The use of conformal Regge theory for high-order perturbative calculations

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based on [2506.05132]
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What do we want to compute?

- Twist-2 flavor nonsinglet operators in QCD (i, j – different flavors) [Gross, Wilczek'1973]

$$\mathcal{O}^{q}_{\mu_1 \ldots \mu_s} = \bar{q}_i \gamma_{(\mu_1} D_{\mu_2} \ldots D_{\mu_s)} q_j - \text{traces}$$

  - Canonical scaling dimension $\Delta_0(s) = 3 + (s-1)$
  - Spin s
  - Twist ($=$ scaling dimension $-$ spin) $\tau = 2$

- RG-equation ($a = \alpha_s / 4\pi$)

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} + \gamma(s) \right) \left[\mathcal{O}^{q}_{\mu_1 \ldots \mu_s} \right]_R = 0$$

- Goal $\longrightarrow$ perturbative calculations

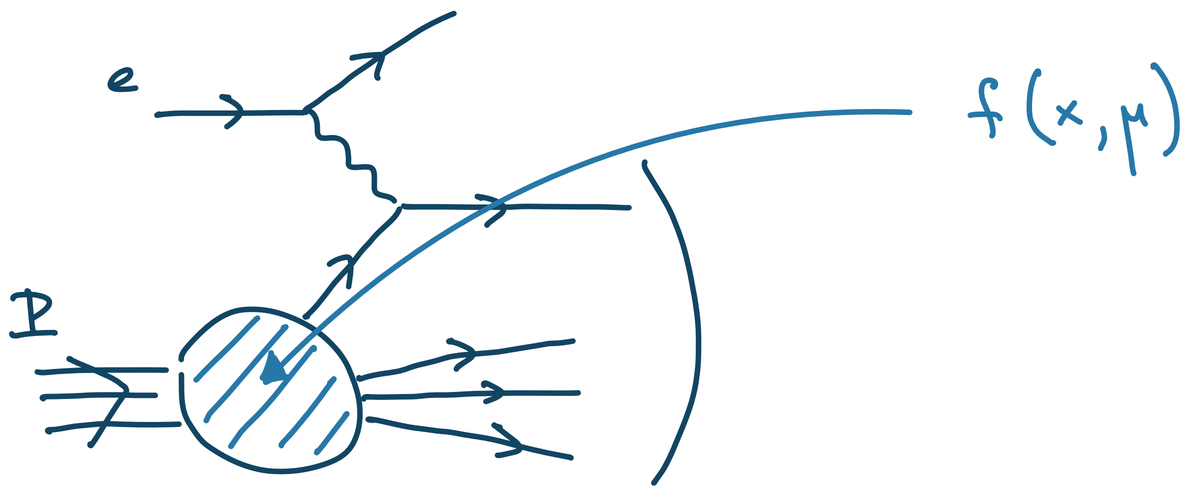
$$\gamma(s) = a \gamma^{(1)}(s) + a^2 \gamma^{(2)}(s) + a^3 \gamma^{(3)}(s) + \ldots$$

Relevance for the phenomenology

- Splitting functions (DGLAP kernels)

$$\gamma(s) = -\int_0^1 dx\, x^{s-1} P(x)$$

- Parton distribution functions [Feynman'1969] [Björken, Pachos'1969]



- PDF scale dependence [Gribov, Lipatov'1972] [Altareli, Parisi'1977] [Dokshitzer'1977]

$$\frac{d}{d\ln\mu} f(x,\mu) = \int_x^1 \frac{dy}{y} P(y)\, f\left(\frac{x}{y},\mu\right)$$

What do we have?

- Results in perturbative calculations

3 loops
  - All-s three-loop result in the singlet case [Moch, Vermaseren, Vogt' 2004]
  - All-s four-loop result for planar limit in the non-singlet case [Moch, Ruijl, Ueda, Vermaseren, Vogt' 2017]

4 loops
  - $s \lesssim 20$ four-loop result in the singlet case [Falconi, Herzog, Moch, Pelloni, Vogt' 2023-2024]
  - All-s ζ_3-contribution to four-loop result in the non-singlet case [Kniehl, Velizhanin' 2025]
  - All-s n_f-contribution to four-loop result in the non-singlet case [Kniehl, Moch, Velizhanin, Vogt' 2025]

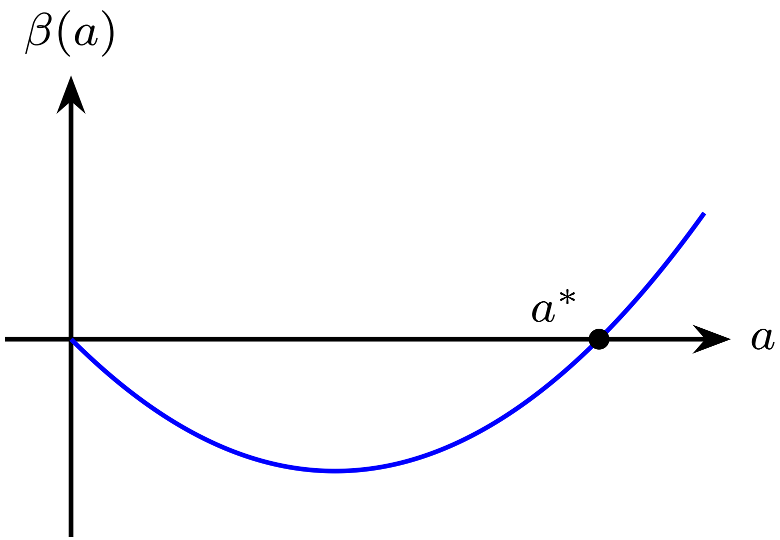
- Needs for collider phenomenology

Four loops $\longleftrightarrow$ $\sim 1\%$ precision

High-Luminocity LHC

Why CFT?

- Consider **critical point** $(d = 4 - 2\epsilon)$



$$\boxed{\beta(a^*) = 0 \quad \Longrightarrow \quad a^* = a^*(\epsilon)}$$

  - MS-like scheme

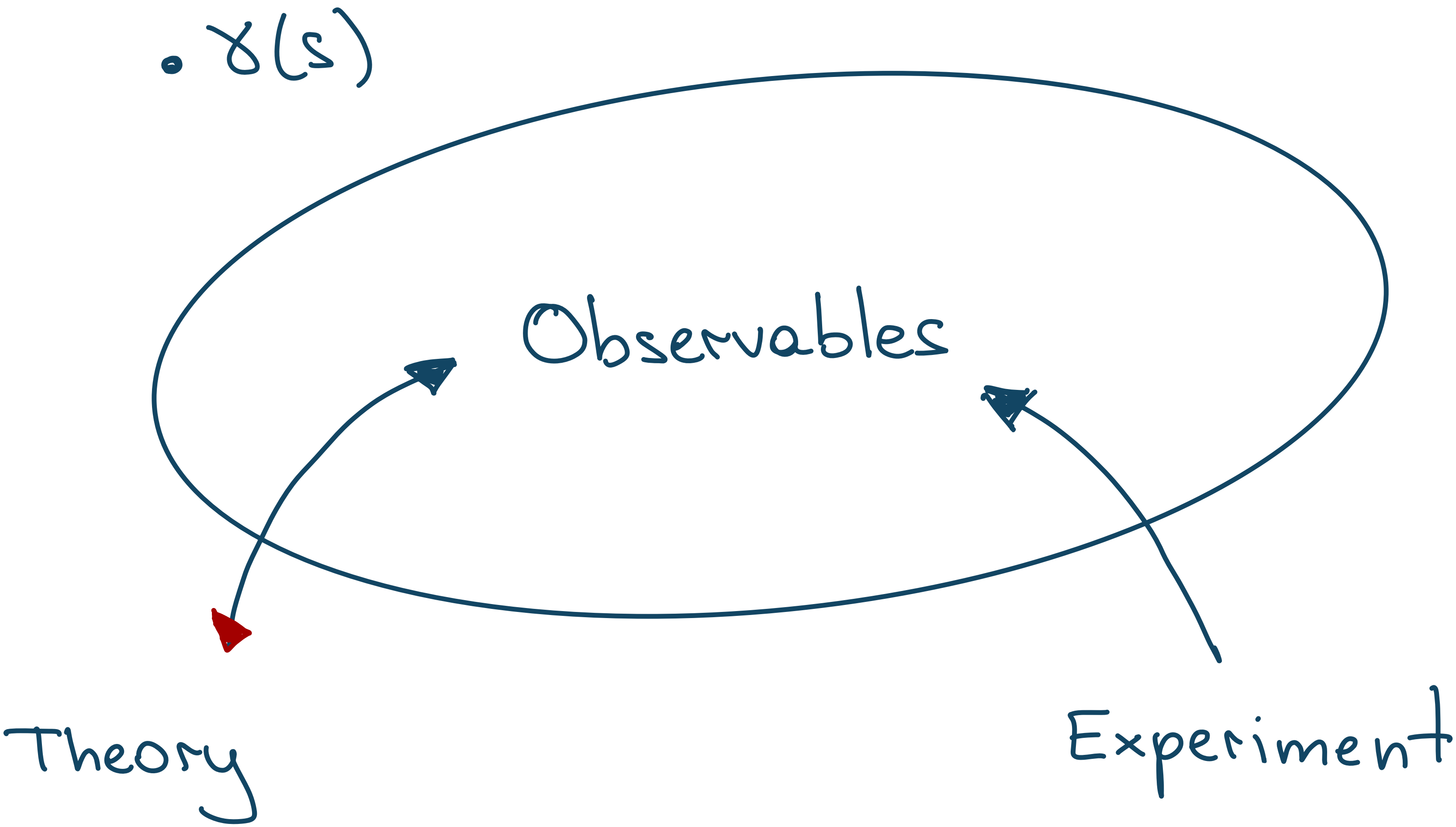
$$\gamma(s) = a\gamma^{(1)}(s) + a^2\gamma^{(2)}(s) + \ldots$$

  - Critical value

$$\gamma^*(s) = a^*\gamma^{(1)}(s) + (a^*)^2\gamma^{(2)}(s) + \ldots$$

- Anomalous dimensions in CFTs
  - $\gamma(s) \subset \{\text{CFT data}\}$
  - "Initial data" for numerical conformal bootstrap [Rattazzi, Rychkov, Tonni, Vichi'2008]

How does conformal invariance help?



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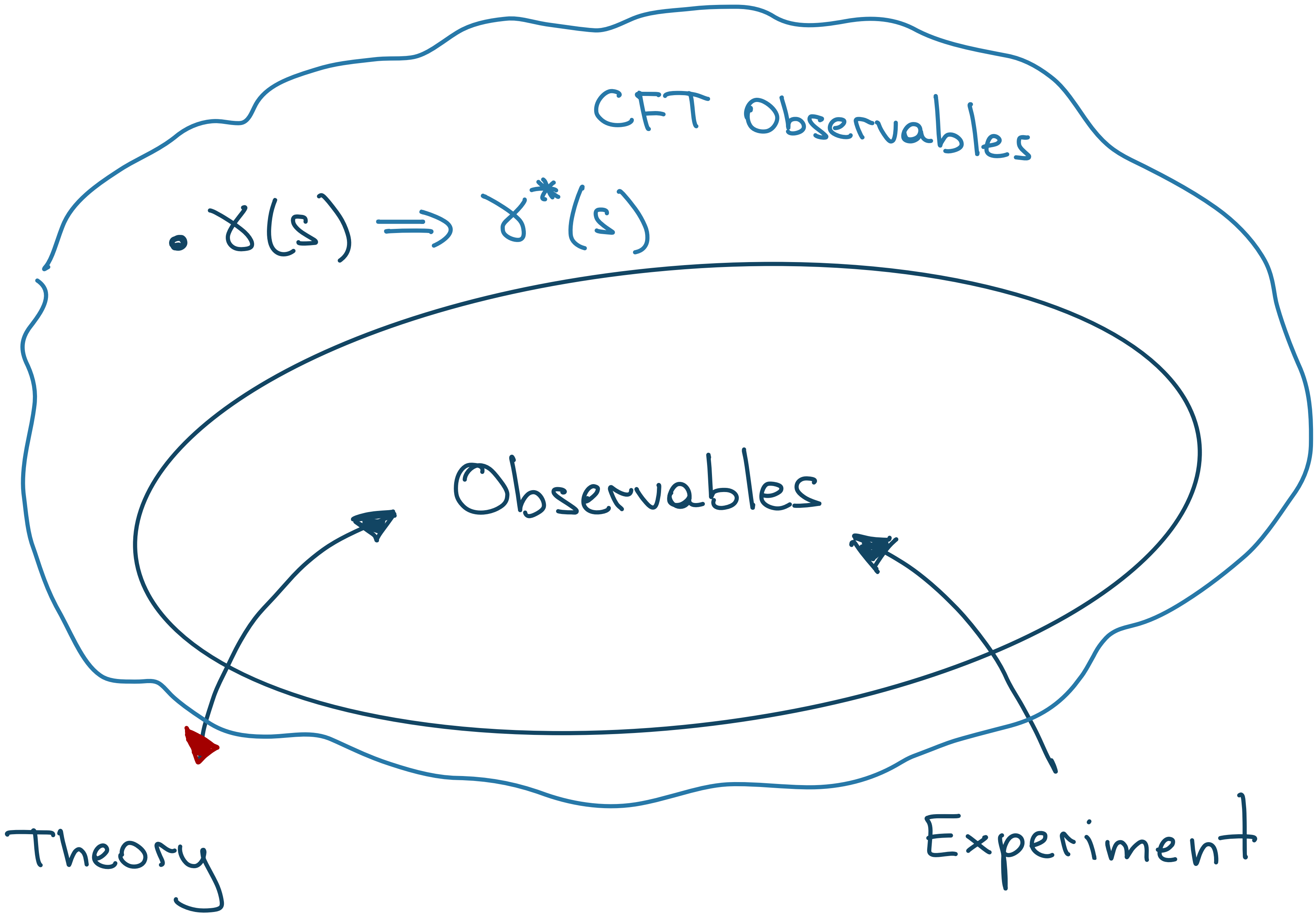
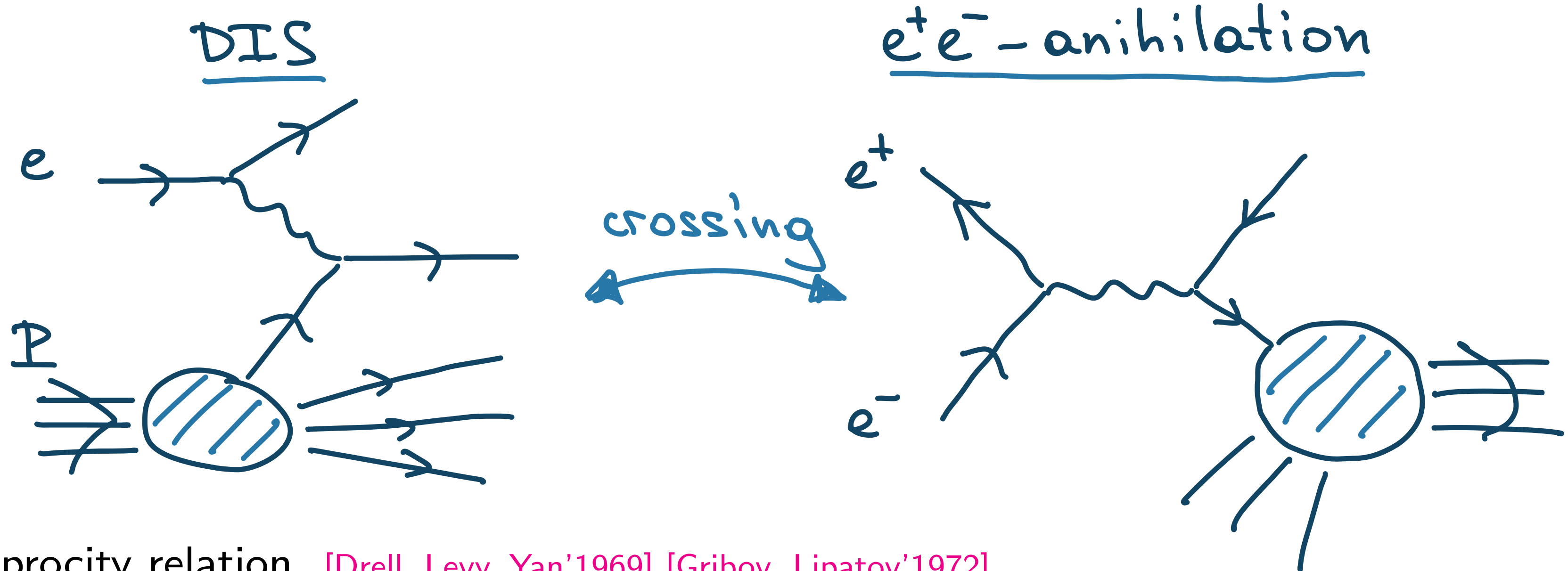




- Reciprocity relation [Drell, Levy, Yan'1969] [Gribov, Lipatov'1972]

$$P(x) = -\frac{1}{x} P\left(\frac{1}{x}\right) \quad \longrightarrow \quad \gamma(s) = \gamma(-s-1)$$

- **Generalized** Gribov-Lipatov reciprocity [Basso, Korchemsky'2007] [Dokshitzer, Marchesini'2007]

$$f^{(\ell)}(j) = 2\Gamma^{(\ell)}_{\text{cusp}} \ln j + \frac{\#}{j(j+1)} + \frac{\#}{j^2(j+1)^2} + O\left(\frac{1}{j^3(j+1)^3}\right)$$

  - $\gamma(s) = f\underbrace{\left(s + \bar{\beta}(a) + \tfrac{1}{2}\gamma(s)\right)}_{=j\text{ -- conformal spin}}$
  - $s(s+1)$ — Casimir of the collinear conformal $(SL(2,\mathbb{R}))$ group

Reciprocity form the CFT side

- Leading-twist operators $\subset$ OPE

$$\Phi(x)\Phi(0) = \sum_s C_{\tau_0,s}|x|^{\tau_0-2\Delta_\Phi} x^{\mu_1\cdots\mu_s}\mathcal{O}_{\mu_1\ldots\mu_s}(0) + \text{(higher twists)}$$

- Consider four-point correlator

$$\Big\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\Big\rangle = \frac{G(u,v)}{x_{12}^{2\Delta_\Phi}x_{34}^{2\Delta_\Phi}}$$

- Conformal partial waves expansion

$$G(u,v)\Big|_{\text{leading twist } \tau_0}$$

s-channel

t-channel

$$\sum_s a_s u^{\tau_0/2+\gamma(s)/2}(1-v)^s(C_s \ln v + \ldots) \qquad\qquad u^{\tau_0/2}v^{-\tau_0/2}$$

- Power-like singularity is controlled by $\lim_{s\to\infty}\gamma(s)$ [Alday, Biss, Łukowski'2015]

$O(N)$-symmetric φ^4 model

- φ^4 model with the N-component scalar field φ^a

$$S = \int d^dx \left((\partial\varphi)^2 + \frac{g}{4!}(\varphi^2)^2 \right)$$

- Singlet twist-2 operators

$$\mathcal{O}_s(x) = \sum_{a=1}^{N} \varphi^a(x)\partial_{\mu_1}\dots\partial_{\mu_s}\varphi^a(x) - \text{traces}$$

- Twist-2 anomalous dimensions are known up to four loops $(u = g/(4\pi)^2)$ [Derkachov, Gracey, Manashov'1998] [Manashov, Skvortsov, Strohmaier'2017]

$$\gamma(s) = 2\gamma_\varphi - u^2 \frac{N+2}{9s(s+1)}\left(3 + \frac{u}{3}(N+8)\left(S_1(s) - 5 + \frac{3}{2}\frac{2s+1}{s(s+1)}\right)\right) + O(u^4)$$

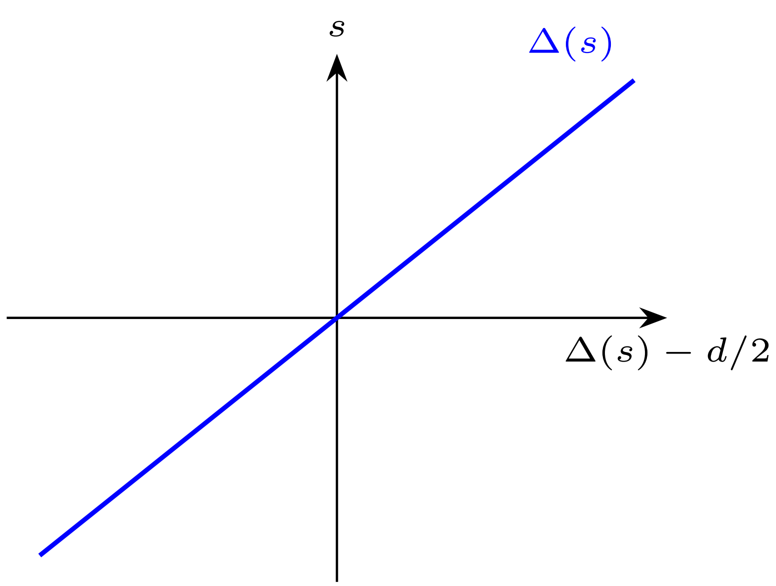
- Evident problem
  - Operator $\mathcal{O}_{s=0} = \varphi^2$ exists
  - Anomalous dimension $\gamma_{\varphi^2} = u\frac{N+2}{3}\left(1 - \frac{5u}{6} + u^2\frac{37+5N}{12}\right) + O(u^4)$
  - $\lim_{s\to 0}\gamma(s) = \infty \neq \gamma_{\varphi^2}$

Analytic continuation

- Continuous limit $s \to 0 \quad \Longrightarrow \quad$ **Analytic continuation** in s

$$s \in \mathbb{N}_0 \quad \longrightarrow \quad s \in \mathbb{C} \backslash \{0, -1, -2, \ldots\}$$

- Analytic continuation in CFTs $(\Delta(s) = d - 2 + s + \gamma(s))$
  - $\Delta(s)$ determines the Regge limit of 4pt functions [Costa, Goncalves, Penedones'2012]
  - Analyticity in spin [Caron-Huot'2017]
  - Analytic continuation on the operator level [Kravchuk, Simmons-Duffin'2018]
- Chew-Frautschi plot around $s = 0$: [Caron-Huot, Koloğlu, Kravchuk, Meltzer, Simmons-Duffin'2022]



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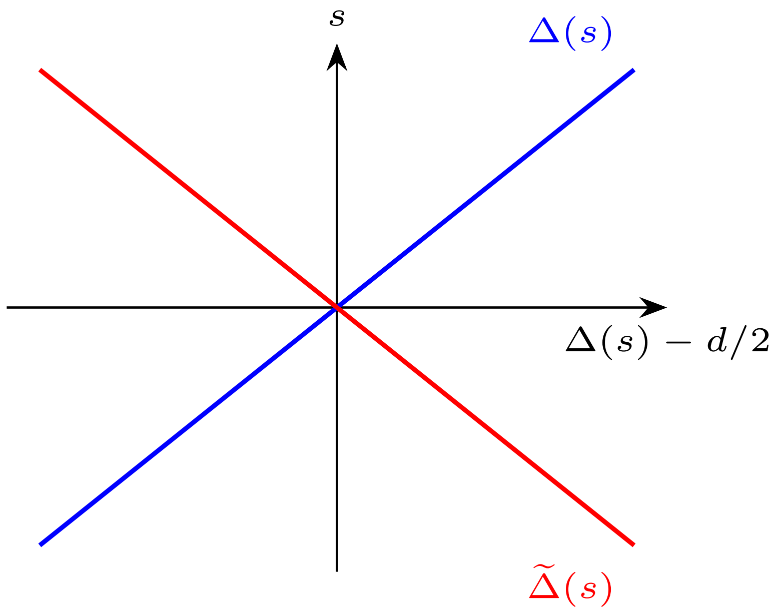

  - Shadow trajectories $\mathbf{S}_J[\mathbb{O}](x, z)$ with $\widetilde{\Delta}(s) = d - \Delta(s)$

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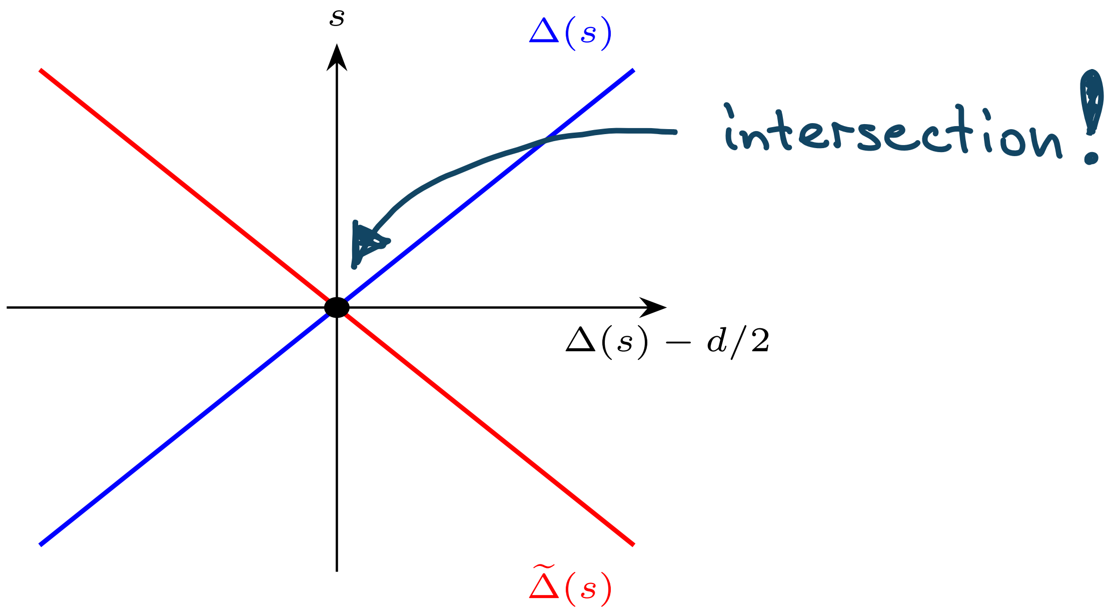
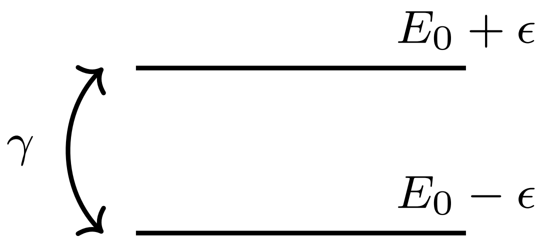


  - Shadow trajectories $\mathbf{S}_J[\mathbb{O}](x, z)$ with $\widetilde{\Delta}(s) = d - \Delta(s)$

No crossing

- Can trajectories cross?
  - **Highly unlikely** in QM [von Neumann, Wigner'1929]
  - **Highly unlikely** in CFT [Korchemsky'2015]
- Hamiltonian of the two-state system

$$H = \begin{pmatrix} E_0 + \epsilon & \gamma \\ \gamma^* & E_0 - \epsilon \end{pmatrix}$$



- Exact solution of the spectral problem

$$E_\pm(\epsilon) = E_0 \pm \sqrt{\epsilon^2 + |\gamma|^2}$$

- Perturbative expansion $|\gamma|^2 \ll \epsilon^2$

$$\begin{cases} E_+ = \underbrace{E_0 + \epsilon}_{\text{higher level}} + \dfrac{|\gamma|^2}{2\epsilon} - \dfrac{|\gamma|^4}{8\epsilon^3} + O(|\gamma|^6) \\ E_- = \underbrace{E_0 - \epsilon}_{\text{lower level}} - \dfrac{|\gamma|^2}{2\epsilon} + \dfrac{|\gamma|^4}{8\epsilon^3} + O(|\gamma|^6) \end{cases}$$

  - "Shadow" symmetry $\epsilon \mapsto -\epsilon$

No crossing

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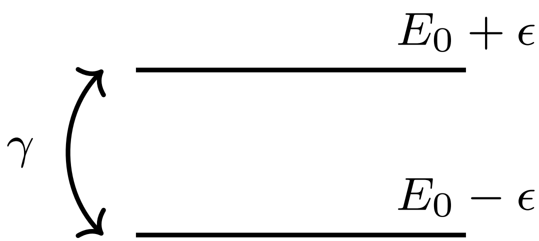
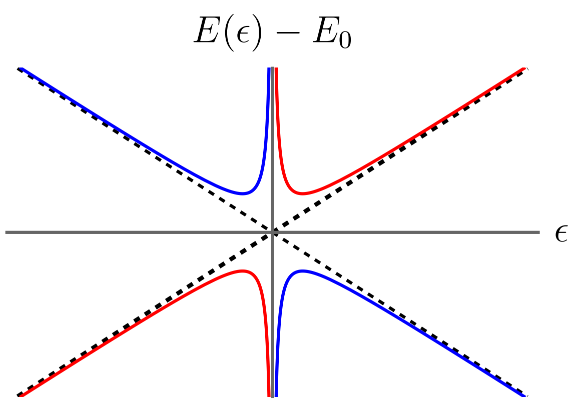


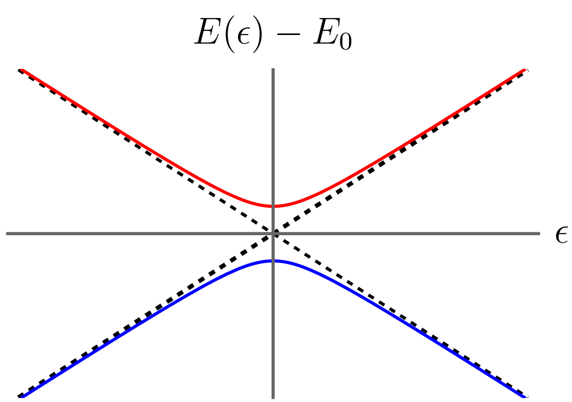
- Exact solution of the spectral problem

$$E_\pm(\epsilon) = E_0 \pm \sqrt{\epsilon^2 + |\gamma|^2}$$

- Plots of perturbative and exact solutions



perturbative



exact

Life around intersection

- Return to twist-2 operators in **Wilson-Fisher** CFT
- What do we expect from the function $\Delta(s)$
  - Regular around $s = 0$
  - Goes to $\Delta(s)$ away from $s = 0$ at one branch
  - Goes to $\widetilde{\Delta}(s)$ away from $s = 0$ at another branch

- Defining equation

$$x^2 - \left(\Delta(s) + \widetilde{\Delta}(s)\right)x + \Delta(s) \cdot \widetilde{\Delta}(s) = 0$$

  - $\Delta(s) + \widetilde{\Delta}(s)$ — regular at $s = 0$
  - $\Delta(s) \cdot \widetilde{\Delta}(s)$ — regular at $s = 0$

- Omitting a priori regular parts

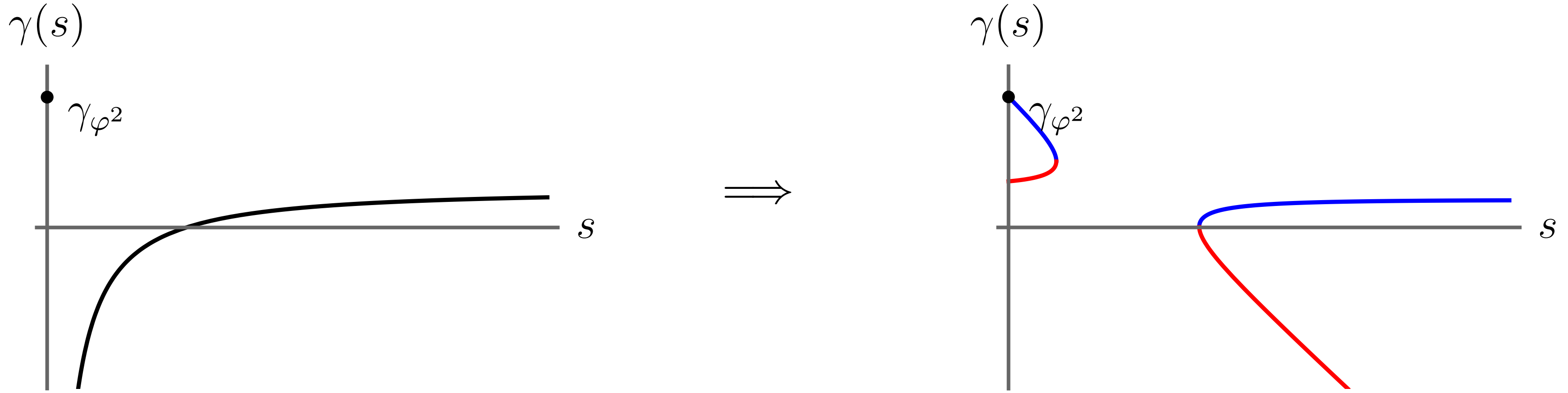
$$\delta m_*^2(s) = 2\gamma^*(s)\left(s - \epsilon + \frac{1}{2}\gamma^*(s)\right)$$

- Back to the physical coupling $\left(\beta(u) = -2u\left(\epsilon + \bar{\beta}(u)\right)\right)$

$$\epsilon = -\bar{\beta}(u^*) \quad \longrightarrow \quad \text{recipe for resummation of singularities}$$

- Resummation around $s = 0$

$$\gamma(s) = -s - \bar{\beta}(u) + \sqrt{(s + \bar{\beta}(u))^2 + \delta m^2(s)}$$



- Up to four-loops for $N = 1$ [Caron-Huot, Koloğlu, Kravchuk, Meltzer, Simmons-Duffin'2022]
- Up to four-loops for all types of operators for arbitrary N [Manashov, Moch, **LS**'2025]

- Large-s asymptotics of the 4pt QCD amplitudes via Bethe-Salpeter equation [Kirschner, Lipatov'1982-1983]
- Double-logaritmic equation (DLE) in QCD [Ermoalev, Manaenkov, Ryskin'1996] [Blümlein, Vogt'1996]

$$2\gamma_{\mathsf{ns}}(s)\left(s+\frac{1}{2}\gamma_{\mathsf{ns}}(s)\right)=-8C_F a$$

  - Resum $(a\ln^2 x)^\ell$ terms in splitting functions
- Generalized DLE in $\mathcal{N}=4$ SYM [Velizhanin'2011]

$$2\gamma_{\mathcal{N}=4}(s)\left(s+\frac{1}{2}\gamma_{\mathcal{N}=4}(s)\right)=\underbrace{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}C_m^k a^k s^m}_{\text{regular in } s}.$$

  - Valid in **planar limit** up to seven loops and in **non-planar** sector up to three loops
- Generalized DLE in QCD [Velizhanin'2017]

$$2\gamma_{\mathsf{ns}}(s)\left(s+\bar{\beta}(a)+\frac{1}{2}\gamma_{\mathsf{ns}}(s)\right)=\underbrace{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}C_m^k a^k s^m}_{\text{regular in } s}$$

  - Valid in **planar limit** up to four loops and in **non-planar** sector up to two loops

Intersection of communities (why δm^2?)

- Holographic duality [Polyakov, Klebanov'2002]

$$\text{Critical } O(N)\text{-symmetric } \varphi^4 \text{ in } d=3 \quad \longleftrightarrow \quad \text{Higher-spin theory on AdS}_4$$

- Masses of the higher-spin fields on AdS_4 [Girardello, Porati, Zaffaroni'2003]

$$m^2(s) = \Delta(s)\big(\Delta(s) - d\big) - s$$

  - $\Delta(s)$ — scaling dimensions of the twist-2 operators ($=$ higher-spin currents)

- Radiative corrections to the mass

$$m^2(s) = (d-2+s)(s-2) - s + \underbrace{2\gamma(s)\left(s + \frac{d}{2} - 2 + \frac{1}{2}\gamma(s)\right)}_{=\delta m^2(s)}$$

  - Radiative corrections to the scalar field ($s=0$) are finite

From the technical point of view

- We can predict singular structure at higher-loop orders:
  - $\delta m^2(s)$ is regular around $s = 0$
  - $\gamma(s)\big|_{\text{1st sheet}} \xrightarrow{s\to 0} \gamma_{\varphi^2}$
- Singular structure of five-loop anomalous dimension [Manashov, Moch, **LS**'2025]

$$\gamma^{(5)}(s) = (N+2)\Bigg\{-\frac{(N+8)(N^2+34N+100)}{648s^4} + \frac{(N+8)(5N^2+236N+776)}{972s^3}$$
$$-\left(\frac{(14+N)(22+5N)}{81}\zeta_3 + \frac{(8+N)(N^2+28N+88)}{486}\zeta_2\right.$$
$$\left.+\frac{16N^3+2067N^2+24270N+63152}{3888}\right)\frac{1}{s^2} + \left(\frac{40N^2+1100N+3720}{243}\zeta_5\right.$$
$$+\frac{(32+N)(22+5N)}{810}\zeta_2^2 - \frac{5N^3-365N^2-5278N-16880}{972}\zeta_3$$
$$\left.+\frac{(N+8)(N+26)(8N+35)}{972}\zeta_2 + \frac{371N^2+6286N+18192}{648}\right)\frac{1}{s}\Bigg\} + O(s^0).$$

  - $N = 1$ case [Caron-Huot, Koloğlu, Kravchuk, Meltzer, Simmons-Duffin'2022]

- N-component fermion field $q_i(x)$ with $SU(N)$ flavor group and scalar field $\sigma(x)$.
  - Gross-Neveu-Yukawa model

$$S_{\text{GNY}} = \int d^dx \left(\bar{q}\not{\partial}q + \frac{1}{2}(\partial\sigma)^2 + g\bar{q}\sigma q + \frac{\lambda}{4!}\sigma^4 \right)$$

  - Gross-Neveu model

$$S_{\text{GN}} = \int d^dx \left(\bar{q}\not{\partial}q + \sigma\bar{q}q - \frac{N}{2g}\sigma^2 \right)$$

- GNY and GN are **critically** equivalent [Zinn-Justin'1991].
- Leading twist (singlet) operators

$$\mathcal{O}^q(s) = -s\sum_{i=1}^{N} \bar{q}_i(x)\gamma_{(\mu_1}\partial_{\mu_2}\dots\partial_{\mu_s)}q_i(x)$$
$$\mathcal{O}^\sigma(s) = \sigma(x)\partial_{\mu_1}\dots\partial_{\mu_s}\sigma(x)$$

- Anomalous dimensions

$$\gamma_{\text{GNY}}(s) = \underbrace{\gamma^{(1)}_{\text{GNY}}(s)\epsilon + \gamma^{(2)}_{\text{GNY}}(s)\epsilon^2}_{\text{[Manashov, Moch, LS'2025]}} + O(\epsilon^3)$$

$$\gamma_{\text{GN}}(s) = \gamma^{(1)}_{\text{GN}}(s)\frac{1}{N} + \gamma^{(2)}_{\text{GN}}(s)\frac{1}{N^2} + O\left(\frac{1}{N^3}\right)$$

Regge trajectories in both models

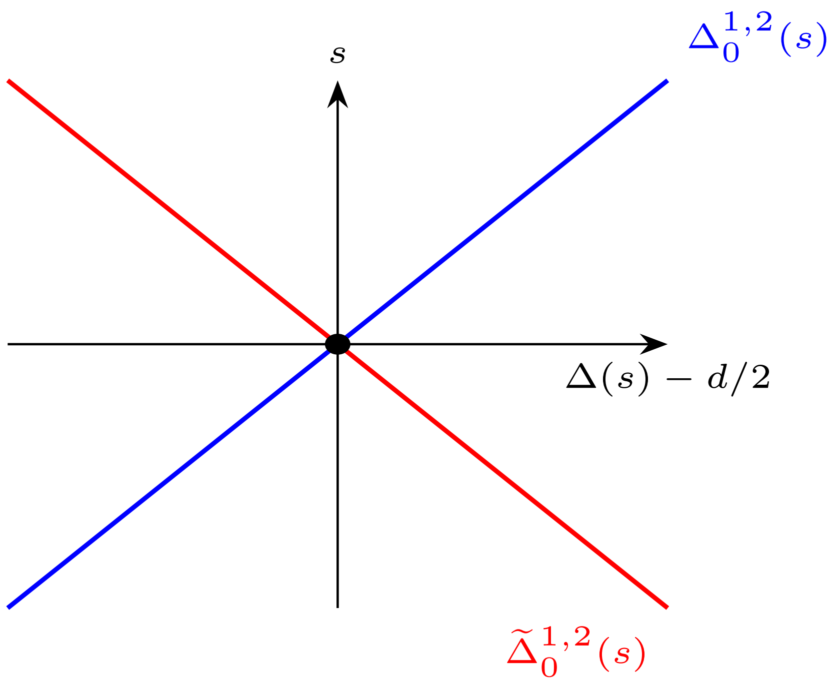
- Different scaling dimensions
  - Gross-Neveu-Yukawa model

$$\Delta_q = \frac{d-1}{2} \qquad \Delta_\sigma = \frac{d}{2} - 1$$

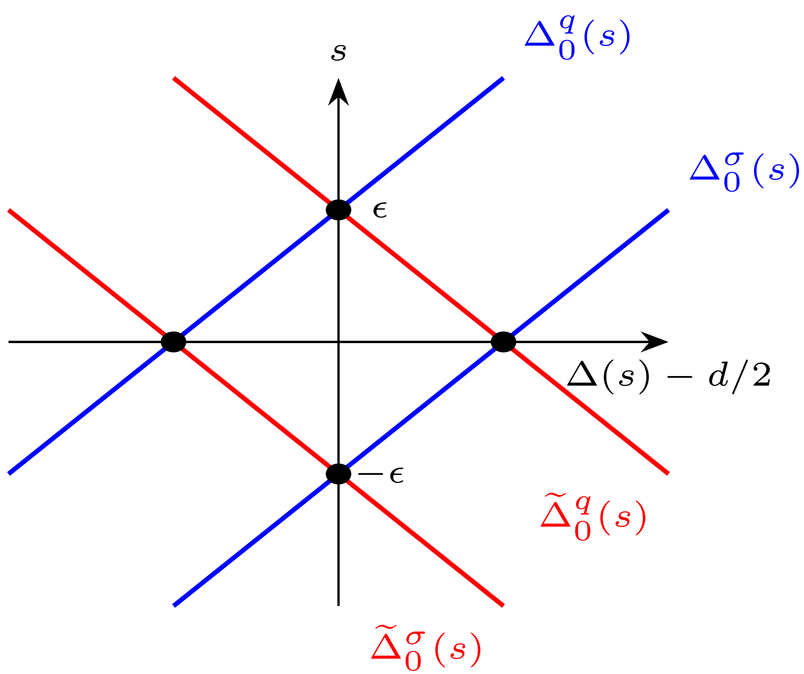
  - Gross-Neveu model

$$\Delta_q = \frac{d-1}{2} \qquad \Delta_\sigma = 1$$

- Chew-Frautschi plot around $s = 0$:



fixed N, small ϵ



fixed ϵ, large N

GN at large N

- Anomalous dimensions of twist-2 operators $(n = 4N, \mu = d/2)$ [Muta, Popovic'1977] [Manashov, Skvortsov' 2017]

$$\gamma_\sigma(s) = -\frac{\eta_1}{n}\frac{2(2\mu-1)}{(\mu-1)}\left(1 - \frac{\mu}{2\mu-1}\frac{\Gamma(\mu)}{\Gamma(s+\mu)}\frac{\Gamma(s+2-\mu)}{\Gamma(3-\mu)}\right) + O(1/n^2)$$

$$\gamma_q(s) = \frac{\eta_1}{n}\left(1 - \frac{\mu(\mu-1)}{(s+\mu-1)(s+\mu-2)}\left(1 + \frac{\Gamma(2\mu-1)\Gamma(s+1)}{(\mu-1)\Gamma(2\mu-3+s)}\right)\right) + O(1/n^2)$$

- Consequences of the intersections:
  - Singularities at $s = \pm\epsilon$

$$\lim_{s\to-\epsilon}\gamma_\sigma(s) = \infty$$

$$\lim_{s\to\epsilon}\gamma_q(s) = \infty$$

  - Operator $\mathcal{O}^\sigma_{s=0} = \sigma^2$ does not lie on the trajectory $\gamma_\sigma(s)$ (even without a divergency) [Giombi, Kirilin, Skvortsov'2017]

$$\lim_{s\to 0}\gamma_\sigma(s) \neq \gamma_{\sigma^2}$$

- Resummation around $s = 0$:

$$\gamma_{q,\sigma}(s) = \pm\omega_1(s) - s + \sqrt{s^2 + \omega_2(s) + \omega_1^2(s)}$$

  - Regular functions $\omega_{1,2}(s)$:

$$\omega_1(s) = \frac{1}{2}\big(\gamma_q(s) - \gamma_\sigma(s)\big), \quad \omega_2(s) = s\big(\gamma_q(s) + \gamma_\sigma(s)\big) + \gamma_q(s)\gamma_\sigma(s)$$

- Operator $\sigma^2(x)$ is back on the trajectory

$$\lim_{s\to 0} \gamma_\sigma(s) = \gamma_{\sigma^2}$$

  - Need to take into account that at $1/N^2$:

$$\gamma_{q,\sigma}^{1/N^2}(s) = \left(-\frac{\eta_1^2}{n^2} \frac{2\mu(2\mu-1)(2\mu-3)}{(\mu-2)^2} \times \frac{1}{s} + O(s^0) \right)$$

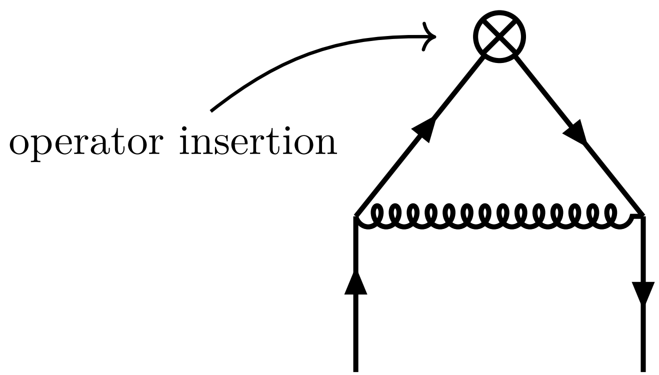
- Singularities of $\gamma_q(s)$ and $\gamma_\sigma(s)$ coincide at **all** orders in perturbation theory

How we perform calculations

- Additional UV-singularities of the composite operator

$$\Big\langle \mathcal{O}(x)\Phi(x_1)\ldots\Phi(x_n)\Big\rangle \quad\longrightarrow\quad \frac{1}{\epsilon}\ \text{poles}$$

- One-loop example in QCD



$$\sim \int \frac{d^d l}{(2\pi)^d}\frac{\gamma^\mu \not{l}\gamma_{\mu_1}\not{l}\gamma_\mu}{l^4(p-l)^2}\,{\color{red}(l_{\mu_2}\ldots l_{\mu_s} - \text{traces})}$$

- Diagram $\xrightarrow{\text{projection}}$ Scalar Integrals $\xrightarrow{\text{IBP}}$ Master Integrals $\xrightarrow{1/\epsilon\text{ pole}}$ Answer
  - $s = 1, 2, \ldots \quad\longrightarrow\quad \gamma(1), \gamma(2), \ldots$
  - Large s $\longrightarrow$ large # of scalar integrals

Result of calculations

- Usual output of calculations

$$\gamma_{\mathsf{ns}}^{(3)}(2) = C_F^3\left(\frac{128\zeta_3}{3} - \frac{560}{243}\right) + \ldots$$

$$\gamma_{\mathsf{ns}}^{(3)}(4) = C_F^3\left(\frac{2878\zeta_3}{75} - \frac{714245693}{48600000}\right) + \ldots$$

$$\gamma_{\mathsf{ns}}^{(3)}(6) = C_F^3\left(\frac{139724\zeta_3}{3675} - \frac{854652999073}{51051262500}\right) + \ldots$$

$$\ldots$$

- Usual answer

$$\gamma_{\mathsf{ns}}^{(3)}(s) = C_F^3\Bigg(\ldots 256S_{1,2,-2}(s) - 64S_{-3,2}(s) - 128S_{1,-2,2}(s) + 1280S_{1,1,-3}(s) + 256S_{1,2,-2}(s)$$

$$+ 192S_5(s) + \frac{352+144s+144s^2}{s(s+1)}S_{-4}(s) + \frac{448+528s+336s^2}{s^2(s+1)^2}S_{-3}(s) +$$

$$\left(\frac{32+64s+32s^2+144s^3+144s^4+144s^5+48s^6}{s^3(s+1)^3} - 192\zeta_3\right)S_{-2}(s) - 64S_5(s)$$

$$+ \frac{64+152s+96s^2+6s^3+12s^4+6s^5}{s^3(s+1)^2}S_2(s) + \frac{-512+96s+96s^2)}{s(s+1)}S_{-3,1}(s)\ldots\Bigg) + \ldots$$

  - Harmonic sums $S_{a,w}(s) = \sum_{k=1}^{s}\frac{\text{sign}(a)^k}{k^{|a|}}S_w(k)$
  - The transcendental weight $|w| \leq 2\ell - 1$

Bootstrapping the answer

- Ansatz $= \{F_1(s), \ldots, F_k(s)\}$ $(N < k)$

$$\begin{cases} a_1 F_1(1) + \ldots + a_k F_k(1) = \gamma(1) \\ \quad \ldots \\ a_1 F_1(N) + \ldots + a_k F_k(N) = \gamma(N) \end{cases} \longrightarrow \quad \text{set of } \textbf{Diophantine} \text{ equations}$$

- Lattice basis reduction algorithm [Lenstra, Lenstra, Lovász'1982]

set of **Diophantine** equations $\xrightarrow{\text{LLL}}$ Analytic answer

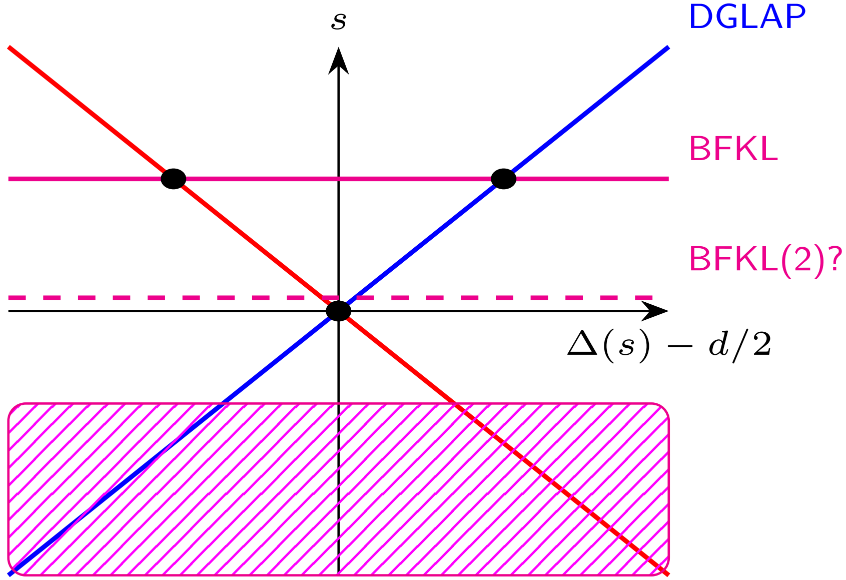
- Every coefficient in the expansion $\longrightarrow$ + **additional** equation

- Generalized Gribov-Lipatov reciprocity
  - Reciprocity-respection (RR) harmonic sums [Beccaria, Forini'2009]

$$2 \cdot 3^{w-1} \longrightarrow 2^{w-1}$$

What about QCD?

- Chew-Frautschi plot in QCD (in one-loop) [Chang, Chen, Simmons-Duffin, Zhu'2025]



  - DGLAP/BFKL mixing

$$\lim_{s\to 1} \gamma_{gg}(s)$$

  - DGLAP/$\widetilde{\text{DGLAP}}$ mixing

$$\lim_{s\to 0} \gamma_{\text{ns}}(s)$$

- Violation of DLE at three-loops in the **non-planar** sector [Velizhanin'2017]

Subleading BFKL(s)?

- What about $s \leq -1$?

Conclusions and further questions

- Conclusions:
  - The structure of anomalous dimensions can be studied at the **critical** point.
  - To study functional properties we perform **analytic continuation in spin**
  - Points of **intersection** of Regge trajectories correspond to the**singularities** in anomaolus dimensions
  - Resolving the singularities we predict coefficients in the expansion around singularity
  - Coefficients can be used to bootstrap the **higher-loop** answer
- Questions:
  - How to **compute** trajectories most effectively (detector operators, Lorentzian inversion formula, ...)?
  - What kind of **information** from the trajectories we need?
  - How to include h**igher-twist** operators?
  - How to completely predict what **mixes** with what (signature, $1/N$, ...)?