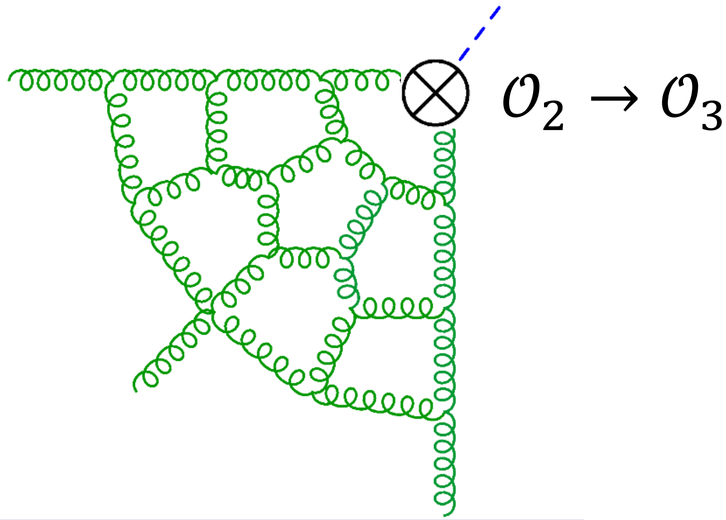
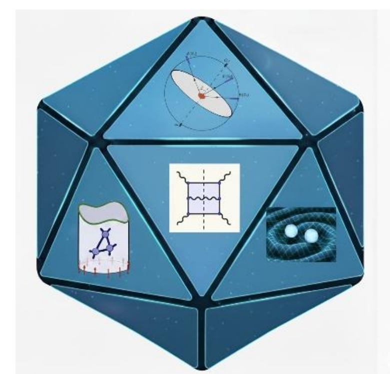
Another 8 Loop Form Factor in Planar N=4 SYM





Lance Dixon (SLAC)

B. Basso, LD, A. Tumanov, 2410.22402

LD, Zhenjie Li, to appear

"Observables in Gauge Theory and Gravity"

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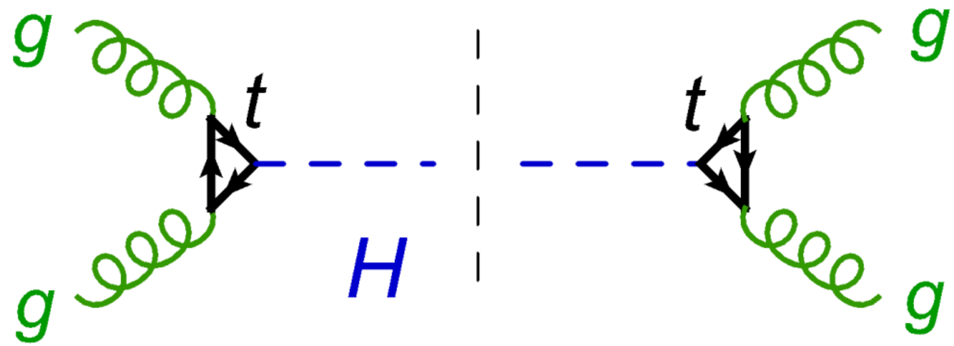
NATIONAL ACCELERATOR LABORATORY

Producing Higgs bosons at LHC

Higgs boson dominantly produced by **gluon fusion**, a quantum process at "one loop", mediated by **top quark,** because ***t*** couples strongly to both gluons and Higgs

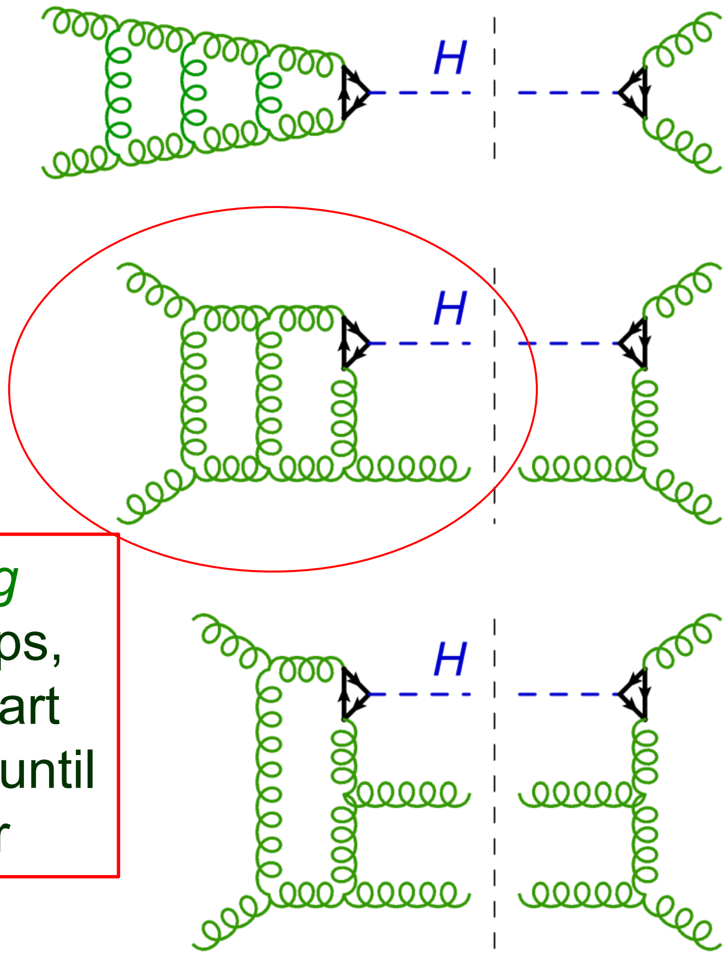
Leading Order (LO) cross section
= $|\text{one} - \text{loop amplitude}|^2$

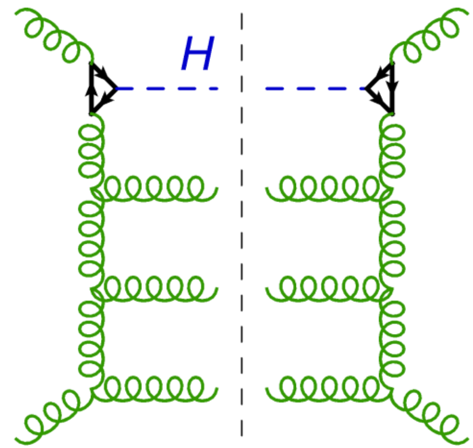


- Since $\boldsymbol{2m_{top}} = 350\ \text{GeV}$
  $\gg \boldsymbol{m_{Higgs}} = 125\ \text{GeV},$

interaction between gluons and Higgs is approximately local (mediated by an operator $H G_{\mu\nu}^{a} G^{\mu\nu\, a}$)

Very few of the NNNLO QCD diagrams





gg → *Hg*
@ 2 loops,
state of art
in QCD until
this year

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

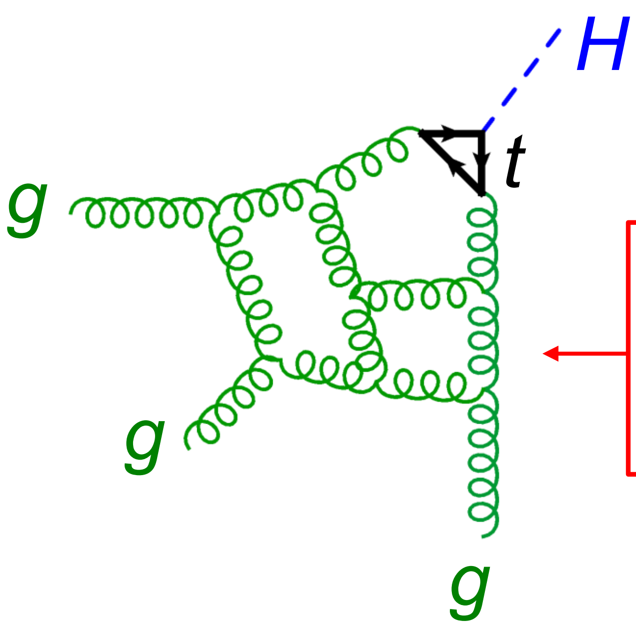
Scattering amplitudes are underlying building blocks

"Goldilocks Process" ~ $gg \rightarrow Hg$

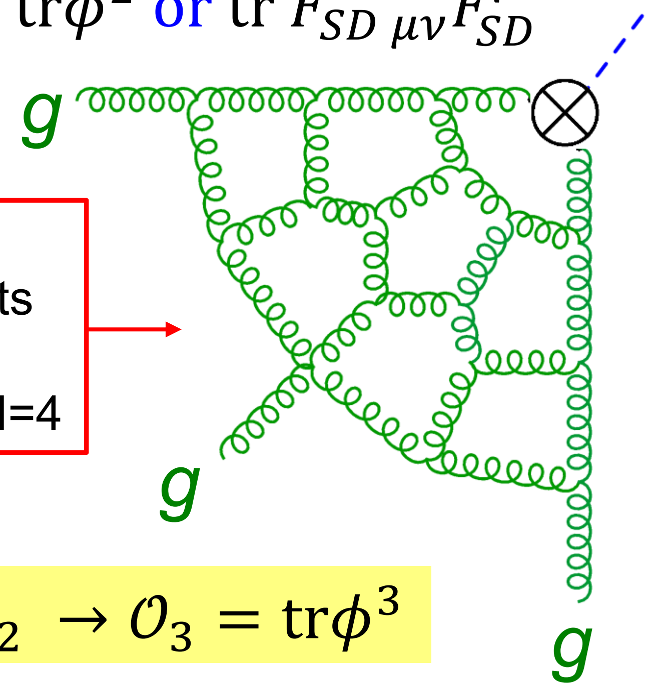
QCD state of art is now three loops @ large N_c (not counting top quark loop)
Chen, Guan, Mistlberger, 2504.06490

Can get to **eight** loops – in analog process: 3-point form factor of $\mathrm{tr}\phi^2$ operator in planar N=4 SYM
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm
2012.12286, 2112.06243, 2204.11901

$$\mathcal{O}_2 = \mathrm{tr}\phi^2 \text{ or } \mathrm{tr}\, F_{SD\,\mu\nu} F_{SD}^{\mu\nu}$$



Leading weight $2L$
transcendental parts
match between
QCD ←→ planar N=4



Today: $\mathcal{O}_2 \rightarrow \mathcal{O}_3 = \mathrm{tr}\phi^3$

Planar N=4 SYM

- Teaches us "something" about the leading transcendental part of QCD amplitudes
- For simple processes, the answer is a polylogarithmic function of known weight ($2L$ at loop order L) and high loop order computations can be relatively simple.
- We know how to remove all infrared divergences, either via "framing" dual Wilson loops, or via a BDS(-like) ansatz.
- Finite remaining functions $\mathcal{E}_3^{(L)}$ for $\mathrm{tr}\phi^2$; $\mathcal{E}_{3,3}^{(L)}$ for $\mathrm{tr}\phi^3$

$\mathrm{tr}\phi^3$ form factor was known to 6 loops

Basso, LD, Tumanov, 2410.22402

- 3 loops agrees with integrand-based computation

  Henn, Lim, Torres Bobadilla, 2410.22465

- 6 loops based on amplitude bootstrap, with exactly the same function space $\mathcal{C}$ as for the $\mathrm{tr}\phi^2$ form factor, and boundary conditions provided by the FFOPE

  Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569; Basso, Tumanov, 2308.08432

- The fact that the same function space works for $\mathrm{tr}\phi^2$ and $\mathrm{tr}\phi^3$ is not obvious

3-point form factor kinematics is two-dimensional

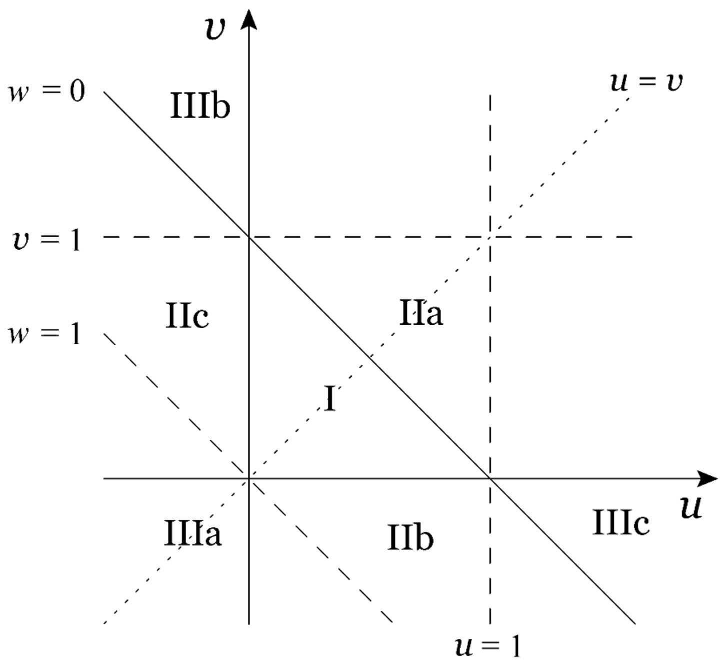
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

$\mathrm{tr}\phi^2$ and $\mathrm{tr}\phi^3$ form factors are both S_3 invariant

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i+1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

Map polylogs to **symbols**

- Iterative differentiation of a polylogarithmic function F

n times for a weight n function,
can be used to define its **symbol:**

["ln" is implicit in s_{i_k}]

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}} \, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where s_{i_k} are **letters** in the symbol **alphabet** $\mathcal{L}$
and $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers (often integers!)
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Common alphabet for $\mathrm{tr}\phi^2$ *and* $\mathrm{tr}\phi^3$

Let $u = \frac{s_{12}}{s_{123}}$, $v = \frac{s_{23}}{s_{123}}$, $w = \frac{s_{31}}{s_{123}} = 1 - u - v$

Then $\mathcal{L} = \{a, b, c, d, e, f\}$

where $a = \sqrt{\frac{u}{vw}},\ b = \sqrt{\frac{v}{wu}},\ c = \sqrt{\frac{w}{uv}},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w}$

- Symbols of $\mathrm{tr}\phi^2$ form factor $\mathcal{E}_3^{(L)}$ simplify (remarkably) at $L = 1$ and 2 loops, to just 6 and 12 terms:

$$\mathcal{S}\left[\mathcal{E}_3^{(1)}\right] = (-2)[\, b\otimes d + c\otimes e + a\otimes f + b\otimes f + c\otimes d + a\otimes e]$$

$$\mathcal{S}\left[\mathcal{E}_3^{(2)}\right] = 8[\, b\otimes d\otimes d\otimes d + c\otimes e\otimes e\otimes e + a\otimes f\otimes f\otimes f + b\otimes f\otimes f\otimes f + c\otimes d\otimes d\otimes d + a\otimes e\otimes e\otimes e]$$
$$+16\,[b\otimes b\otimes b\otimes d + c\otimes c\otimes c\otimes e + a\otimes a\otimes a\otimes f + b\otimes b\otimes b\otimes f + c\otimes c\otimes c\otimes d + a\otimes a\otimes a\otimes e]$$

(really only 1 and 2 terms, plus images under dihedral symmetry)

Low loop $\mathrm{tr}\phi^3$ symbols

- Symbols of $\mathrm{tr}\phi^3$ form factor $F_{3,3}^{(L)}$ at $L = 1, 2$ loops are just 2 and 18 terms, plus D_3 dihedral images:

$$\mathcal{S}\left[\mathcal{E}_{3,3}^{(1)}\right] = a\otimes b - a\otimes a + \mathrm{dihedral}$$

$$\begin{aligned}\mathcal{S}\left[\mathcal{E}_{3,3}^{(2)}\right] = {} & 6\, a\otimes a\otimes a\otimes a - 6\, a\otimes a\otimes b\otimes a - 6\, a\otimes b\otimes a\otimes a + 5\, a\otimes b\otimes b\otimes a \\ & + a\otimes b\otimes c\otimes a - 3\, a\otimes e\otimes a\otimes a + 3\, a\otimes e\otimes c\otimes a + 8\, a\otimes a\otimes b\otimes b \\ & - 6\, a\otimes a\otimes a\otimes b - 6\, a\otimes b\otimes b\otimes b + 5\, a\otimes b\otimes a\otimes b \\ & - 2\, a\otimes a\otimes c\otimes b - 2\, a\otimes c\otimes b\otimes b + 3\, a\otimes f\otimes a\otimes b - 3\, a\otimes f\otimes b\otimes b \\ & + a\otimes b\otimes c\otimes b + a\otimes c\otimes a\otimes b + a\otimes c\otimes c\otimes b \\ & + \mathrm{dihedral}\end{aligned}$$

The bootstrap

Combines:

1. knowledge of the polylogarithmic function space (steal from $\mathrm{tr}\phi^2$)
2. empirical multi-final entry conditions (similar to $\mathrm{tr}\phi^2$)
3. leading and next-to-leading discontinuities (in bulk!)
4. the FFOPE at level of "one flux tube excitation" (fixes collinear limit of the function)
   Basso, Tumanov, 2308.08432

1. Function space from $\mathrm{tr}\phi^2$

- Symbol alphabet $\{a, b, c, d, e, f\}$
- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- 12 pairs of adjacent letters are forbidden:

$$\begin{array}{lll} \ldots a \otimes d \ldots, & \ldots b \otimes e \ldots, & \ldots c \otimes f \ldots \\ \ldots d \otimes a \ldots, & \ldots e \otimes b \ldots, & \ldots f \otimes c \ldots \\ \ldots d \otimes e \ldots, & \ldots e \otimes f \ldots, & \ldots f \otimes d \ldots \\ \ldots e \otimes d \ldots, & \ldots f \otimes e \ldots, & \ldots d \otimes f \ldots \end{array}$$

- Resemble constraints from causality: Steinmann relations
  Steinmann, Helv. Phys. Acta (1960)
- But not really, which mystified us for a while…
- However, the relations are antipodally dual to (extended) Steinmann relations for the 6-gluon amplitude!!

2. (multi-)final entries

- $\mathrm{tr}\phi^2$ symbol only ends in $d, e,$ or f (3 final entries)
- $\mathrm{tr}\phi^3$ symbol only ends in $a, b,$ or c;
  but also satisfy $\mathcal{E}^a_{3,3} + \mathcal{E}^b_{3,3} + \mathcal{E}^c_{3,3} = 0$ (2 final entries)
- At each loop order, we learn another layer of multi-final entry conditions, e.g. by 3 loops we know that: (→ 6 double-final entries)

$$\mathcal{E}^{d,a}_{3,3} = \mathcal{E}^{e,b}_{3,3} = 0, \qquad \mathcal{E}^{f,b}_{3,3} = -\mathcal{E}^{f,a}_{3,3}, \qquad \mathcal{E}^{c,a}_{3,3} = -\mathcal{E}^{a,a}_{3,3} - \mathcal{E}^{b,a}_{3,3}$$

$$\mathcal{E}^{c,b}_{3,3} = -\mathcal{E}^{a,b}_{3,3} - \mathcal{E}^{b,b}_{3,3}, \qquad \mathcal{E}^{d,b}_{3,3} = \mathcal{E}^{e,a}_{3,3} - \mathcal{E}^{f,a}_{3,3} + \mathcal{E}^{a,b}_{3,3} - \mathcal{E}^{b,a}_{3,3}.$$

3. Leading discontinuities

- Compact formula from FFOPE:

$$\mathcal{W}^{L}_{3,3}|_{T^1\,\ln^L T} = \int du \frac{\pi}{\cos(\pi i u)} S^{2iu} \frac{1}{L!}[2(\psi(\tfrac{1}{2}+iu)+\psi(\tfrac{1}{2}-iu)-2\psi(1))]^L$$

- Normally such formulas only hold in the collinear limit $v \to 0$, but this one actually holds in the bulk (generic u, v) for a suitable extrapolation.
- There is only a 2-letter symbol for this discontinuity, with letters $A \equiv \frac{u}{v} = \frac{1+T^2}{S^2}$, $C \equiv \frac{1-w}{v} = \frac{1+S^2+T^2}{S^2}$ and it is possible to predict it recursively to all loop orders.

3. Leading discontinuity (cont.)

After converting the leading discontinuity from $\mathcal{W}^{L}_{3,3}$ to $\mathcal{R} \equiv \exp(R_{3,3})$ normalization, we find for the symbols of $\mathcal{R}^{(L)}$:

$$c_L(\mathtt{XCA}^k) = 2L\, c_{L-1}(\mathtt{XA}^k) - \xi_k \frac{L!}{(L-k-1)!}\, c_{L-k-1}(\mathtt{X}),$$

$$\mathcal{R}^{(0)} = 1,$$

$$\mathcal{R}^{(1)} = 0,$$

$$\mathcal{R}^{(2)} = -\,\mathtt{AA}, \qquad (1.9)$$

$$\mathcal{R}^{(3)} = -2\,\mathtt{CAA} - 6\,\mathtt{ACA} + 4\,\mathtt{A} \ldots \qquad (1.10)$$

$$\mathcal{R}^{(4)} = -16\,\mathtt{CCAA} - 16\,\mathtt{CACA} \ldots \qquad (1.11)$$

$$\mathcal{R}^{(5)} = -160\,\mathtt{CCCAA} - 160\,\mathtt{C} \ldots \mathtt{A}$$
$$+160\,\mathtt{CAACA} + 320\,\mathtt{A} \ldots \mathtt{A}$$
$$-190\,\mathtt{AACAA} - 150\,\mathtt{A} \ldots \qquad (1.12)$$

$$\xi_k = \sum_{m=0}^{k} \frac{(-2)^{m+1}}{(m+1)!} \frac{k!}{m!(k-m)!}$$
$$= (-1)^k\, 2\; {}_1F_1(-k, 2; 2)$$
$$= 0, -\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{2}{5}, \tfrac{4}{45}, \tfrac{10}{63}, -\tfrac{32}{105}, \tfrac{142}{405}, \ldots .$$

$$\mathcal{R}^{(6)} = -1920\,\mathtt{CCCCAA} - 1920\,\mathtt{CCACCA} - 1920\,\mathtt{CCCACA} - 1920\,\mathtt{CACCCA} - 5760\,\mathtt{ACCCCA}$$
$$+1920 \ldots$$
$$+1920 \ldots$$
$$-1224 \ldots$$
$$-2280\,\mathtt{ACACAA} - 1800\,\mathtt{ACAACA} - 2280\,\mathtt{AACCAA} - 2280\,\mathtt{AACACA} - 1800\,\mathtt{AAACCA}$$
$$+608\,\mathtt{CAAAAA} + 672\,\mathtt{ACAAAA} + 912\,\mathtt{AACAAA} + 992\,\mathtt{AAACAA} + 672\,\mathtt{AAAACA}$$
$$-185\,\mathtt{AAAAAA}\,. \qquad (1.13)$$

$$\frac{c_L(\mathtt{A}^L)}{L!} = \sum_{m=0}^{L} \frac{(-1)^{L+m}}{m!} \frac{L!}{m!(L-m)!} = (-1)^L\; {}_1F_1(-L, 1; 1)$$

Solve a large system of linear equations

- At 8 loops, 1495 x 403 = 602,485 unknowns
- Solve mod p
- See Zhenjie's: https://github.com/munuxi/SparseRREF
- One 10 digit prime was enough to reconstruct over the rational numbers.
- Symbol level unknowns, 1251 x 403 = 504,153 of them, are all integers!

Higher loop trϕ^2 and trϕ^3symbols are lengthy!

loop order L	trϕ^2 symbol terms	trϕ^3 symbol terms
1	6	9
2	12	105
3	636	1,773
4	11,208	44,391
5	263,880	747,837
6	4,916,466	14,637,501
7	92,954,568	274,014,315
8	1,671,656,292	$\sim 5 \times 10^9$?

LD, Gürdoğan, McLeod, Wilhelm, 2204.11901;
Basso, LD, Tumanov, 2410.22402; LD, Z. Li, to appear

NOTE: Even bigger symbols for 5-loop 7-point amplitude!
He, Jiang, Li, Liu, 2511.09669

trϕ^2: number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

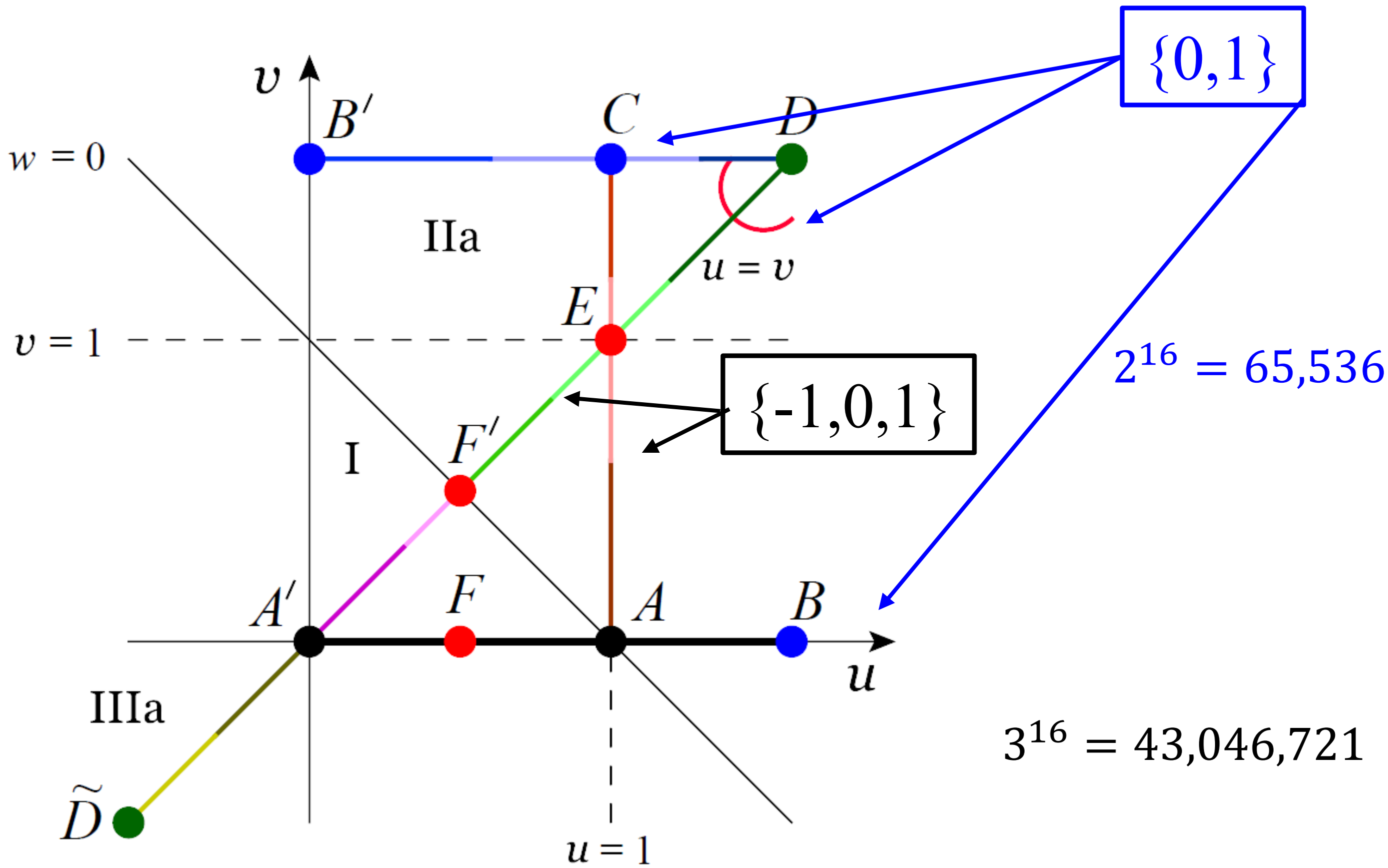
- Properly normalized L loop trϕ^2 form factors $\mathcal{E}_3^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}_3^{(L)}$ also obeys multiple-final-entry relations, saturation on right

trϕ^3: number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	2	1														
$L=2$	1	3	3	2	1												
$L=3$	1	3	9	13	6	2	1										
$L=4$	1	3	9	21	29	13	6	2	1								
$L=5$	1	3	9	21	48	57	29	13	6	2	1						
$L=6$	1	3	9	21	48	105	112	57	29	13	6	2	1				
$L=7$	1	3	9	21	48	108	242	206	112	57	29	13	6	2	1		
$L=8$	1	3	9	21	48	108	242	519	375	206	112	57	29	13	6	2	1

- Properly normalized L loop trϕ^3 form factors $\mathcal{E}_{3,3}^{(L)}$ belong to **same** small space $\mathcal{C}$
- $\mathcal{E}_{3,3}^{(L)}$ obeys **different** multiple-final-entry relations, saturation on right

Plot functions on various lines



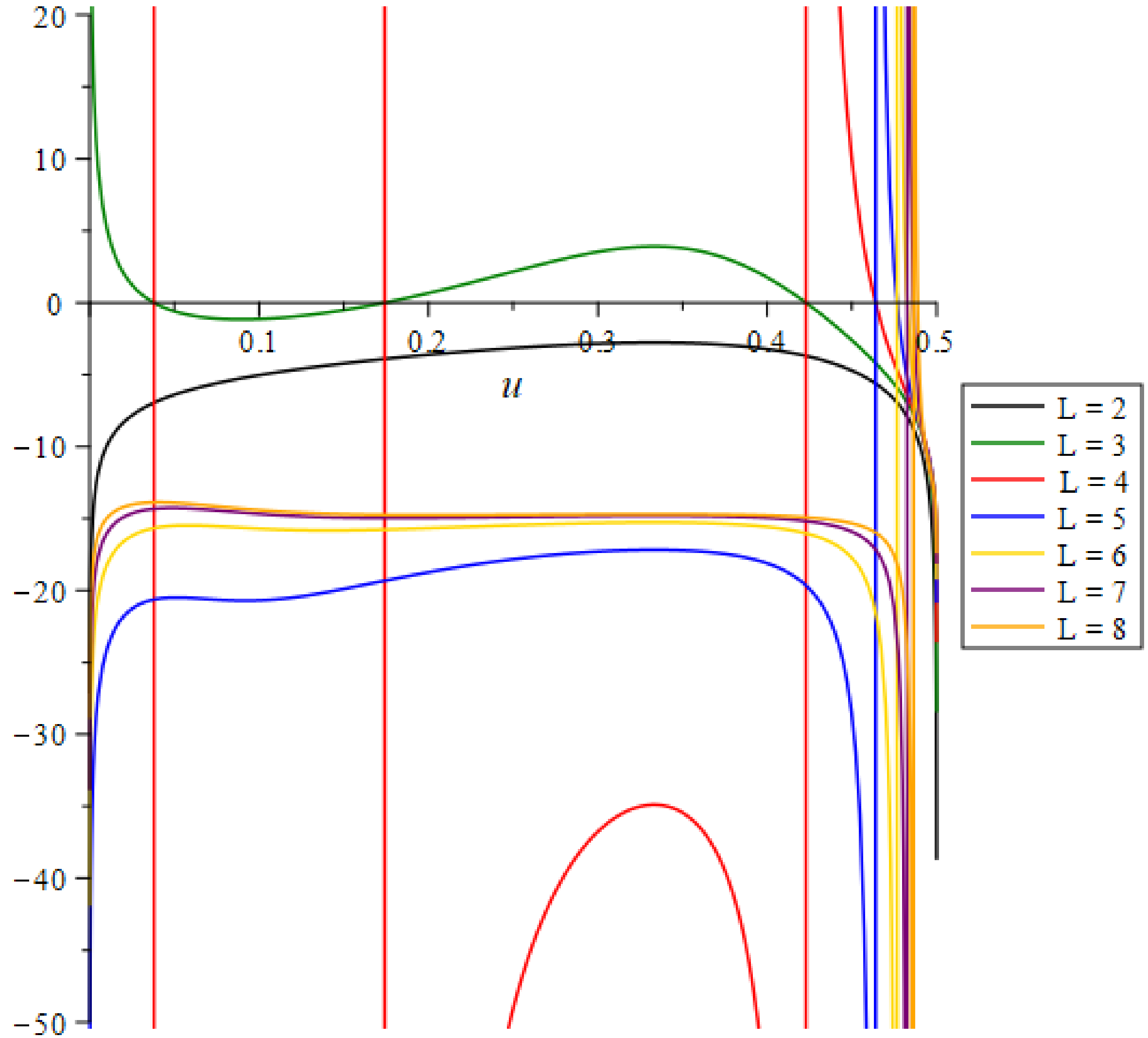
Series expand on multiple overlapping segments

Successive loop ratios reveal finite radius of convergence

$\mathrm{tr}\phi^3$

$$\frac{\mathcal{E}_{3,3}^{(L)}(u,u,1-2u)}{\mathcal{E}_{3,3}^{(L-1)}(u,u,1-2u)}$$

$$g_c = \frac{1}{4}$$
$$\leftrightarrow ratio \rightarrow -16$$



High energy at fixed angles

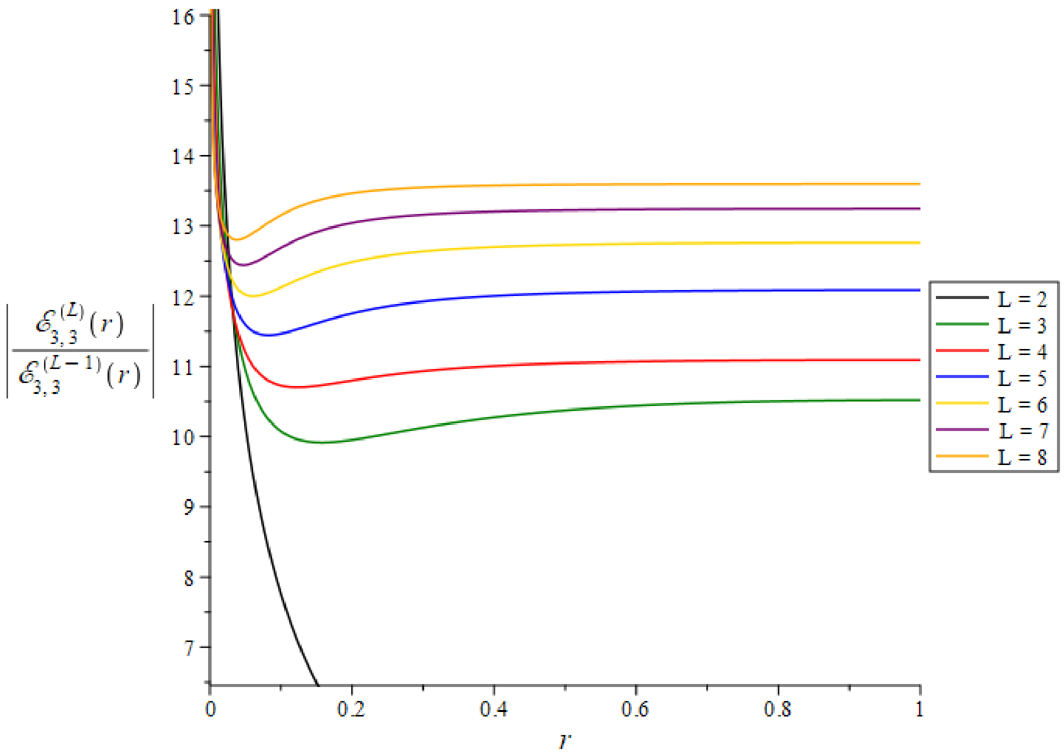
- $u, v \rightarrow \infty$, with $r = u/v$ held fixed
- trϕ^2 form factor

$\mathcal{E}_3 \rightarrow constant$

(indep. of r)

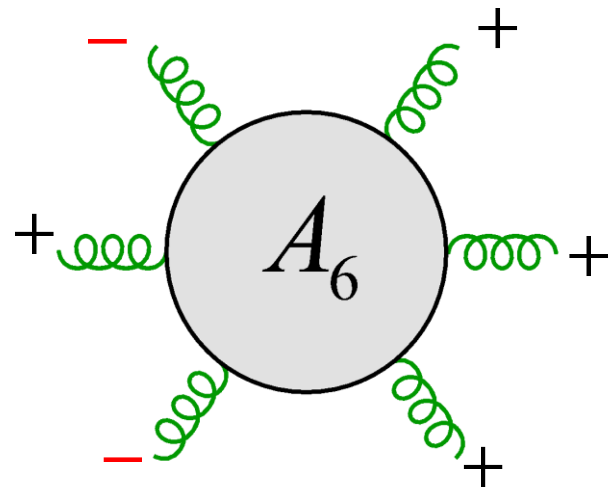
- trϕ^3 form factor

$\mathcal{E}_{3,3}$ is nontrivial →



Six-gluon MHV amplitude

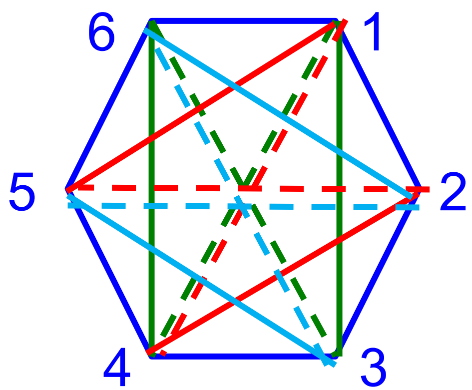
- Dual to Wilson hexagon, invariant under dual conformal transformations; it only depends on 3 dual conformal cross ratios, $\hat{u}, \hat{v}, \hat{w}$:



$$\hat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$\hat{v} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$\hat{w} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



D_6 dihedral symmetry: cycle $(\mathrm{mod}\ 6)$ and flip, but it acts on $\hat{u}, \hat{v}, \hat{w}$ as $D_3 = S_3$

Parity-preserving surface: (lower-dimensional kinematics)

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

Antipodal duality (AD) for trϕ^2

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$\boxed{F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w}\right)}$$

- Antipode map S, is a "coinverse" for the Hopf algebra.
- At symbol level, it simply reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

- Kinematic map in terms of underlying variables is:

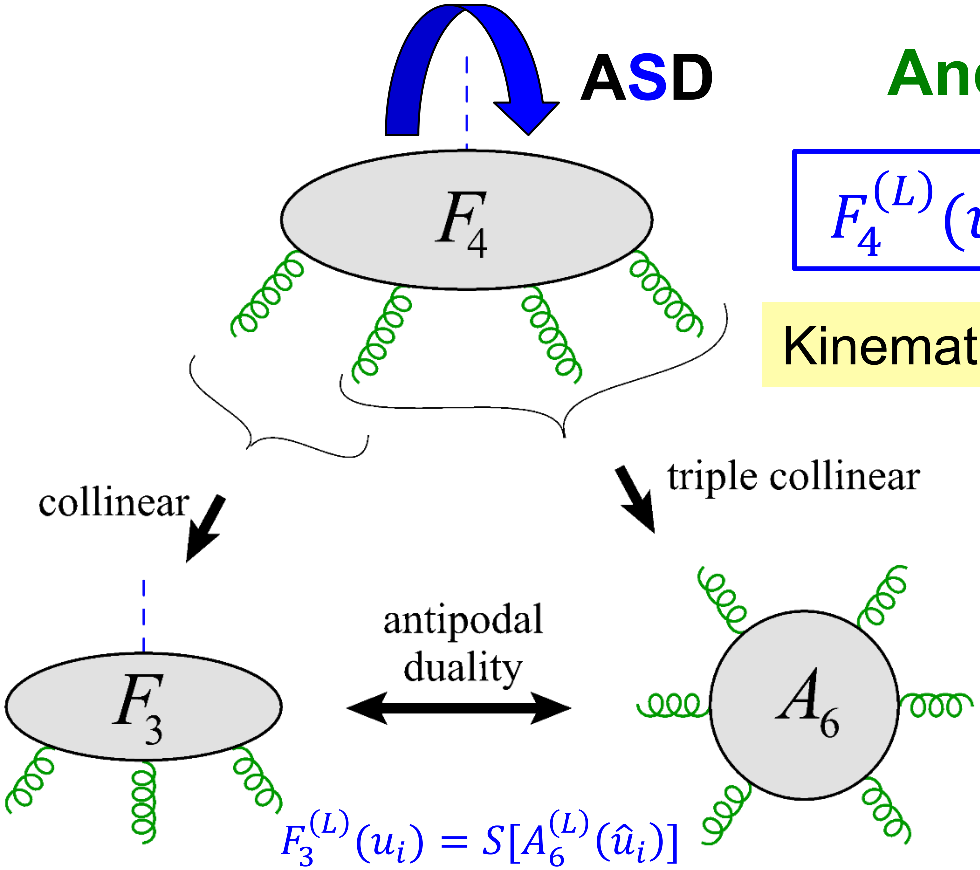
$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u+v+w=1$ to $\Delta = 0$ parity-preserving surface for A_6

- Role of branch cuts and derivatives exchanged by S

Antipodal Self Duality

Given antipodal duality relating 2-collinear and 3-collinear limits of F_4, natural to search for self-duality of F_4 that holds for all parity-preserving bulk kinematics



And it's there!

$$F_4^{(L)}(u_i, v_i) = S[F_4^{(L)}(\mathbf{C}(u_i), \mathbf{C}(v_i))]$$

Kinematic map $\mathbf{C}$ simple in FFOPE variables:

$$\mathbf{C}: \quad T_2 \to \frac{T}{S}, \quad S_2 \to \frac{1}{TS}$$

$$T \to \sqrt{\frac{T_2}{S_2}}, \qquad S \to \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

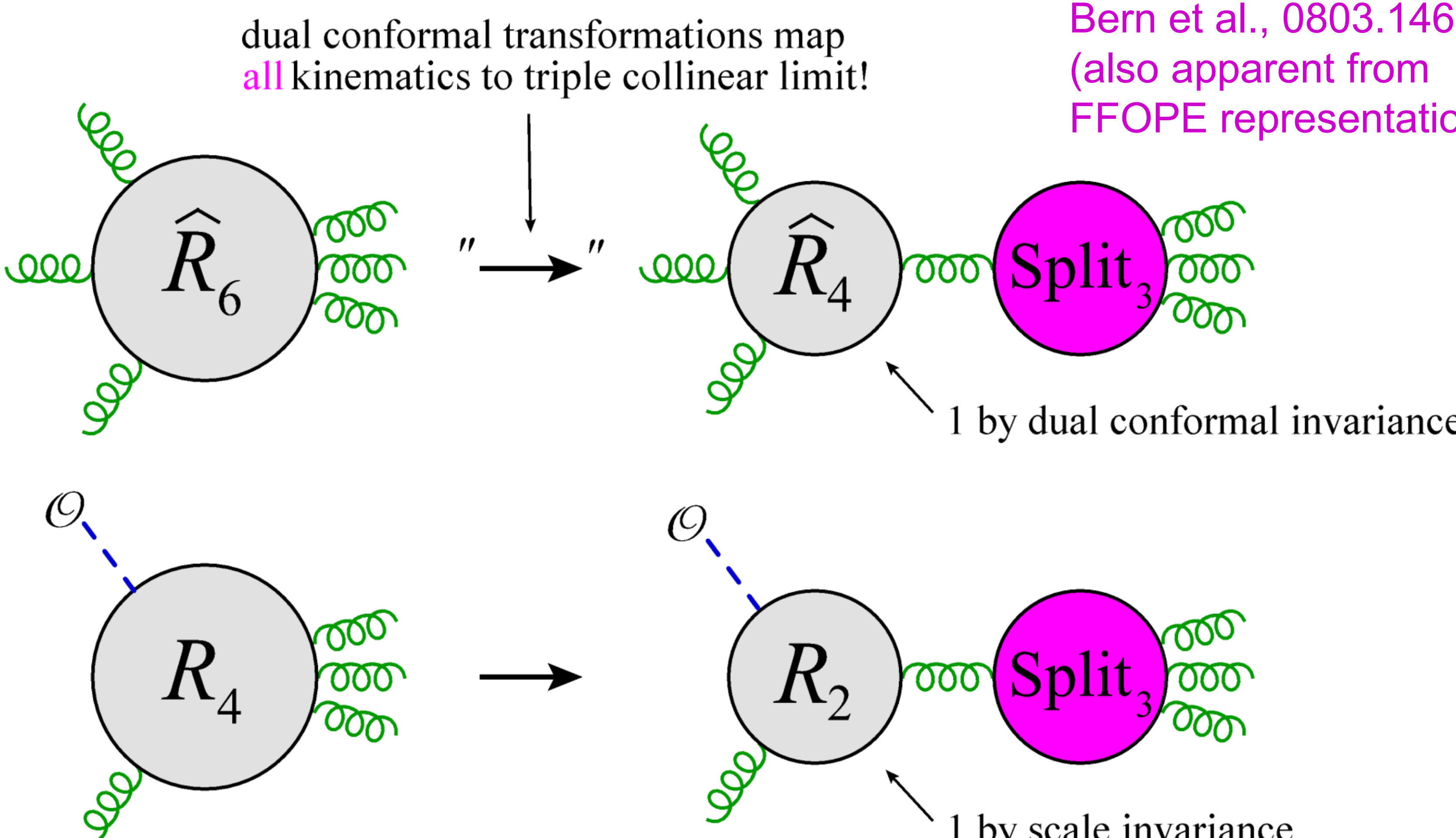
Main Open Question

- Why does trϕ^3 3-point form factor live in same space as the trϕ^2 one?
- This space has powerful adjacency conditions whose only explanation is via antipodal duality.
- But the 2 final entries for trϕ^3, $\mathcal{E}_{3,3}^{a/b}$ and $\mathcal{E}_{3,3}^{b/c}$,
  are very different from the 3 final entries for trϕ^2,
  $\mathcal{E}_3^d$, $\mathcal{E}_3^e$, and $\mathcal{E}_3^f$,
  and so they map to the "wrong" first entries for the hexagon functions (A_6 amplitude).
- ???

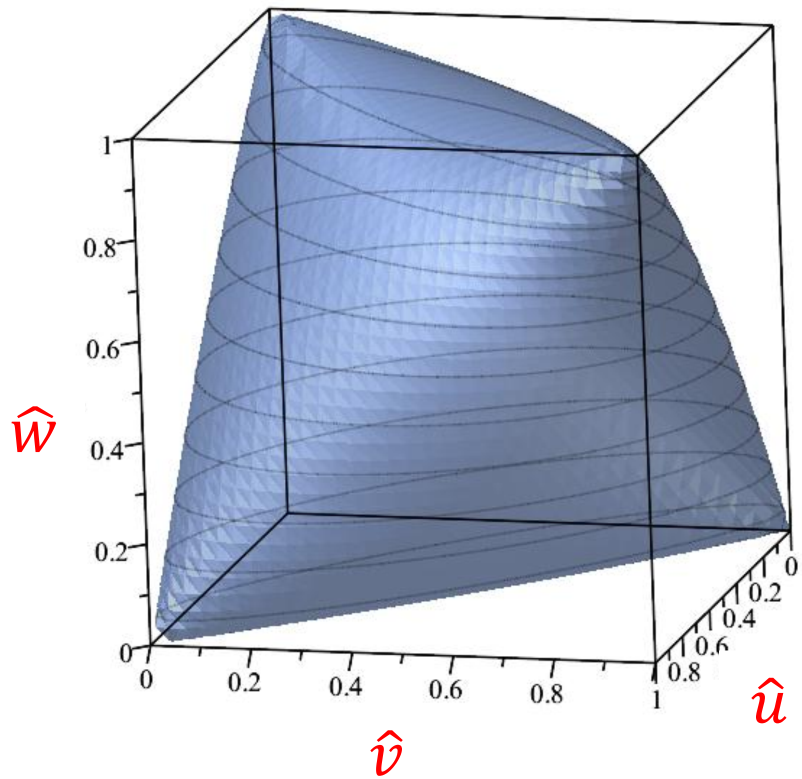
Extra Slides

Triple Collinear Limit of 4-point form factor → 6-gluon amplitude

Bern et al., 0803.1465;
(also apparent from
FFOPE representation)



Parity-preserving surface



$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

$\Delta = 0$ means that kinematics lies in a 3d subspace of 4d spacetime → parity invariant

FFOPE kinematical variables for F_4

$$u_i = \frac{s_{i,i+1}}{s_{1234}}, \qquad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$$

$$-u_1 + u_3 + v_4 + v_1 = 1$$
$$-u_2 + u_4 + v_1 + v_2 = 1$$
$$-u_3 + u_1 + v_2 + v_3 = 1$$

$$u_1 = \frac{T^2 T_2^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_2 = \left\{1 + T^2 + \frac{S^2[(1+F_2^2)S_2 T_2 + F_2(1+S_2^2+T^2+T_2^2)]}{F_2 S_2^2}\right\}^{-1}$$

$$u_3 = \frac{S^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_4 = \frac{S^2 T^2}{S_2^2}\, u_2$$

$$v_1 = \frac{T_2^2+1}{S^2+T^2+T_2^2+1}$$

- OPE limit takes $T, T_2 \to 0$, interpolates between 2-collinear limit $T_2 \to 0$ and 3-collinear limit $T \to 0$

ASD beyond 2 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410v2;
LD, Z. Li, to appear

- Bootstrapped symbol of F_4 at 3 loops, using same 113 letter (2-loop) alphabet.
- **Unique result**, which obeys all the FFOPE predictions we could check.
- 2 loop symbol uses only 34 of the letters [3,784 terms]
- 3 loop symbol uses only 88 of the letters [3,621,202 terms]
- 4 loop symbol uses only 88 of the letters [???? terms]
- No OPE checks yet at 4 loops, but few constraints needed to fix it.
- **ASD holds in 3D at 4 loops!**