$$n = 0 \implies f(x) \geq 0 \quad \equiv \quad \text{positive,}$$

$$n = 1 \implies f'(x) \leq 0 \quad \equiv \quad \text{monotonically decreasing,}$$

$$n = 2 \implies f''(x) \geq 0 \quad \equiv \quad \text{convex,}$$

$$\vdots$$

- Leads to typically boring graphs for the function and all its signed derivatives in the region



- Taylor coefficients around any point inside the region alternate in sign.

- Property does not depend on having a convergent Taylor series.

- The property as defined crucially depends on the **choice of variable** *and the* **region**.

Examples

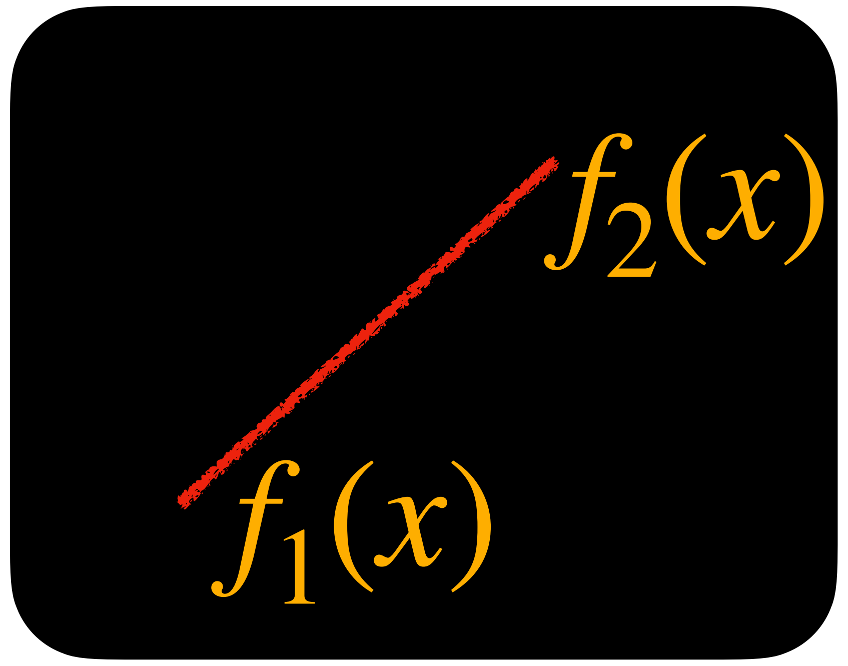
(1) $\dfrac{1}{x+\alpha}$ is CM in $R = (0,\infty)$.

(2) $-\log x$ is CM in $R = (0,1)$.

(3) $\dfrac{\log x + A}{x+1}$ *and* $\dfrac{1}{1-x}$ are not CM on any (0,B).

Properties of CM functions

If $f_1(x)$ and $f_2(x)$ are CM in R then for any $\lambda_i \geq 0 \; i = 1,2$ then the following are also CM in R



Space of CM functions in R

1. Convex Cone: $\lambda_1 f_1(x) + \lambda_2 f_2(x)$
2. Closed under products: $f_1(x)\, f_2(x)$
3. Closed under signed derivatives: $(-\partial_x)^n f_i(x)$

Also closed under *some* compositions and limits.

Multivariate version

- Definition generalizes readily to several variables.
- If $f(x_1, \cdots, x_n)$ is Completely monotone (CM) on $R \subset \mathbb{R}^n$ if it satisfies ∀ points in R

$$(-\partial_{x_1})^{m_1} \cdots (-\partial_{x_n})^{m_n} f(x_1, \cdots, x_n) \geq 0 \qquad \forall m_i = 0,1,2,\cdots$$

Nontrivial examples are harder to construct but a simple yet important example is

$$\frac{1}{(c_1 x_1 + \cdots + c_n x_n + d)^\alpha}$$ is CM in $R = \mathbb{R}^n_+$ for $c_i, d, \alpha > 0$.

Quantity $(-1)^L A^{(L)}$		*Region*	*Loop order*
angle dep. cusp anomalous dimension	$-\Gamma^{(L)}_{cusp}(x)$	$x \in (0,1)$	L=3 in QCD L=4 in QED and SYM
4 point coloumb branch amplitudes	$M^{(L)}(u,v)$	$u, v > 0$	L=3
MHV 6-particle BDS like reminder function	$\mathcal{E}^{(L)}(u,v,w)$	$u, v, w > 0$ $u + v + w < 1$ $(1-u-v-w)^2 - 4uvw \geq 0$	L=4
Scalar Feynman Integrals	$I(\{s_{T,R}, m_i^2\})$	$-s_{T,R}\ ,\ m_i^2 \geq 0$	arbitrary L
⋮			

Outline

- ~~What?~~
- Why/How ?

- Elementary arguments from suitable representations.

- From causality and analyticity.

- From positive geometry.

- Applications

- S-matrix bootstrap.

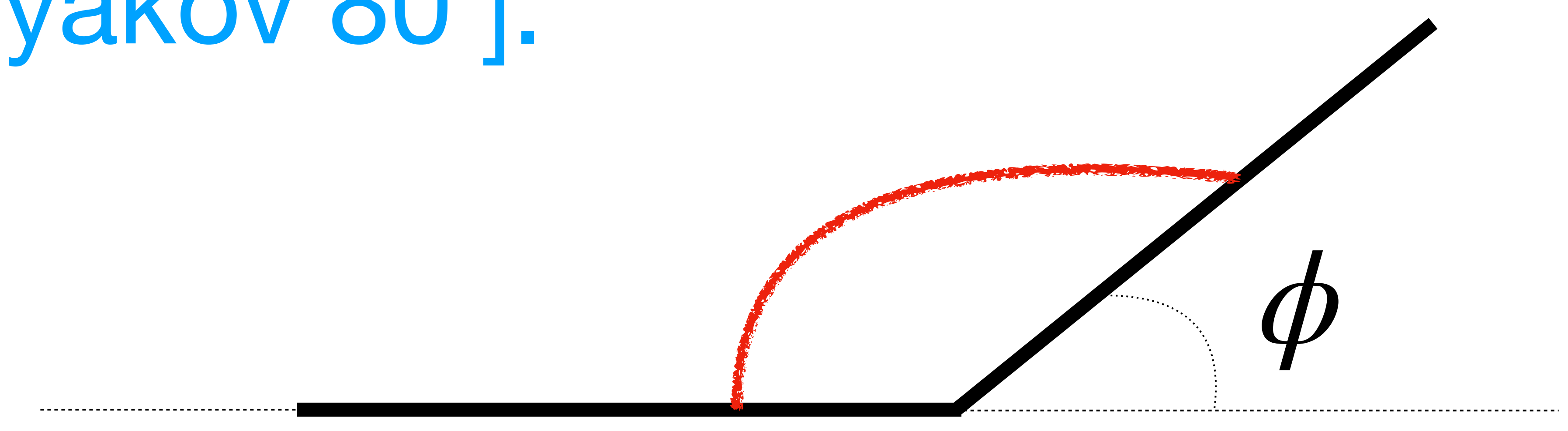
- Positive geometry.

-Numerical Bootstrap of Feynman integrals

Elementary arguments from suitable representation

- The expectation value of a Wilson loop with a cusp is UV divergent [Polyakov 80'].

$$\langle W_c \rangle \sim \frac{1}{\epsilon} \Gamma_{cusp}(\phi)$$



$$x = e^{i\phi}$$

- Γ_{cusp} is also related to:

- IR divergences in scattering of massive W-bosons in coulomb branch of N=4 SYM.

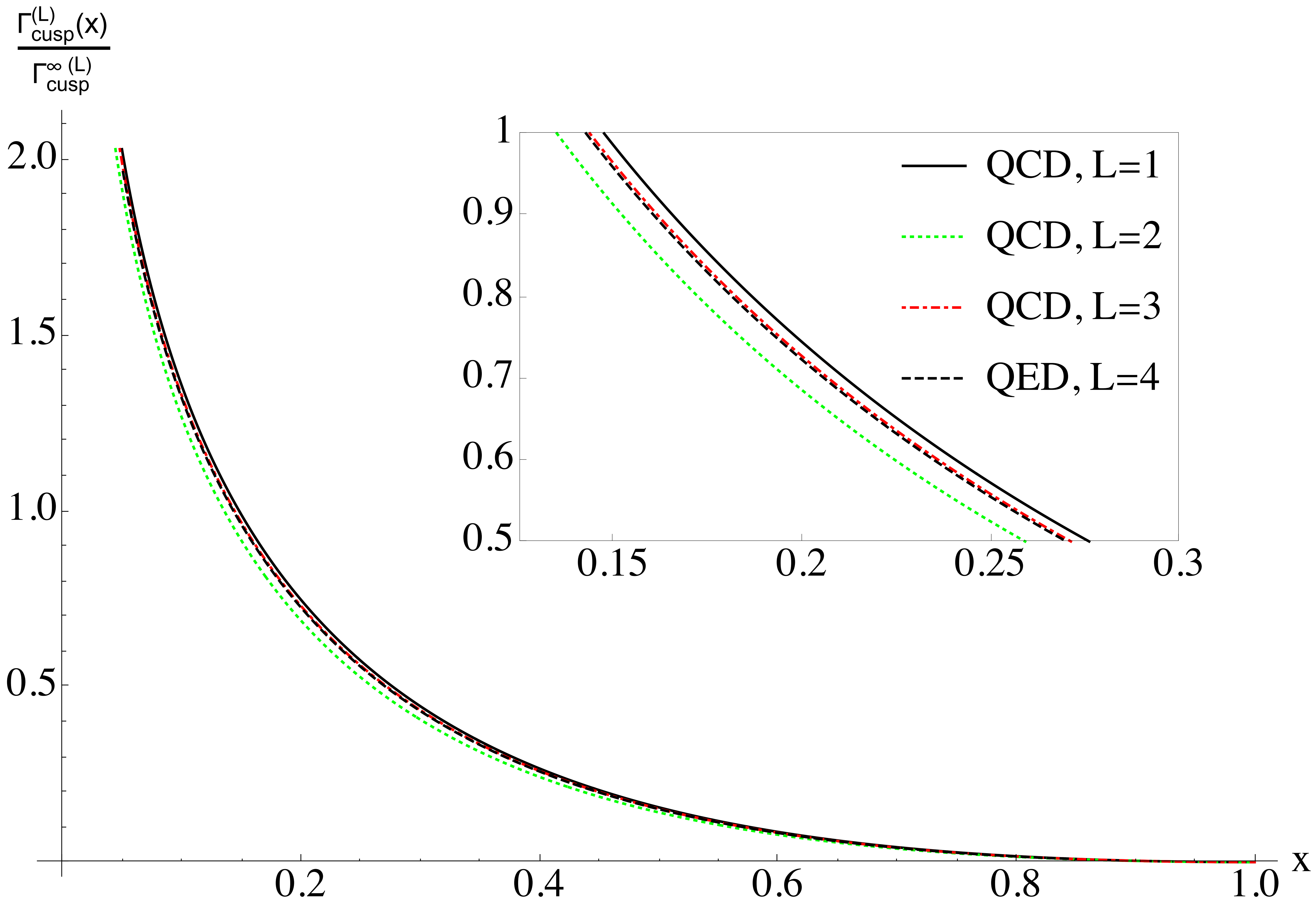
- anomalous dimension in correlators with high spin composite operators.

- quark-anti quark potential on the sphere etc.

Complete monotonicity of angle dependent cusp anomalous dimension

Claim: $\Gamma_{cusp}(x) = \sum_{L\geq 1} \Gamma^{L}_{cusp}(x) g^{2L}$ **and** $(-1)^{L+1}\Gamma^{(L)}_{cusp}(x)$ **is a CM for** $x \in (0,1)$.

- Some proofs are by using elementary arguments, for example $\Gamma^{(1)}_{cusp}(x) = \dfrac{1-x}{1+x}(-\log x)$



Checked up to L=3 in QCD and L=4 in N=4 SYM and in QED.

Hausdorff theorem and Stieltjes functions

- **[Hausdorff-Bernstien-Widder Thm]** A function $f(x_1, \cdots, x_n)$ is completely monotone for $R = \mathbb{R}^n_+$ if and only if it can representable as

$$f(x_1, \cdots, x_n) = \int_{\mathbb{R}^n_+} d\mathbf{y}\ e^{-x_1 y_1 - \cdots - x_n y_n} \quad \underbrace{\mu(y_1, \cdots, y_n)}_{\geq 0}$$

- A nice subclass of CM functions are when $\mu(\mathbf{y})$ is itself CM called Stieltjes functions

$$f(x_1, \cdots, x_n) = \int_{\mathbb{R}^n_+} d\mathbf{z}\ \frac{1}{(x_1 + z_1)(x_2 + z_2)\ldots(x_n + z_n)}\ \underbrace{\nu(z_1, \cdots, z_n)}_{\geq 0}$$

A complex property unlike CM which is a real property.

From Analyticity/ Causality

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- CM property is closely related to dispersion, Källen-Lehmann/spectral and Mandelstram representations.
- Consider, a function that satisfies an *unsubtracted* dispersion relation

$$A(s) = \int_{4m^2}^{\infty} ds' \frac{1}{s' - s} \text{ Disc } A(s')$$

$$A(x) = \int_0^{\infty} dt\ e^{-tx} \underbrace{\int_{4m^2}^{\infty} ds'\ e^{-ts'} \text{ Disc } A(s')}_{\mu(t)}. \qquad \text{(For } x = -s\text{)}$$

- $A(x)$ is

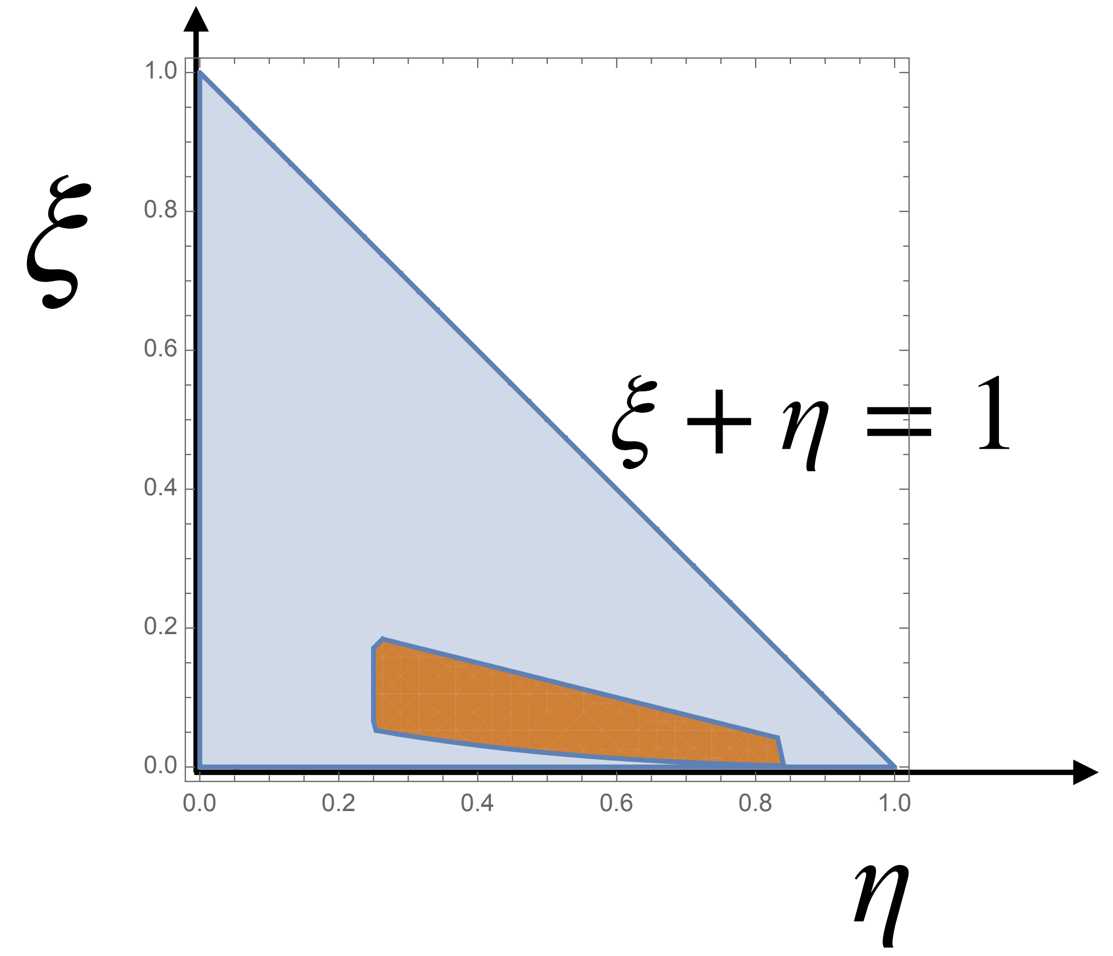
  - Stieltjes if Disc $A(s') \geq 0$, Stronger (can follow from **Unitarity/Optical theorem**)

  - CM for $x > 0$, if $\mu(t) \geq 0$. Weaker

Coulomb branch amplitudes in N=4 SYM.

- The 4-pt Coloumb branch amplitudes admits a particularly simple looking Mandelstram representation [Caron-Huot, Henn 13']

$$M(u,v) = \int_\Delta d\xi d\eta \frac{\rho(\xi,\eta)}{(\xi+u)(\eta+v)}$$

$\Delta \equiv$ 

with a double spectral function $\rho(\xi,\eta)$.

- $M(u,v)$ is

[Mahoux, Martin 64'

- Stieltjes for $\rho(\xi,\eta) \geq 0$ **(L=1)**

Correia, Sever, Zhiboedov 20']

- CM for $u, v > 0$, if $\int_\Delta d\xi d\eta\ \rho(\xi,\eta)\ e^{-\xi p - \eta q} \geq 0\ \forall\ p, q \geq 0$ **(L=2)**

- Remarkably the CM property also seems to hold at finite coupling.

[Alday, Armanini,Häring, Zhiboedov; 25]

From Positive geometries (oversimplified)

- The positive geometry program associates a *putative* positive geometry to certain theories and directly computes scattering amplitudes from the geometry. [Arkani-Hamed, Trnka 12', Arkani-Hamed, Bai,He,Yam 17',…]
- Many radical features:

- Locality and unitary are emergent.

- Amplitudes (for tree level) and loop Integrands (for loop level) are associated with differential forms.

- Examples:

All loop, all multiplicity —— Planar N = 4 SYM, $Tr(\Phi^3)$ theory, ABJM.

up to 1-loop, All multiplicity —— Scalar theories with colour Φ^p

Integrands as volumes and Positivity

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- For tree level amplitudes have an interpretation as a volume of the dual polytope. [Hodges 13',….]

- For loop level integrands this is a conjecture and the "dual amplituhedron" is yet to be found. [Arkani-Hamed, Hodges, Trnka 14']

- Integrands are dual volumes $\Longrightarrow$ positivity inside the geometry.

- Positivity of integrand $\rightarrow$ Integrated results ?

  Non-trivial but empirical evidence exists. [Dixon, Hippel, Mcleod, Trnka 17']

CM in projective space

- Let, V be a finite dimensional vector space

  A cone is defined a subset that satisfies $C = \{\lambda x \in C \,|\, \forall\, \lambda > 0, x \in C\}$.

  A dual cone $C^* = \{y \in V^* \,|\, \langle y, x \rangle \geq 0 \,\forall\, x \in C\}$ where $\langle y, x \rangle = \sum_i x_i\, y_i$.

- A function real valued function $f : C \to \mathbb{R}$ is CM on C if for all points in C and

$$(-1)^k D_{v_1} D_{v_2} \cdots D_{v_k} f(\mathbf{x}) \;\; \geq 0 \;\; \forall\, v_1, \cdots v_k \in C$$

  where $D_v = v \,.\, \nabla$ is the directional derivative.

Choquet's theorem

- Let, f be a real valued function then f is CM on an open cone C, then $f(\mathbf{x})$ is CM **iff**

$$f(\mathbf{x}) = \int_{C*} e^{-\langle y,x \rangle} \mu(\mathbf{y}) \quad \text{with } \mu(\mathbf{y}) \geq 0 \text{ supported on the dual cone.}$$

- When $C = \mathbb{R}^n_+ \; then \; C* = \mathbb{R}^n_+$ which is a Laplace transform.

- If $\mu(\mathbf{y})$=1 then the integral just computes the volume of $C*$.

- Happens whenever the positive geometry is a polytope [Arkani-Hamed,Bai,Lam 17']

Integrands are CM (not just positive) and admit representations as dual volumes.

[Henn, PR;24']

- Thus, every CM function in projective space is naturally a (generalised) dual volume.

6-particle BDS-like MHV Reminder function

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-6-particle MHV normalised IR finite amplitude in N=4 SYM, $\mathcal{E}(u,v,w) = \sum_{L\geq 1} \left(\frac{g_{YM}}{16\pi^2}\right)^L \mathcal{E}^{(L)}(u,v,w)$

- Bootstrapped to high loop orders [Lance's talk today]

-Evidence for positivity up to L = 4 inside tree amplituhedron [Arkani-Hamed, Hodges, Trnka 14',Dixon, Hippel, Mcleod, Trnka; 17']

$$P_{\text{MHV}} : \left\{ \begin{array}{l} u,v,w > 0,\ u+v+w < 1, \\ (u+v+w-1)^2 < 4uvw \end{array} \right\}$$

Claim: $(-1)^L\ \mathcal{E}^{(L)}(u,v,w)$ is CM for kinematics inside P_{MHV}.

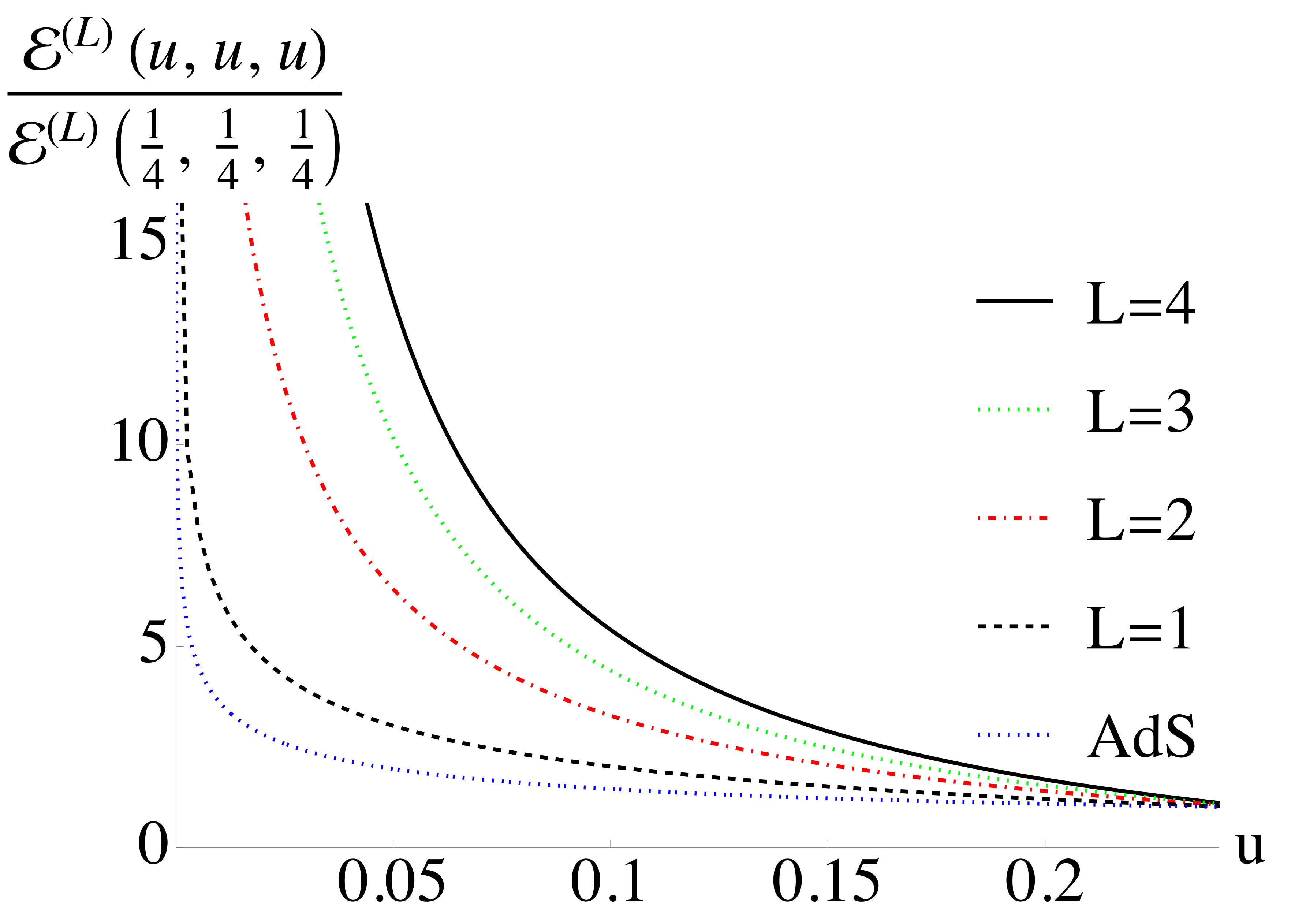
[Henn, PR; 24']

- Proof up to L = 2.

- Numerical checks for L= 3, 4.

- Strong coupling result from AdS/CFT for $u = v = w$ slice is CM.

[Alday,Giotto,Maldacena ; 09', Basso, Sever, Vieira; 14', Basso,Dixon, Papathansiou; 20]



Scalar Feynman Intgerals

- Let us consider the Feynman parametrization for a scalar Feynman graph G with L-loops in D-dimensions, in the Feynman parametrization

$$I(x) = \frac{\Gamma(\sum_i \nu_i - LD/2)}{\prod_i \Gamma(\nu_i)} \int_{\alpha_i \geq 0} \frac{\prod_i d\alpha_i \alpha_i^{\nu_i - 1}}{\mathrm{GL}(1)} \frac{U(\alpha)^{\sum_i \nu_i - (L+1)D/2}}{F(\alpha, x)^{\sum_i \nu_i - LD/2}}$$

where, U and F are the Symanzik graph polynomials and $x = \{-s_{T,R}, m_i^2\}$.

- Key point: U does not depend on external kinematics and only F depends on $\{-s_{T,R}, m_i^2\}$ but dependence is linear !

- The region $\mathbb{E} = \{x \,|\, F(\alpha, x) > 0 \;\forall\; \alpha > 0\}$ is called the Euclidean region.

- Finite Scalar Feynman integrals (without numerators) $I(x)$ satisfy

- CM for $x \in \mathbb{E}$. [Henn, PR; 24]

- Stieltjes for $0 < \sum_i \nu_i - \frac{LD}{2} \leq 1.$ [To appear Ditsch,Henn, PR]

- Both statements can be proved by arguing the same for the integrand and using the convexity properties of the space of CM, Stiletjes functions.

Applications

Martin inequalities for pion amplitudes

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- Consider, a 2-2 amplitude of identical massive scalars

$$M(s,t) = c(t) + \frac{1}{\pi}\int_{4m^2}^{\infty}\left(\frac{s^2}{s'^2(s'-s)} + \frac{u^2}{s'^2(s'-u)}\right)A(s',t)$$

dispersion + unitarity implies the following

$$\frac{d^n}{ds^n}M(s,t) > 0 \quad \text{for fixed} \quad t \quad \text{and} \quad 2m^2 - t/2 \le s \le 4m^2$$

+2 more relations due to crossing

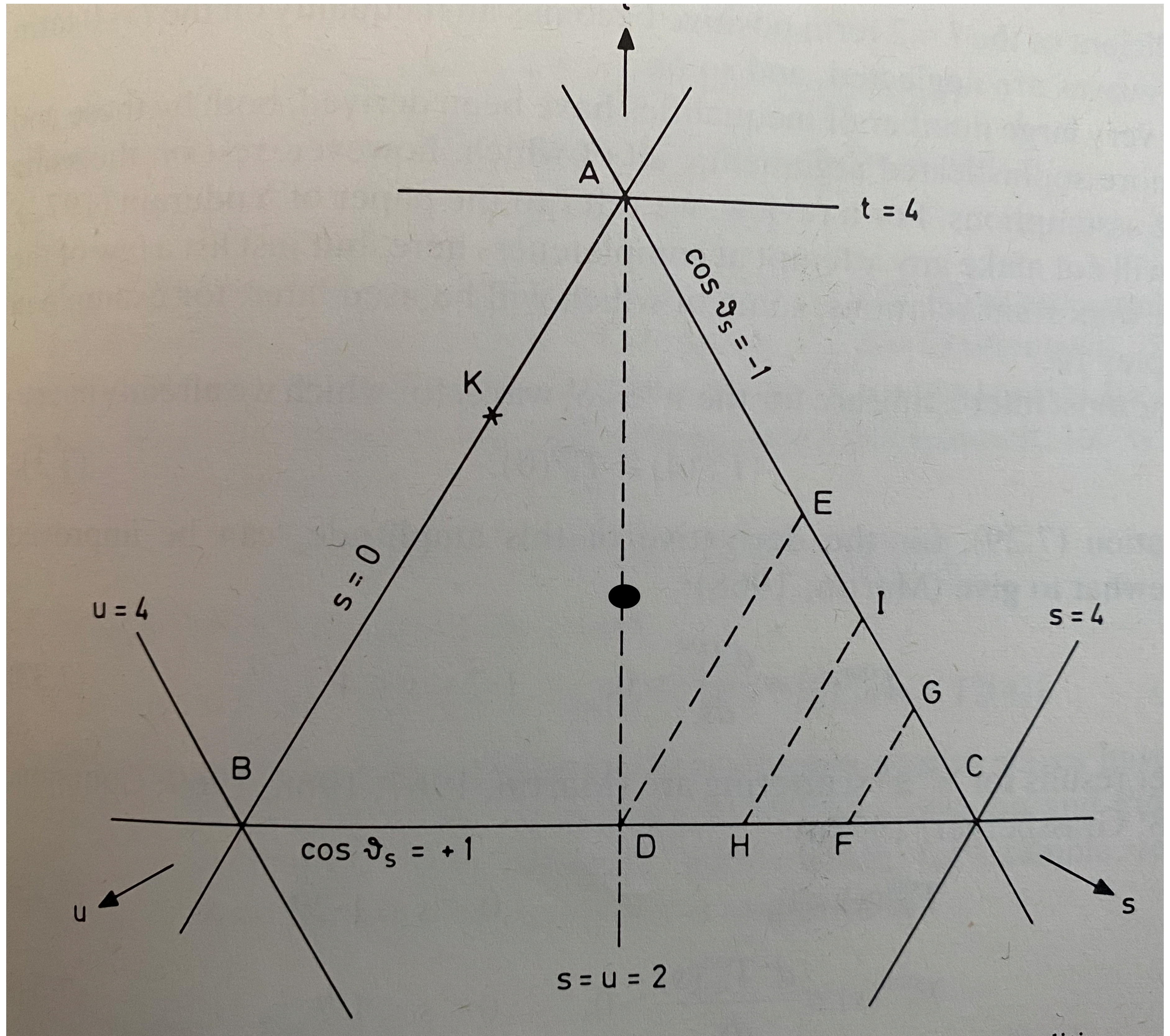

FIG. 7.1. Diagram to illustrate the derivation of some rigorous inequalities.

- Result 1: $M(s,t,u) > M(4m^2/3, 4m^2/3, 4m^2/3)$

- Result 2: Rigorous inequalities for partial waves.

- Result 3: -100< M(4/3,4/3,4/3)< 16.

[Martin 64', Martin and Cheung 67']

Numerically Bootstrapping Feynman Integrals

[To appear with Sara Ditsch and Johannes Henn]

- Feynman Integrals satisfv first order linear differential equations

$$\frac{d}{dx}\mathbf{f}(x) = A(x)\mathbf{f}(x)$$

- Using IBPs one can usually choose a CM basis $\mathbf{f}(x)$

$$(-1)^n \frac{d^n}{dx^n} f(x) \geq 0 \quad \text{for all } n \in \mathbb{N}_0, \ \forall \ x \in R$$

- Higher derivatives can be obtained recursively using the DE

$$(-1)^n \frac{d^n}{dx^n} f(x) = Q_n(x)\mathbf{f}(x)$$

Preparing the CM Bootstrap

- With the Q_n's being defined as

$$Q_0 = \mathbb{I}, \quad Q_1 = -A$$

$$Q_n = -\partial_x Q_{n-1} + Q_{n-1} \cdot Q_1$$

- The CM condition now is given by

$$Q_n(x) \,.\, \mathbf{f}(x) \geq 0 \quad \forall\ n \geq 0 \ \rightarrow \text{Linear constraints on } \mathbf{f}(x)$$

Linear Program: Maximize/Minimize $\left.\dfrac{\mathbf{f}_i(x)}{\mathbf{f}_{i_0}(x)}\right|_{x=x_0}$ subject to

$$Q_n(x) \,.\, \mathbf{f}(x) \geq 0 \text{ for } 0 \leq n \leq n_0.$$

$\mathbf{f}_{i_0}$ can be usually be chosen to be some single scale ints /tadpoles

Simple Example: The Massive Bubble Integral in D=2

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$$f(x) = \frac{2}{\sqrt{x(4+x)}} \log\left(\frac{\sqrt{1+4/x}+1}{\sqrt{1+4/x}-1}\right),$$

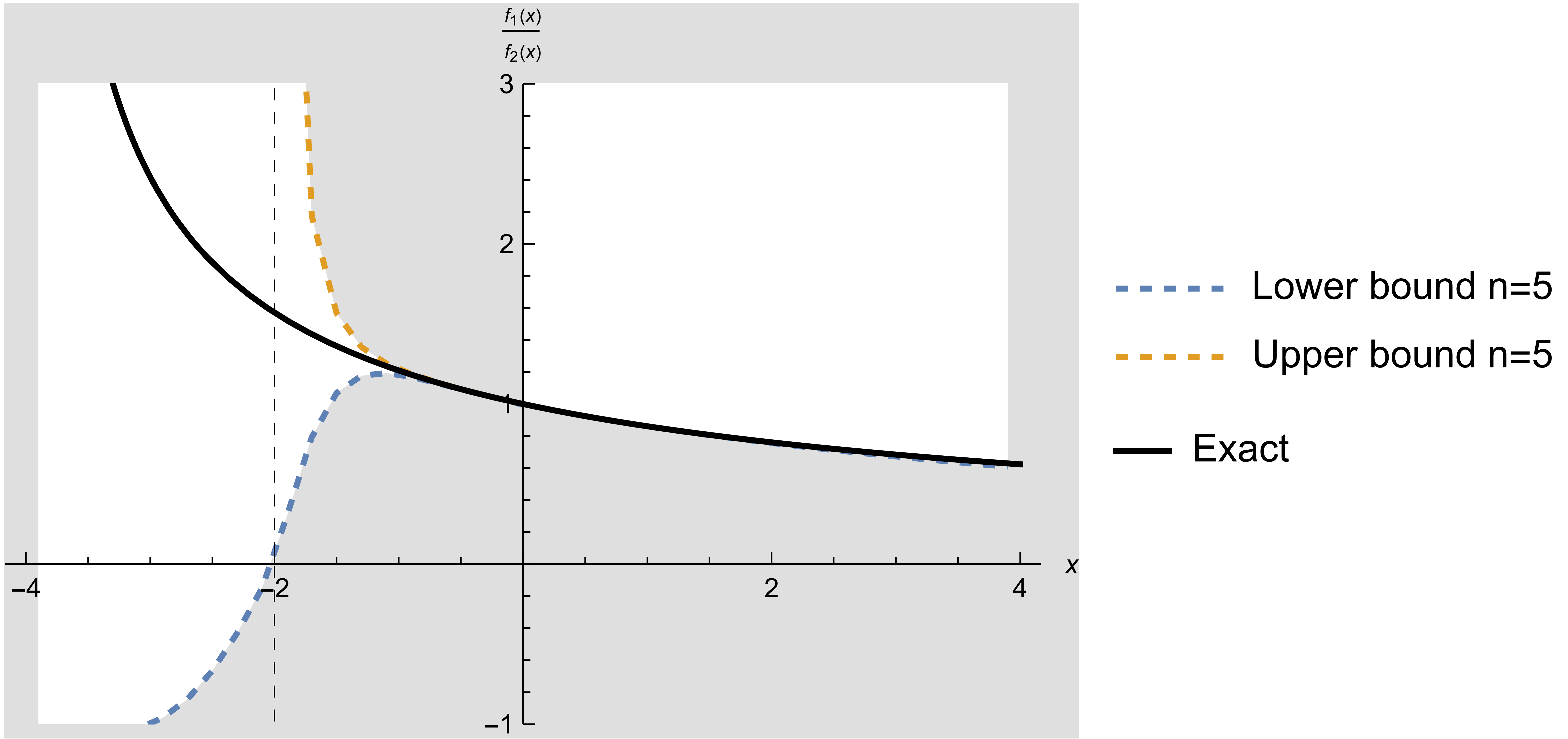
1. Obtain DE:

$$A_x = -\begin{pmatrix} 0 & 0 \\ -\frac{2}{(4+x)x} & \frac{2+x}{(4+x)x} \end{pmatrix}$$

2. Compute Derivatives recursively

3. Linear Program: fix first Integral

x=-2 x=0



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-Region 2 has very strong constraints with rapid convergence.

Padé approximations

- The Padé approximation (PA) is simple and useful alternative to polynomial approximation of analytic functions.

- Suppose, we are given a Taylor expansion of a function convergent in $|z| \leq R$,

$$f(z) = a_0 + a_1(z - x_0) + a_2(z - x_0)^2 + \cdots a_n(z - x_0)^K$$

- Find a rational function $P^N_M(z; x_0)$ with numerator degree N and denominator degree M that agrees with the truncated Taylor expansion $N + M \leq K$ i.e.,

$$P^N_M(z; x_0) \equiv \frac{P_N(z)}{Q_M(z)} = f(z) + O\left(x^{N+M+1}\right),$$

- Very useful for applications, but still not understood why they work for generic functions.

Stieltjes functions and Padé approximations

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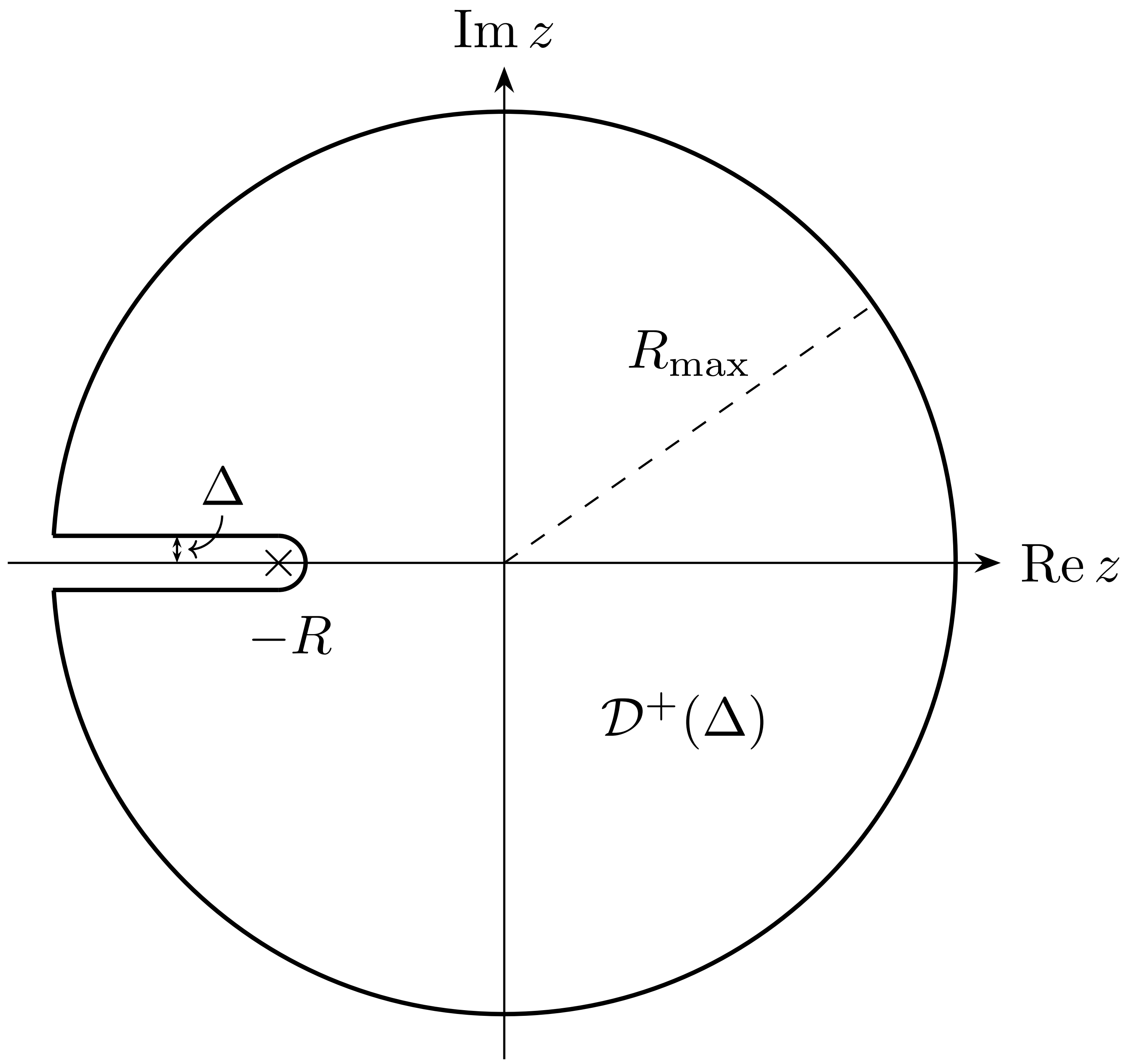
- The Padé approximants of Stieltjes functions:
- **Convergence on the real axis:** $P_N^{N-1}(x;x_0) \leq f(x) \leq P_N^N(x;x_0), \quad x \geq x_0\,.$
- **Convergence in the cut plane:** The Padé approximants $P_N^{N-1}(x;x_0)$ and $P_N^N(x;x_0)$ both converge in the cut plane to the function $f(z)$.

- **Error bounds:** For any $z \in \mathscr{D}_+$ and $\forall\, J \geq -1, M \geq 1$,

$$|f(z) - P_M^{M+J}(z;x_0)| < c\left|\frac{(z-x_0)}{\rho}\right|^{J+1}\left|\frac{\sqrt{\rho+z-x_0}-\sqrt{\rho}}{\sqrt{\rho+z-x_0}+\sqrt{\rho}}\right|^{2M},$$

where, $\rho = R + x_0 - \Delta$ and c is a constant.



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Stieltjes bootstrap

Idea: Combine results from CM bootstrap with Padé, (guaranteed to work well for Stieltjes functions)

-**Step 1: Select a starting point:** In the subset of the Euclidean region where CM bootstrap works very well.

-**Step 2: Compute the basis integrals at the starting point:** Using the CM bootstrap.

-**Step 3: Compute the Taylor expansion:** To a given order with the required precision.

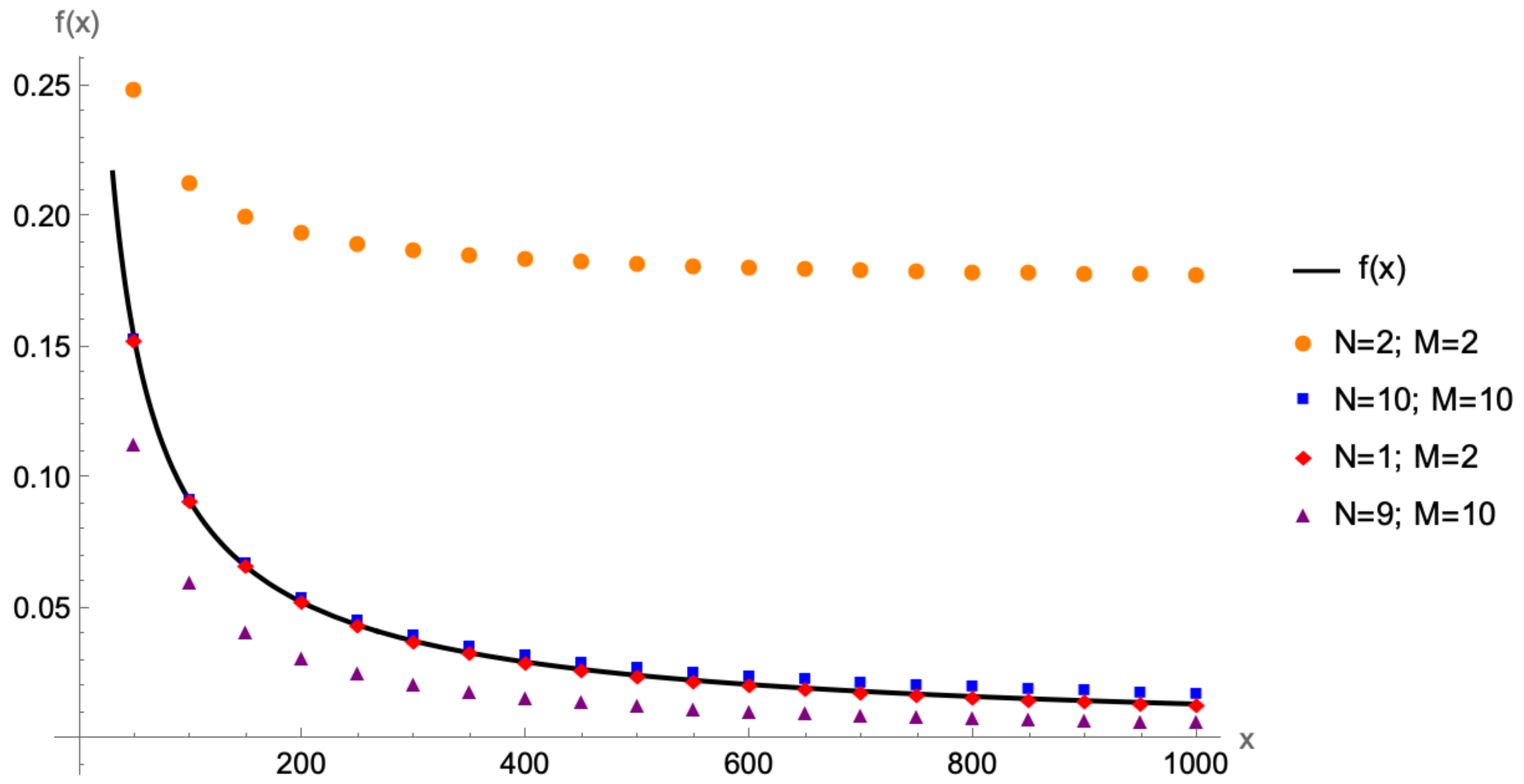
-**Step 4: Construct the Padé approximants and store them.**

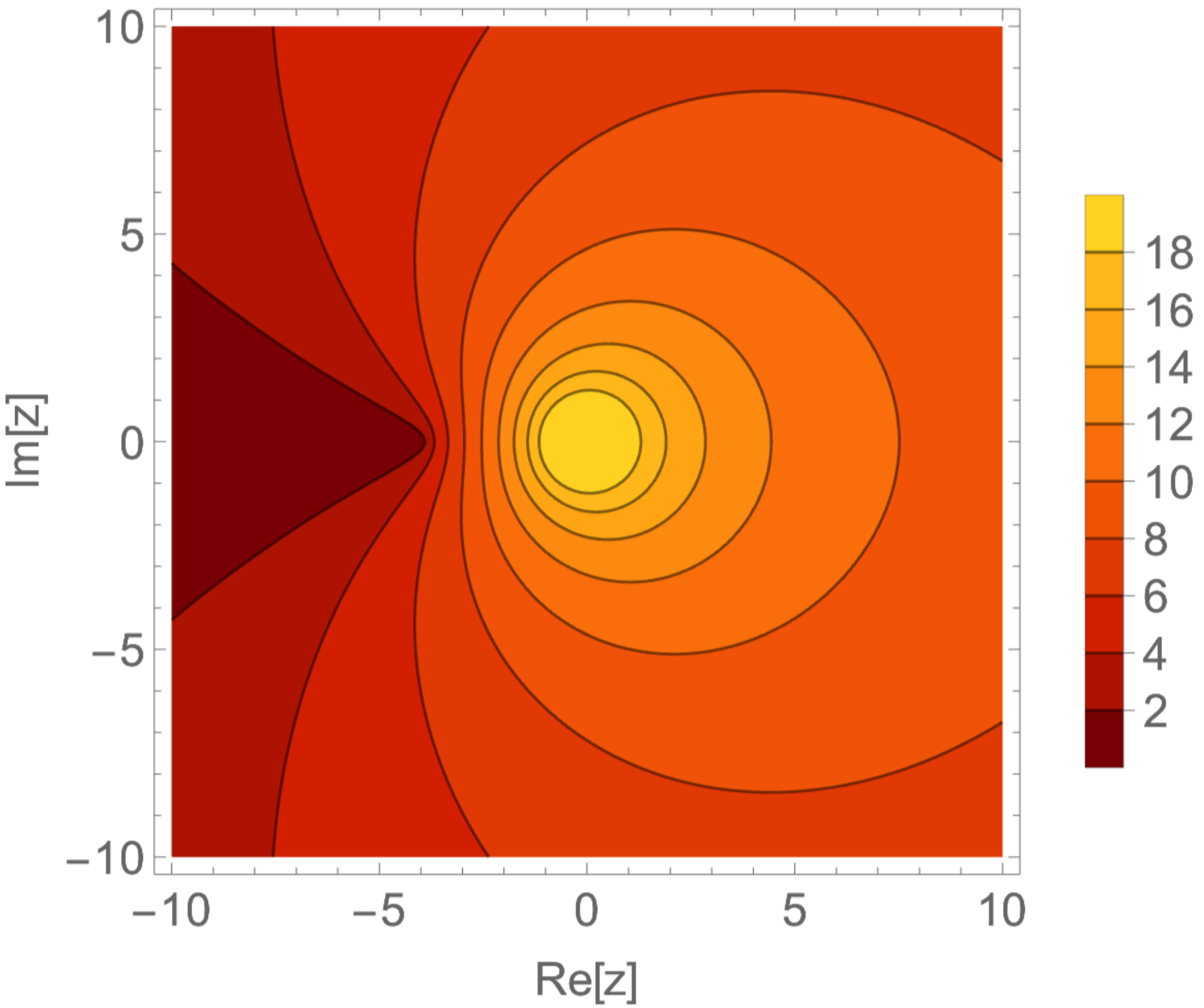
-**Step 5: Evaluate the Padé approximants.**

- As a proof of principle we can apply it Sunset/Banana integrals, which are Stieltjes in D=2.

One-loop massive bubble example 2.0

-Comparison between Padé approximants $P^{9}_{10}(z;-0.1), P^{10}_{10}(z;-0.1)$ and the actual function for real and complex points.



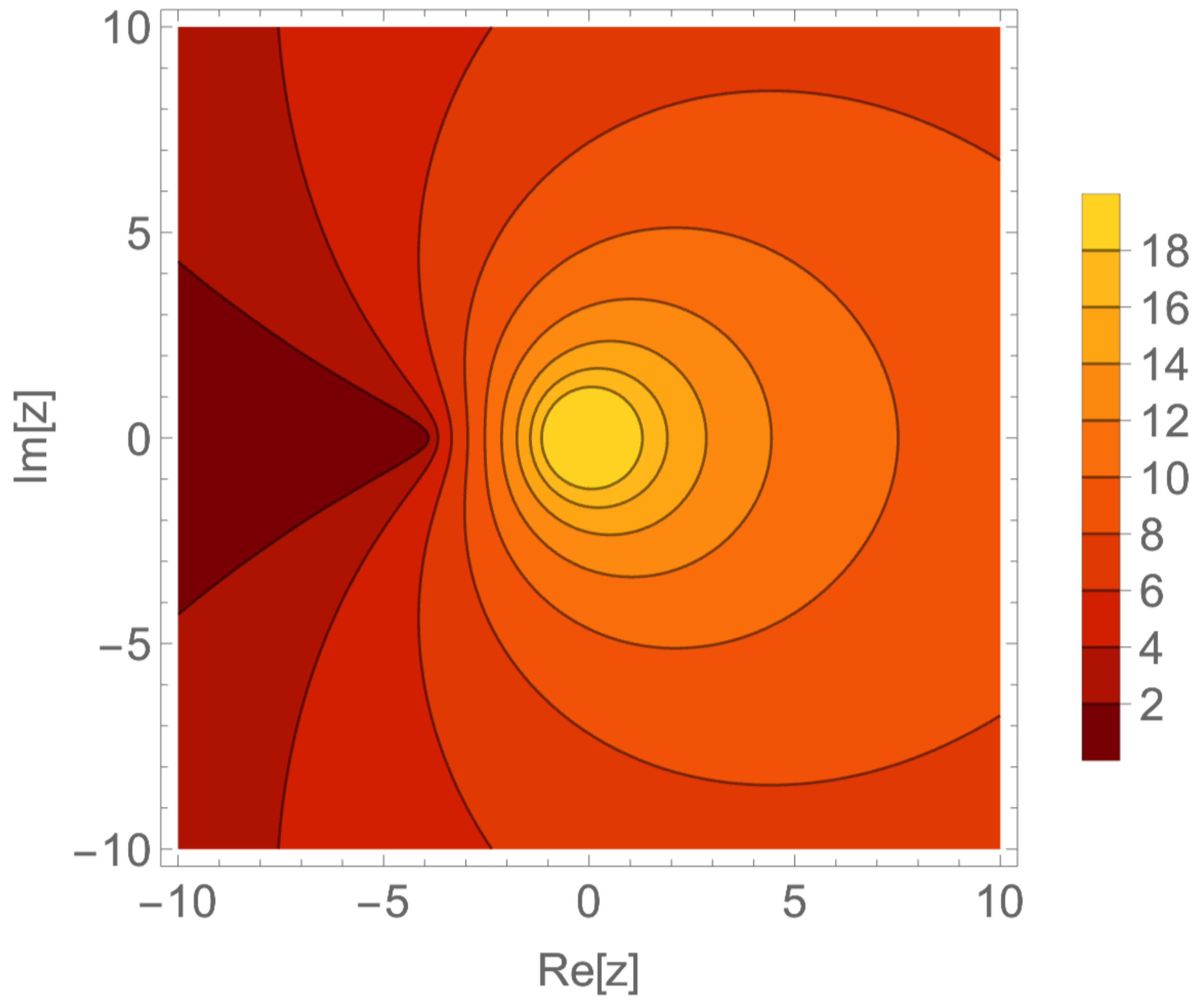


Example without using DE

- The interplay between Padé and Stieltjes functions does not depend on the knowledge of a DE, provided we can obtain the Taylor coefficients using another method.

- We can do this for L-**loop banana type integrals** using representation as a Bessel integral valid in the Euclidean region I.e., for $x \geq -(L+1)^2$ and computing the Bessel moments numerically. [Groote,Korner, Pivovarov 05'; Vanhove 14']

z	Upper bound	Actual value	Lower bound
10^3	1.53585×10^{13}	1.53585×10^{13}	1.53585×10^{13}
10^4	1.53489×10^{13}	1.53489×10^{13}	1.53489×10^{13}
10^5	1.52272×10^{13}	1.52272×10^{13}	1.52272×10^{13}



# of digits agreement Between the Padé

Conclusion and Outlook

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- We looked a novel kind of positivity properties on a function and all it derivatives called complete monotonicity and Stieltjes property which several physical quantities and building blocks (like generic scalar Feynman integrals) satisfy in the ``Euclidean region”.

- We discussed why these functions are natural from the perspective of volumes and in the positive geometry program and also saw it can be used for efficiently for numerical bootstrap.
- These properties also holds seems to hold in the coupling for some exact observables (admit Fredholm determinants) [Talk to Maximilian Haensch]

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