Bootstrapping six-point Yang-Mills amplitudes

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Annecy-le-Vieux

based on works `2505.01245` and `2510.20565` in collaboration with

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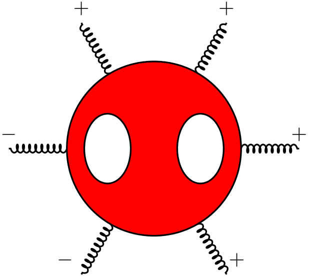
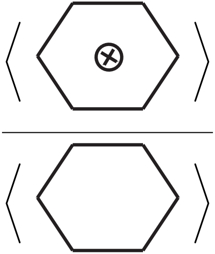
IPhT, CEA-Saclay
12 December 2025

Bootstrap six-point two-loop finite observables in super Yang-Mills and QCD



planar MHV amplitude
in YM and QCD



Corrlation function
of null Wilson loops and
Lagrangian (half-BPS operator)
in planar $\mathcal{N} = 4$ sYM
$\Updownarrow$
all-plus amplitude in YM

Enormous success of the **bootstrap** for six-, seven-, …point amplitudes and three-, four-point form factors of half-BPS operators in $\mathcal{N} = 4$ super-Yang-Mills theory

The Steinmann Cluster Bootstrap for $\mathcal{N} = 4$ Super Yang-Mills Amplitudes

Simon Caron-Huot,[1] Lance J. Dixon,[2] James M. Drummond,[3] Falko Dulat,[2] Jack Foster,[3] Ömer Gürdoğan,[3,4] Matt von Hippel,[5,6] Andrew J. McLeod[2,6] and Georgios Papathanasiou[*7]

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Table 1: The dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

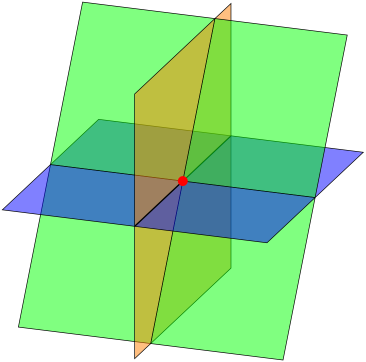
How much can be done for **Yang-Mills/QCD**?

- Six-point amplitudes are natural candidates for attempting bootstrap beyond one-loop in non-DCI gauge theories

- Reach kinematics of six-point amplitudes in non-DCI gauge theories

- However, six-point alphabet for non-DCI theories is very complicated — **245** letters from two-loop Feynman integrals

Success of Bootstrap depends on Ansatz size and strength of Physical Constraints



A finite ansatz for L-loop six-point four-dimensional observable

$$\sum_{ij} c_{ij}\,\mathcal{R}_i\,f_j$$

with rational prefactors $\mathcal{R}_i$, transcendental functions f_j, and unknown numbers c_{ij}

Our bootstrap is informed by the Feynman integral calculation

Scenario A

Symbol alphabet

$\Downarrow$

Functions
for the alphabet
& Adjacencies

$\Downarrow$

Bootstrap

$\Downarrow$

Observable

<u>Scenario B</u>

Function space
of Master Integrals

$\Downarrow$

Bootstrap

$\Downarrow$

Observable

Scenario C

Master Integrals
& The integrand

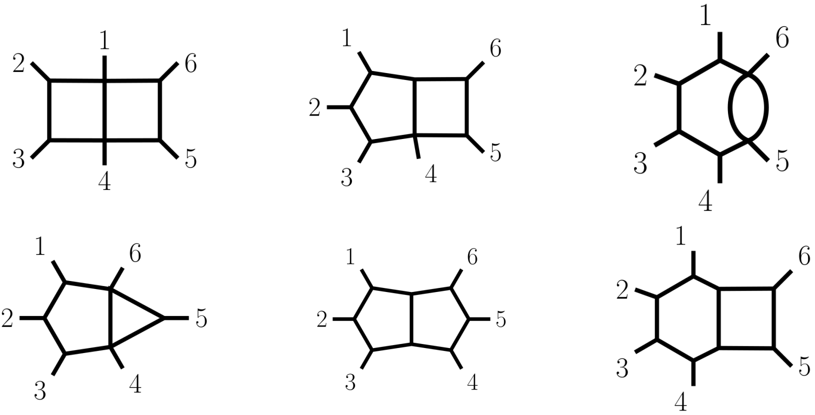
$\Downarrow$

$\boxed{\text{IBP}}$

$\Downarrow$

Observable

Two-loop Six-point Planar Feynman integrals



[S. Abreu, P. F. Monni, B. Page, J. Usovitsch '24]

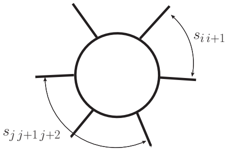
[J. Henn, A. Matijašić, J. Miczajka,T. Peraro, Y. Xu, Y. Zhang '25]

- 7 kinematic variables
- 245 alphabet letters (in four dimensions)
- 639 transcendental functions* for finite part of amplitudes

* weight-4 symbols of the Feynman integrals

Six-particle massless kinematics in 4d

- 9 Mandelstam variables $s_{i\,i+1}$, $s_{j\,j+1\,j+2}$



$$p_i^2 = 0\,, \quad i = 1, \ldots, 6$$

$$\sum_i p_i^\mu = 0$$

- Gram-determinant constraint

$$\det G(p_1, \ldots, p_5) = 0$$

- parity odd combinations

$$\mathrm{i}\epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma$$

- 7 variables parametrize kinematics

Canonical DE, Iterated integrals, Symbols

Master integrals $\vec{I} \equiv \Big($  ,  $, \ldots \Big)$ in $d = 4 - 2\epsilon$ satisfy **canonical DE** with alphabet $\{W_i\}$ and $\mathbb{Q}$-valued matrices A_i, [J. Henn '13]

$$\left.\begin{aligned} d\vec{I} &= \epsilon\, dA.\vec{I} \\ dA &\equiv \sum_i A_i\, d\log W_i \\ \vec{I} &= \sum_{w\geq 0} \epsilon^w\, \vec{I}^{(w)} \end{aligned}\right\}$$

$$\vec{I}^{(w)} = \sum_{i_1\ldots i_w} \#\int d\log W_{i_1} \circ \ldots \circ d\log W_{i_w}$$

+lower order iterated interations

Symbol projection

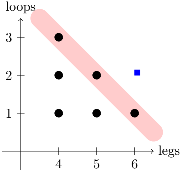
$$\mathcal{S}(\vec{I}^{(w)}) = \sum_{i_1\ldots i_w} \# W_{i_1} \otimes \ldots \otimes W_{i_w}$$

Transcendental weight w	1	2	3	**4**
Two-loop six-point symbols	9	62	266	**639**

I. Six-point MHV amplitudes in Yang-Mills and QCD

Loop amplitudes in gauge theories

- Phenomenology of high energy physics



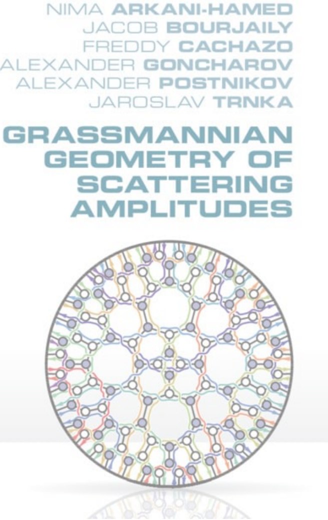
- Hidden structures of gauge theories and deep connection to mathematics

$$\mathcal{A}^{(L)} = \sum_i \mathcal{R}_i f_i$$

Transcendental weight w	Rational prefactors $\mathcal{R}$	Transcendental functions f
$w = 0$	Hard	Easy
$\vdots$	$\vdots$	$\vdots$
$w = 2L$	Easy	Hard

Rational prefactors is a major bottleneck at high miltiplicity

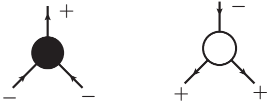
On-shell diagrams calculate 4d leading singularities



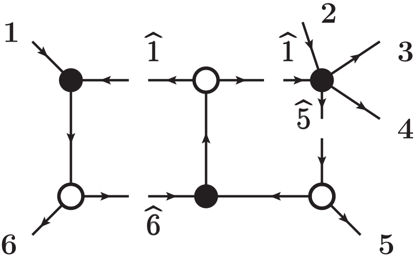
Glue white
and black 3-vertices
by BCFW-bridges

$$|\widehat{i}\rangle = |i\rangle + z|j\rangle$$

$$|\widehat{j}] = |j] - z|i]$$

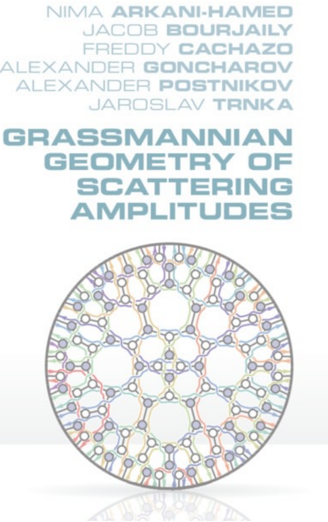


MHV_3 $\overline{\mathrm{MHV}}_3$



$$\text{Parke-Taylor} \times \oint \frac{(\langle 12\rangle + z\langle 26\rangle)^3 \langle 56\rangle^4}{\langle 12\rangle^4 (\langle 15\rangle + z\langle 56\rangle)^4} dz$$

On-shell diagrams calculate 4d leading singularities



Glue white
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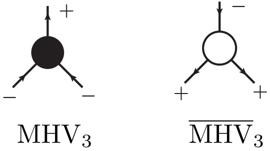
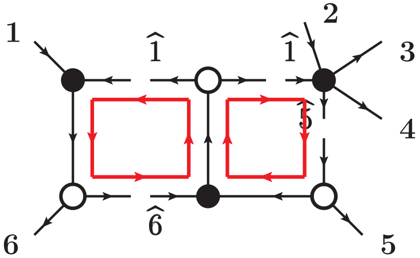
$$|\widehat{i}\rangle = |i\rangle + z|j\rangle$$

$$[\widehat{j}| = [j| - z[i|$$

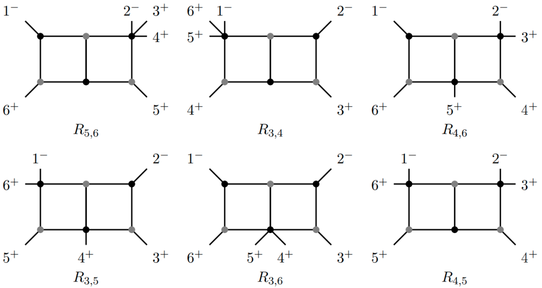




$$\text{Parke-Taylor} \times \oint \frac{(\langle 12\rangle + z\langle 26\rangle)^3 \langle 56\rangle^4}{\langle 12\rangle^4 (\langle 15\rangle + z\langle 56\rangle)^4} dz$$

All rational prefactors of the hard function are captured by 4d on-shell diagrams

- infrared-subtracted MHV $--++++$ amplitude
- Parke-Taylor $R_1 \equiv \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle 61\rangle}$ at tree level and one loop
- Six new rational prefactors at two loops



rational prefactors are explicitly **conformal**

$$u_{ij} \equiv \frac{\langle 1i\rangle\langle 2j\rangle}{\langle 12\rangle\langle ij\rangle}$$

$$R_{i,j}/R_1 = -1 + 12u_{ij} - 30u_{ij}^2 + 20u_{ij}^3$$

Infrared finite two-loop hard function

$$\mathcal{H}^{(2)}_{\mathrm{YM}} = \lim_{\epsilon\to 0}\left(\mathcal{A}^{(2)} - I^{(1)}\mathcal{A}^{(1)} + \frac{1}{2}\left(I^{(1)}\right)^2\mathcal{A}^{(0)}\right)$$

- dipole IR subtraction $I^{(1)} = -\frac{1}{\epsilon^2}\sum_i (-s_{ii+1}/\mu^2)^{-\epsilon}$ [Catani '98]

- Two-loop six-gluon ansatz

$$\mathcal{H}^{(2)}_{\mathrm{YM}} = R_1 G_1 + \sum_{2<i<j\leq 6} R_{i,j} G_{i,j}$$

  - ☐ Parke-Taylor R_1 and six two-loop leading singularities $R_{i,j}$
  - ☐ **639/712** dimensional space of two-loop symbols with **167** letters

$$\ldots + W_{k_1}\otimes W_{k_2}\otimes W_{k_3}\otimes W_{k_4} + \ldots \text{ for } G_1,\, G_{i,j}$$

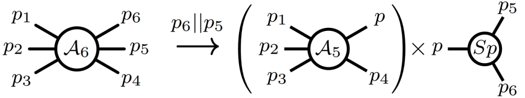
- Flip symmetry cuts the ansatz size down to **2412** unknowns

$$1^- 2^- 3^+ 4^+ 5^+ 6^+ \quad \leftrightarrow \quad 2^- 1^- 6^+ 5^+ 4^+ 3^+$$

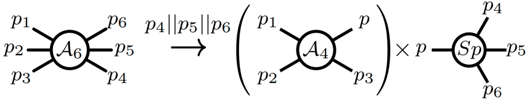
Bootstrap the observable from physical limits

ansatz $\mathcal{H}^{(2)}_{\mathrm{YM}}$	2412
scaling	996
no spurious poles	
$\langle 36 \rangle = 0$	333
$\langle 35 \rangle = 0$	614
$\langle 34 \rangle = 0$	629
$\langle 45 \rangle = 0$	343
collinear	
$--++++$	1785
$+--+++$	1307
$++--++$	1785
$+++--+$	1646
$-++++-$	1646
triple collinear	
$--++++$	1836
$++--++$	724
Total	**2412**

Inhomogeneous input
— splitting functions $h \to h_5 h_6$

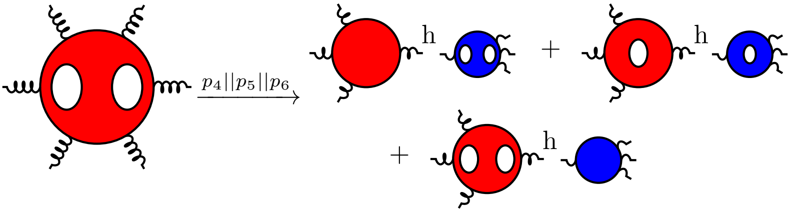


Consistency in the triple collinear limit
— kinematics factorizes



167 letters in the ansatz
but **137** letters in the observable

Two-loop triple collinear splitting functions



$$\mathcal{H}_6^{(2)}/\mathcal{A}_6^{(0)} \to \mathcal{H}_4^{(2)}/\mathcal{A}_4^{(0)} + \mathcal{C}_{h_4h_5h_6}^{(1)h} \times \mathcal{H}_4^{(1)}/\mathcal{A}_4^{(0)} + \mathcal{C}_{h_4h_5h_6}^{(2)h}$$

One of IR-subtracted triple collinear functions is the two-loop six-point remainder function of $\mathcal{N}=4$ sYM

$$\mathcal{C}_{+++}^{(2)+} = \mathcal{C}_{\mathcal{N}=4\ \mathrm{sYM}}^{(2)}$$
$$\mathcal{C}_{-++}^{(2)-} = \mathcal{C}_{\mathcal{N}=4\ \mathrm{sYM}}^{(2)} + \text{correction}$$

Outlook

- Bootstrap easily extended to the planar QCD hard function $--++++$

$$\mathcal{H}^{(2)}_{\mathrm{QCD}} = \mathcal{H}^{(2)}_{\mathrm{YM}} + \left(\frac{N_f}{N_c}\right)\mathcal{H}_{\text{quark loop}}$$

- More six-point two-loop amplitudes: other MHV, NMHV, and quark legs?
- Rich kinematic limits: triple splitting and double-soft functions at two loops
- Beyond symbol terms and lower transcendentalities?
- Better understanding of the function space? Cluster adjacencies?

245 4d letters

167 two-loop letters

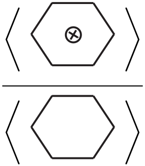
137 physical letters

II. Hexagonal Null Wilson loop with Lagrangian insertion in $\mathcal{N} = 4$ super-Yang-Mills

Hexagonal Null Wilson loop with Lagrangian insertion

- Light-like polygonal contour $x_{i\,i+1}^2 = 0$
- Wilson loop $W = \mathrm{tr}_{\mathrm{F}} \mathrm{Pexp}(\oint dx^\mu A_\mu(x))$
- chiral form of the Lagrangian $\mathcal{L}$ is half-BPS operator of dimension $\Delta = 4$

$$F_6 \equiv \frac{\langle W[x_1, \ldots, x_6] \mathcal{L}(x_0) \rangle}{\langle W[x_1, \ldots, x_6] \rangle} \equiv$$



introduced in [Alday, Buchbinder, Tseytlin '11]

Conformal (co)variant function well-defined in four dimensions,
no regularization/subtraction required!

The finite observable brings together the 4d loop integrands and loop functions

Loop integrands

- On-shell diagrams & generalized unitarity
- Positive grassmannians
- Amlituhedron geometry
- Color-kinematic duality

Loop functions

- Steinmann/Cluster adjacency
- Near-collinear OPE
- Multi-Regge kinematics
- Anomalous dual superconformal symmetry

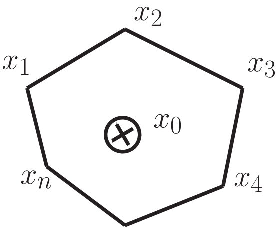
Thanks to the Wilson loop/Amplitude duality the L-loop integrand of F is the $(L+1)$-loop integrand of the logarithm of the MHV amplitude

$$g^2 \partial_{g^2} \log(\langle W_n \rangle) \sim \int d^D x_0 \, F(x_0)$$

Kinematics of n-cusp Null Wilson loop with Lagrangian insertion is the same as of n-particle scattering amplitude

$$x^2_{i\,i+1} = 0$$

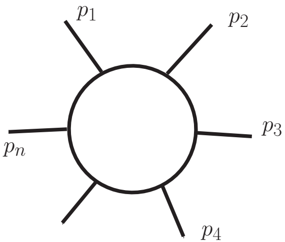
$$\sum_i p_i = 0\,, \quad p_i^2 = 0$$



$$p_i^\mu = x_{i+1}^\mu - x_i^\mu$$

$$\Longleftrightarrow$$

$$x_0 \to \infty$$



$3n - 11$ independent dual-conformal cross-ratios

$$u_{ijkl} = \frac{x^2_{i0} x^2_{k0} x^2_{jl}}{x^2_{j0} x^2_{l0} x^2_{ik}} \quad \xrightarrow[x_0 \to \infty]{} \quad \frac{x^2_{jl}}{x^2_{ik}}$$

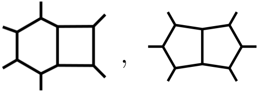
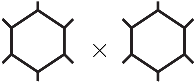
[7 variables in the hexagonal case]

Function space for the hexagonal Wilson loop with Lagrangian insertion

Three-loop integrand of $\log(\mathcal{A}_6^{\mathrm{MHV}})$ contains products of lower loops:

$$\mathcal{A}^{(3)}(x_0) + \mathcal{A}^{(2)}(x_0) \times \mathcal{A}^{(1)} + \mathcal{A}^{(1)}(x_0) \times \mathcal{A}^{(2)} + \mathcal{A}^{(1)}(x_0) \times \mathcal{A}^{(1)} \times \mathcal{A}^{(1)}$$

where $\mathcal{A}^{(L)}$ is an L-loop integrand and loop dual-momentum $x_0 \to \infty$

Transcendental weight	1	2	3	**4**
	9	62	266	**639**
	9	59	221	**428**
Total	9	62	319	**945**

Leading singularities (rational prefactors) are Kermit one-loop forms

[T.V. Brown, J. Henn, E. Mazzucchelli, J. Trnka '25]

- Kermits $[a_1 b_1 c_1; a_2 b_2 c_2]$ triangulate one-loop MHV Amplituhedron

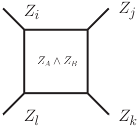
$$x_0 \Leftrightarrow (AB) \equiv Z_A \wedge Z_B$$

$$\Omega_n^{(1)}(AB) = \sum_{i,j} [1\, i\, i+1; 1\, j\, j+1]$$

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka '10]

- Nonlocal poles in Kermits $[a_1 b_1 c_1; a_2 b_2 c_2]$ for generic labels

$[ijk; jkl] =$



- Conformal invariance (in momentum space) at $x_0 \to \infty$ $\qquad p_i^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \lambda_i^\alpha \widetilde{\lambda}_i^{\dot{\alpha}}$

$$\underbrace{\sum_i \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot{\alpha}}}}_{\mathbb{K}^{\alpha\dot{\alpha}}} [a_1 b_1 c_1; a_2 b_2 c_2] = 0$$

Dihedral invariant ansatz

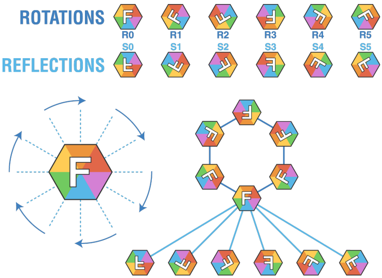


fig. from hexnet.org

Two-loop symbol ansatz

$$F^{(2)} = \sum_{i,j} c_{ij} \, R_i \, f_j$$

with

- 20 leading singularities R_i
- 945 weight-four symbols f_j
- 20×945 unknowns c_{ij}

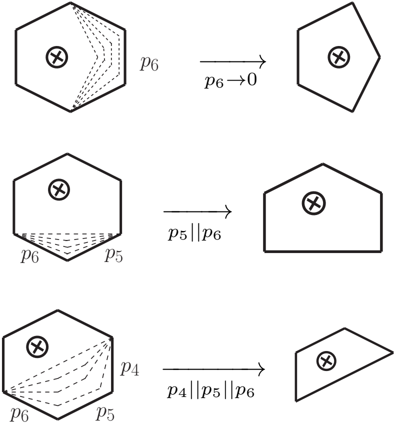
Dihedral transformations of the leading singularities and symbols

$$\sigma \circ R_i = R_{\sigma(i)} \, , \quad \sigma \circ f_j = f_{\sigma(j)} \, , \quad \sigma \in D_6$$

1718 unknowns in the dihedral invariant ansatz $\sigma \circ F^{(2)} = F^{(2)}$

Bootstrap the observable from physical limits

ansatz	20×945
D_6 invariance	1718
soft	1439
collinear	1539
triple collinear	353
no spurious $p_2 \cdot p_4$	1044
no spurious $p_2 \cdot p_5$	483
scaling	585
unknowns	0

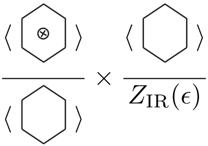


245 letters in the ansatz
but **137** letters in the observable

The same physical subalphabet for YM hard functions!

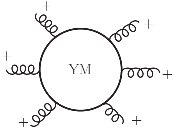
Conjectured duality with All-Plus amplitude in pure Yang-Mills

[DC, Henn '22]



$$F \times \mathcal{R}^{\text{MHV}} \quad \sim \quad \mathcal{H}^{\text{YM, all-plus}}$$

$\leq L$-loop in super-YM — $(L+1)$-loop in pure YM



- duality ($\sim$) of the maximal transcendental parts, in the leading color limit
- IR divergences of gauge theory amplitudes

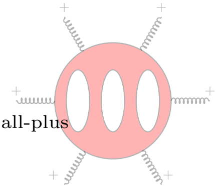
$$\mathcal{R}^{\text{MHV}} = \lim_{\epsilon\to 0} Z_{\text{IR}}^{-1}(\epsilon)\, \mathcal{A}^{\text{MHV}}(\epsilon)$$

$$\mathcal{H}^{\text{YM, all-plus}} = \lim_{\epsilon\to 0} Z_{\text{IR}}^{-1}(\epsilon)\, \mathcal{A}^{\text{YM,all-plus}}(\epsilon)$$

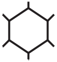
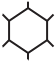
are subtracted $Z_{\text{IR}} = \exp(g^2 I(\epsilon))$ by the dipole $I(\epsilon) = -\frac{1}{2\epsilon^2}\sum_i (s_{i\,i+1}/\mu^2)^{-\epsilon}$

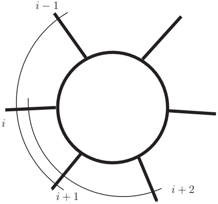
- All-plus amplitude vanishes at tree level, and $(L+1)$-loop amplitude is of transcendentality $2L$

Conjectured duality with Three-Loop All-Plus amplitude in pure Yang-Mills and Steinmann relations



$$F_6^{(2)} + F_6^{(1)}\mathcal{R}_6^{(1)\text{MHV}} + F_6^{(0)}\mathcal{R}_6^{(2)\text{MHV}} \sim \mathcal{H}_6^{(3)\,\text{YM, all-plus}}$$

Both products  $\times$  from $F_6^{(2)}$ and $F_6^{(1)}\mathcal{R}_6^{(1)\text{MHV}}$ are incompatible with Steinmann relations. However, Steinmann relations hold for their sum

$$\underset{s_{i-1\,i\,i+1}}{\text{Disc}}\;\underset{s_{i\,i+1\,i+2}}{\text{Disc}}$$  $$= 0$$

Outlook

- Positivity/Negativity of loop functions inside the one-loop MHV Amplituhedron region

[Arkani-Hamed,Henn,Trnka '21][DC, Henn '22]
[Abreu, DC, Sotnikov, Zoia '24][DC, Henn, Trnka, S-Q.Zhang '24]

$$F_6^{(0)} < 0\,, \qquad F_6^{(1)} > 0\,, \qquad F_6^{(2)} < 0$$

✓ ✓ ?

- $F_6^{(2)}$ captures **137**-letter two-loop six-point **physical** alphabet

- Negative geometry expansion reorganizes $F^{(L)}$ according to "connectivity" among loops specified by the Amplituhedron [Arkani-Hamed,Henn,Trnka '21]

Two-loop Six-point and Three-loop Five-point Ladder Negative geometries

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Thank you!