Correlation functions in $\mathcal{N} = 2$ superconformal long-quiver theories

Alessandro Testa

IPhT, Saclay

Observables in gauge theory and gravity, IPhT, 11/12/2025

based on 2501.17223 in collaboration with G. P. Korchemsky

What's to come

Introduction and motivations

Twisted and untwisted operators

Correlation functions at weak coupling

Correlation functions at strong coupling

Three-point functions

Conclusions and future perspectives

Introduction and motivations

Introductions and motivations

This work is part of a broad program that aims to solve $4d$ $\mathcal{N}=4$ and superconformal $\mathcal{N}=2$ **planar** Yang-Mills theories for arbitrary $\lambda = g^2_{\mathrm{YM}} N_c$

Several techniques (e.g. localization, integrability, ...) allow us to realize this program for

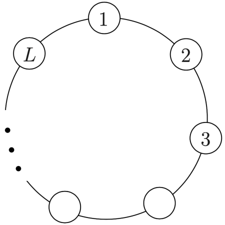
- v.e.v. of half-BPS circular Wilson loops in $\mathcal{N}=4$ SYM [Pestun]
- correlation functions of infinitely heavy half-BPS operators (octagon) [Korchemsky et al.]
- correlation and partition functions in certain $\mathcal{N}=2$ SYM theories [Billo' et al.]

These observables ($\mathcal{F}_\ell$) can be **universally** expressed as Fredholm determinants of certain **Bessel** operators:

$$\mathrm{e}^{\mathcal{F}_\ell(\hat{\lambda})} = \det\Big(\mathsf{I} - \mathbf{K}_\ell(\hat{\lambda})\Big) \ , \qquad \hat{\lambda} = \frac{\sqrt{\lambda}}{4\pi}$$

$\mathcal{N} = 2$ quiver theories

We consider a $SU(N)$ $\mathcal{N} = 2$ quiver theory with L identical couplings ($\lambda^{(I)} \equiv \lambda$ for $I = 1, \ldots L$), arising from an orbifold projection of $SU(NL)$ $\mathcal{N} = 4$



- Each node denotes a $SU(N)$ gauge-group factor, while lines represent hyper-multiplets in the representation $(\mathbf{N}, \bar{\mathbf{N}})$ of $SU(N) \times SU(N)$. As a result, the theory is **conformal**
- This model posses a cyclic symmetry $I \rightarrow I + 1$
- Planar equivalence of certain observables with $\mathcal{N} = 4$ SYM (e.g. free-energy on $\mathbb{S}^4$, half-BPS Wilson loops) and dual description in terms of a $AdS_5 \times S_5/\mathbb{Z}_L$ geometry

A first look at the long-quiver limit

Novel properties for $L \to \infty$, where the diagram nodes become continuously distributed suggesting an emergent fifth dimension (deconstruction)

Physical questions: Can we approximate the limit $L \to \infty$ by a five dimensional theory ?

If so, what are the properties of this theory ?

Claim: As $L \to \infty$, the properties of the model depend on how we compare L with λ.

When $L \gg \lambda$, the model involves a fifth dimension with massive degrees of freedom

We will show this property by investigating correlators which are **not** planar equivalent to $\mathcal{N} = 4$ SYM. These observables are expressible by ratios of special Fredholm determinants

Twisted and untwisted operators

Twisted and untwisted operators

We consider complex combinations of $\frac{1}{2}$-BPS single-trace gauge-invariant operators

$$T_{\alpha,n}(x) = \frac{1}{\sqrt{L}} \sum_{I=0}^{L-1} \mathrm{e}^{-ip_\alpha I} O_n^I(x) \,, \qquad\qquad O_n^I(x) = \mathrm{tr}\, \phi_I^n(x)$$

where $\alpha = 0, \ldots, L-1$ is the sector index and $\phi^I(x)$ denotes the I-th vector-multiplet scalar

- $p_\alpha = \frac{2\pi\alpha}{L}$ is the quasimomentum of excitations
- $T_{\alpha,n}(x)$ is a conformal primary of dimension and $U(1)$ charge n
- $T_{\alpha,n}(x)$ transforms under $\mathbb{Z}_L$ non-trivially

$$\phi_I \to \phi_{I+1} \qquad \Rightarrow \qquad T_{\alpha,n}(x) \to \mathrm{e}^{ip_\alpha}\, T_{\alpha,n}(x)$$

- $\alpha = 0$ denotes the untwisted sector, which is planar equivalent to $\mathcal{N} = 4$ SYM

Correlators

Conformal symmetry, charge conservation and invariance under $\mathbb{Z}_L$ leads to

$$\langle T_{\alpha_1,n_1}(x_1) T^\dagger_{\alpha_2,n_2}(x_2)\rangle = G^{(2)}_{\alpha,n} \frac{\delta_{n_1,n_2}\delta(p_{\alpha_1}-p_{\alpha_2})}{|x_1-x_2|^{2n_1}} ,$$

$$\langle T_{\alpha_1,n_1}(x_1) T_{\alpha_2,n_2}(x_2) T^\dagger_{\alpha_3,n_3}(x_3)\rangle = G^{(3)}_{\vec{\alpha},\vec{n}} \frac{\delta_{n_1+n_2,n_3}\delta(p_{\alpha_1}+p_{\alpha_2}-p_{\alpha_3})}{|x_1-x_3|^{2n_1}|x_2-x_3|^{2n_2}} ,$$

$G^{(2)}_{\alpha,n}$ and $G^{(3)}_{\vec{\alpha},\vec{n}}$ carry the dependence on (λ, L, n) and can be computed by localization on $\mathbb{S}^4$. Defining these models on $\mathbb{S}^4$ the path-integral reduces to

$$\mathcal{Z}_{\mathbb{S}^4} = \int \prod_{I=0}^{L-1} \mathcal{D}a_I \exp\left(-\sum_{I=0}^{L-1}\left[\text{tr}\, a_I^2 - S_{\rm int}(a_I, a_{I+1})\right]\right)$$

$$S_{\rm int} = \sum_{i,j=2}^{\infty} C_{ij}\left(\text{tr}\, a_I^i - \text{tr}\, a_{I+1}^i\right)\left(\text{tr}\, a_I^j - \text{tr}\, a_{I+1}^j\right)$$

where $\langle\phi_I(x)\rangle_0 = a_I$ is a $su(N)$ matrix subjected to the condition $a_{I+L} = a_I$, $C_{ij} \sim \lambda^{(i+j)/2}$ is a semi-infinite matrix controlling the interaction strength between nodes I and J ($C^{|I-J|}$)

Correlators

The naive expectation based on the structure of the operator $T_{\alpha,n}(x)$ is

$$T_{\alpha,n}(x) \quad \to \quad \mathcal{T}^{\mathrm{naive}}_{\alpha,n} = \sum_{l} \mathrm{e}^{-ip_\alpha l}\, \mathrm{tr}\, a_l^k \ ,$$

which, however, violates violating the $U(1)$ charge conservation. The proper identification involves a Grand-Schmidt procedure

$$T_{\alpha,n}(x) \quad \to \quad \mathcal{T}_{\alpha,n} = \sum_{l} \sum_{2\leq k\leq n} c_k \mathrm{e}^{-ip_\alpha l}\, \mathrm{tr}\, a_l^k$$

with c_k begin fixed by the requiring that the two-point function is diagonal in the indices n, m [Komargodski, Billo' et al.].

The Bessel matrix

The matrix model prediction at the **planar** level for the correlators is

$$G^{(2)}_{\alpha,n} \propto \frac{\det(1 - s_\alpha K_{n+1})}{\det(1 - s_\alpha K_{n-1})} \;, \qquad \frac{G^{(3)}_{\vec{\alpha},\vec{n}}}{\sqrt{\prod_{i=1}^{3} G^{(2)}_{\alpha_i,n_i}}} = \frac{1}{\sqrt{L}N} \prod_{i=1}^{3} n_i \sqrt{1 + \frac{1}{2n_i} \hat{\lambda} \partial_{\hat{\lambda}} \log G^{(2)}_{\alpha_i,n_i}}$$

where $s_\alpha = \sin^2 \frac{p_\alpha}{2}$ and the dependence on the t' Hooft coupling λ sits in the **one-parameter** matrix

$$(K_\ell)_{ij} = \int_0^\infty dt \; \psi_i(t) \psi_j(t) \chi\left(\frac{\sqrt{t}}{2\hat{\lambda}}\right) \;,$$

where $\chi(x)$ is said to be the matrix **symbol**, while ψ_i is an orthonormal basis of Bessel functions

$$\psi_i(x) = (-1)^i \sqrt{2i + \ell - 1}\, \frac{J_{2i+\ell-1}(\sqrt{x})}{\sqrt{x}} \;, \qquad \chi(x) = -\frac{1}{\sinh^2\left(\frac{x}{2}\right)}$$

Relation to Fredholm determinants

We can relate $(K_\ell)_{ij}$ to an **integrable** Bessel operator $\mathbf{K}_\ell$ as $(K_\ell)_{ij} = \langle \psi_i | \mathbf{K}_\ell | \psi_j \rangle$ where

$$(\mathbf{K}_\ell f)(x) = \int_0^\infty \mathrm{d}y \; k_\ell(x,y)\chi\left(\frac{\sqrt{y}}{2\hat{\lambda}}\right) f(y) \; ,$$

$$k_\ell(x,y) = \frac{\sqrt{x}J_{\ell+1}(\sqrt{x})J_\ell(\sqrt{y}) - \sqrt{y}J_{\ell+1}(\sqrt{y})J_\ell(\sqrt{x})}{2(x-y)}$$

As a result,

$$G^{(2)}_{\alpha,n} \propto \frac{\det(1 - s_\alpha K_{n+1})}{\det(1 - s_\alpha K_{n-1})} \propto \frac{\det(1 - s_\alpha \mathbf{K}_{n+1})}{\det(1 - s_\alpha \mathbf{K}_{n-1})}$$

When $\chi(x) = \theta(1-x)$, the determinant $\det(1 - K_\ell)$ coincides with the **Tracy-Widom distribution**, which describes the eigenvalue statistics of a $N \times N$ hermitian matrix in the Laguerre ensemble for $N \to \infty$

Tracy-Widom distribution

We consider an Hermitian matrix a in the Gaussian Unitary Ensemble (GUE)

$$Z_{\mathrm{GUE}} = \int_{-\infty}^{\infty} d^{N\times N}a \; e^{-\frac{1}{2}\,\mathrm{tr}\, a^2} = \int_{-\infty}^{\infty} d\lambda_i \ldots d\lambda_N \prod_{i\neq j}^{N} (\lambda_i - \lambda_j)^2 \, e^{-\frac{1}{2}\Sigma_i \lambda_i^2}$$

and in the Laguerre ensemble (Wishart matrix theory)

$$Z_{\mathrm{Lag}} = \int_{0}^{\infty} d\lambda_i \ldots d\lambda_N \prod_{i\neq j}^{N} (\lambda_i - \lambda_j)^2 \prod_{k}^{N} \lambda_k^{\ell} e^{-\lambda_k}$$

where $\ell > -1$ to guarantee the convergence of the integral and $\lambda_i \in \mathbb{R}_+$

Probability of finding n eigenvalues at the points $x_1, \ldots x_n$

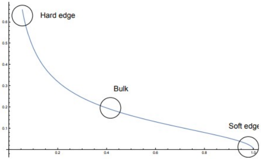
$$R(x_1, \ldots, x_n) = \left\langle \prod_{i}^{n} \delta(x_i - \lambda_i) \right\rangle_{\mathrm{GUE,Lag}} = \det K_N(x_i, x_j)\Big|_{i,j=1,\ldots,N}$$

$$K_N(x,y) = \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y)$$

where $\phi_k(x)$ are the Hermite and Laguerre polynomials

Tracy-Widom distributions

The eigenvalue distribution of a $N \times N$ hermitian matrix in the Laguerre ensemble as $N \to \infty$



$$R_1(4Nx) \sim \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}$$

Behaviour of $K_{N\to\infty}(x,y)$ around $x=0$ (hard edge), $x=1$ (soft edge) and $0<x<1$ (bulk)

bulk: $$K_{\mathrm{Sin}}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$$

soft edge: $$K_{\mathrm{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}$$

hard edge: $$K_{\mathrm{Bessel}}(x,y)\frac{\sqrt{x}J_{\ell+1}(\sqrt{x})J_\ell(\sqrt{y}) - \sqrt{y}J_{\ell+1}(\sqrt{y})J_\ell(\sqrt{x})}{2(x-y)}$$

The probability that there are not eigenvalues in an interval $[0,s]$ is [Tracy, Widom]

$$E(0,s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^s dx_1 \ldots dx_n \det K_{\mathrm{bulk,Airy,Bessel}}(x_i,x_j)\Big|_{1\le i,j\le n} = \det(1-\mathbf{K})_{[0,s]}$$

Correlation functions at weak coupling

Mass gap at weak coupling

As a warm up, we consider the weak coupling regime, where $\lambda \to 0$ and fixed L, n. We can expand the determinants in power of traces

$$\mathcal{F}_\ell = \log\det(1 - s_\alpha \mathbf{K}_\ell) = -s_\alpha \operatorname{Tr}\mathbf{K}_\ell - \frac{1}{2} s_\alpha \operatorname{Tr}\mathbf{K}_\ell^2 + \dots$$

The two-point function takes the form of a double series $\hat\lambda^2 = \lambda/16\pi^2$ and $s_\alpha = \sin^2\left(\frac{p_\alpha}{2}\right)$

$$R_{\alpha,n} = \mathrm{e}^{\mathcal{F}_{n+1} - \mathcal{F}_{n-1}} = 1 + \sum_{m\geq 1} \left(s_\alpha\, \hat\lambda^{2n}\right)^m Q_m(\hat\lambda^2)$$

where $Q_m(\hat\lambda) = c_n^m + \mathcal{O}(\hat\lambda^2)$ with $c_n = -4\binom{2n}{n}\zeta(2n-1)$. The analytical structure of $R_{\alpha,n}$ as a function of p_α becomes evident only by *resumming* the leading corrections

$$R_{\alpha,n} = \frac{1}{1 - s_\alpha \hat\lambda^{2n} c_n} + \dots$$

The closest pole to the origin, i.e. $p_\alpha = \pm i\mu_n$ with $\mu_n \sim \log 1/\hat\lambda$, sets the *mass scale* at weak coupling and is *large* as $\lambda \to 0$

A one-dimensional lattice model

The quiver diagram defines a one-dimensional periodic lattice model with L sites and partition function $\mathcal{Z}_{\mathbb{S}^4}$. Each node has N^2-1 d.o.f. and interact with its nearest neighbourhood via $S_{\rm Int}$

- $\mathcal{T}_{\alpha,n}$ represents an n-particle creation operator of definite quasimomentum $p_\alpha = 2\pi\alpha/L$
- $G^{(2)}_{\alpha,n}$ captures the propagation of an $n-$particle excitation of definite quasimomentum p_α, while $G^{(3)}_{\vec{\alpha},\vec{n}}$ describes the amplitude $n_1+n_2 \to n_3$
- The presence of poles implies exponentially suppressed correlations between the nodes $Y=|I-J|$ or, equivalently, **short-ranged, localized** interactions [Korchemsky, Beccaria]

$$f_Y = \frac{1}{L}\sum_{\alpha=0}^{L-1} \frac{\mathrm{e}^{ip_\alpha Y}}{1-s_\alpha \hat{\lambda}^{2n} c_n} + \dots \qquad \Rightarrow \qquad f_Y = \mathrm{e}^{-\mu_n Y}\tilde{f}(\mu_n, L) + (Y \to L-Y)$$

An effective one-dimensional description

In the limit $L \to \infty$, correlation functions cease to depend on L and approach a finite value. This suggests that the short-ranged interactions can be effectively described by a local effective field theory and the two-point function $\tilde{f}(\mu_n, L)$ behaves as

$$f_Y \sim \int \frac{dp}{2\pi} \frac{e^{ipY}}{p^2 + \mu_n^2} + \mathcal{O}(e^{-\mu_n}) \sim D(Y, \mu_n) + \mathcal{O}(e^{-\mu_n}) \ , \quad \text{where} \quad D(x, m) = \frac{e^{-mx}}{2m^2}$$

The effective description is collapses to a free theory

Remark: the theory behaves as a higher-dimensional model with massive propagations decoupled from the four-dimensional ones, which remain massless. Lorentz invariance is *broken*

$$\langle O_n^I(x) \bar{O}_m^J(0) \rangle \sim \frac{f_Y}{x^{2n}} \qquad \to \qquad \langle O_n(x_1, y_1) \bar{O}_m(x_2, y_2) \rangle \sim \frac{f(y_{12})}{x_{12}^{2n}}$$

Remark: the emergence of an additional dimension does not occur as in standard lattice descriptions, where we send to zero the lattice spacing, but rather as an effective description of a system with an infinite number of nodes and localized interactions

Correlation functions at strong coupling

Fredhom determinants at strong coupling

In our case, the method of differential equations leads to [Bajnok, Boldis, Beccaria, Korchemsky, Tsytlein]

$$\mathcal{F}_\ell = \log\det(1 - s_\alpha \mathbf{K}_\ell)$$
$$= 4\pi\hat{\lambda}\, a(1-a) - \frac{1}{2}(2\ell-1)\log\hat{\lambda} + B_\ell + \Delta\mathcal{F}_\ell(\hat{\lambda}) \,, \qquad a = \frac{\alpha}{L} \,, \qquad B_{\ell+1} - B_\ell = \log\frac{\ell}{\sqrt{4 s_\alpha}}$$

$\Delta\mathcal{F}_\ell$ vanishes as $\hat{\lambda} \to \infty$ and takes the following form

$$\Delta\mathcal{F}_\ell = f_\ell + \Delta f_\ell$$

f_ℓ captures perturbative expansions at strong coupling

$$f_\ell = -\frac{1}{16\hat{\lambda}}(2\ell-1)(2\ell-3)P_1 - \frac{1}{64\hat{\lambda}^2}(2\ell-1)(2\ell-3)P_1^2 + \dots$$
$$P_n = \frac{(-1)^n}{(2\pi)^{2n-1}(2n-2)!}\left[\psi^{(2n-2)}(a) + \psi^{(2n-2)}(1-a) - 2\psi^{(2n-2)}(1)\right] \,.$$

Non-perturbative corrections

The perturbative expansion f_ℓ grows factorially at large orders, requiring the presence of non-perturbative corrections to make the asymptotic expansion well-defined

$$\Delta f_\ell = \sum_{m,n\geq 1} \Lambda_+^{2n}\Lambda_-^{2m} f_\ell^{(m,n)} \ , \qquad \Lambda_+^2 = \mathrm{e}^{-8\pi\hat{\lambda}(1-a)} \ , \qquad \Lambda_-^2 = \mathrm{e}^{-8\pi\hat{\lambda}a}$$

- f_ℓ and $f_\ell^{(m,n)}$ satisfy resurgence relations
- $f_\ell^{(m,n)}$ are asymptotic series

$$f_\ell^{(1,0)} = -i(-1)^\ell \mathcal{S}(a)\left[1 + \frac{(2\ell-3)(2\ell-1)}{16\pi a\hat{\lambda}} + \frac{(2\ell-5)(2\ell-3)(2\ell-1)(2\ell+1)}{512(\pi a\hat{\lambda})^2} + \dots\right]$$

- **An immediate consequence:** for $\hat{\lambda}, L \to \infty$ and $L \gg \hat{\lambda}$ one finds that

$$\Lambda_+^2 \to 0 \ , \qquad \Lambda_-^2 \to 1 \ ,$$

implying that non-perturbative and perturbative contributions become comparable

Two-point functions at strong coupling

We apply this result to calculate the two-point function at strong coupling

$$R_{\alpha,n} = \frac{(n-1)n}{4\hat{\lambda}^2 s_\alpha} R^{(0,0)} \left[1 + e^{-8\pi\hat{\lambda}a} R^{(1,0)} + e^{-8\pi\hat{\lambda}(1-a)} R^{(0,1)} + \dots \right]$$

$$R^{(0,0)} = \left(1 - \frac{1}{2\hat{\lambda}} P_1\right)^{2(n-1)} \left(1 - \frac{(n-1)(2n-3)(2n-1)}{96\hat{\lambda}^3} P_2 + \dots\right)$$

- $R^{(m,n)}$ are again asymptotic series in negative powers of $\hat{\lambda}$
- This result is valid for $\hat{\lambda} \to \infty$ and fixed L and n or, equivalently, $\lambda \gg L, n$
- Insisting in keeping fixed L, the non-perturbative corrections are suppressed and the perturbative series in $1/\hat{\lambda}^m$ is in one-to-one correspondence with the AdS/CFT expansion

Correlations at strong coupling

For $L/\hat{\lambda} \ll 1$, we do not expect an emergent fifth dimension. To see this we note that

$$f_Y = \frac{(n-1)n}{\hat{\lambda}^2 L} \sum_{\alpha=0}^{L-1} e^{ip_\alpha Y} R_{\alpha,n} \simeq \frac{1}{L} + \frac{(n-1)nL}{\hat{\lambda}^2} \left(\frac{1}{3} - \frac{2Y(L-Y)}{2L^2} \right) + \mathcal{O}(L^2/\hat{\lambda}^3)$$

- this expansion is well-defined for $L/\hat{\lambda} \ll 1$
- correlations grow with $Y = |I - J|$, meaning that interactions are no longer localized
- excitations propagate with **finite** quasimomentum and there is not an emergent fifth dimension

The long-quiver limit at strong coupling

We can now study the limit of long quiver at strong coupling where

$$\hat{\lambda}, L \to \infty \ , \qquad \xi = \frac{2\pi\alpha\hat{\lambda}}{L} = \text{fixed} \ , \qquad P_n = \frac{(-1)^{n+1}}{(2\pi a)^{2n-1}} + \dots$$

The corresponding two-point function takes the following form

$$\widetilde{R}_{\alpha,n} = (n-1)n\,\xi^{-2}\,\widetilde{R}^{(0,0)}\left[1 + e^{-2\xi}\,\widetilde{R}^{(1,0)} + O(e^{-4\xi})\right]$$

$$\widetilde{R}^{(0,0)} = 1 - \frac{(n-1)}{\xi} + \frac{1}{4\xi^2}(n-1)(2n-3) - \frac{1}{32\xi^3}(n-1)(2n-5)(2n-3) + O(1/\xi^4)$$

- for $\xi \gg 1$ or, equivalently, $\hat{\lambda} \gg L$ the result is well-defined and in correspondence with the AdS/CFT expansion

- for $\xi \ll 1$ or, equivalently, $L \gg \hat{\lambda}$ all the contributions are equally important, requiring the resummation of an infinite tower of non-perturbative terms

Resummation

We can perform such resummation by observing that in the long-quiver limit at strong coupling

$$s_\alpha \chi\left(\frac{\sqrt{t}}{2\hat{\lambda}}\right) \quad \rightarrow \quad \widetilde{\chi}(t) = -\frac{(2\xi)^2}{t} \qquad\qquad \widetilde{\mathcal{F}}_\ell = \log\Big(\Gamma(\ell)\xi^{1-\ell} I_{\ell-1}(2\xi)\Big)$$

- $I_n(x)$ is the modified Bessel function of order n
- $\widetilde{\mathcal{F}}_\ell$ describes the one-point function of a chiral operator of dimension $\Delta = \ell - 1$ with the insertion of a half-BPS Wilson loop in $\mathcal{N} = 4 SYM$
- the two-point functions can be expressed as a ratio of Bessel functions

$$\widetilde{R}_{\alpha,n} = \frac{n(n-1)}{\hat{\lambda}^2} \frac{I_n(2p_\alpha \hat{\lambda})}{p_\alpha^2 I_{n-2}(2p_\alpha \hat{\lambda})}$$

for fixed $\hat{\lambda} p_\alpha$, in the limit $\hat{\lambda}, L \rightarrow \infty$.

Mass gap at strong coupling

For $L \gg \hat{\lambda}$, the two-point function receives contributions from excitations with small (continuos) quasimomentum

$$f_n(y) = \frac{2(n-1)n}{\hat{\lambda}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipy} \frac{I_n(p)}{p^2\, I_{n-2}(p)}\,, \qquad y_{12} = \frac{Y}{2\hat{\lambda}} = \text{fixed}$$

The dominant contribution comes from the minimal zero m_1 and implies **short-ranged** interactions as $f_n(y) \sim \mathrm{e}^{-m_1 y}$, suggesting an effective description in terms of a local field theory. To see this, we observe that

$$\frac{I_n(p)}{p^2 I_{n-2}(p)} = \sum_{k=1}^{\infty} \frac{4(n-1)}{m_k^2\,(p^2+m_k^2)} \qquad \leftrightarrow \qquad J_{n-2}(m_k) = 0$$

The final result takes the form of a one-dimensional Källén-Lehmann representation

$$f_n(y) = \frac{2n(n-1)}{\hat{\lambda}} \int_0^{\infty} d\mu^2 \rho_n(\mu^2) \frac{e^{-y\mu}}{2\mu}\,, \qquad \rho_n(\mu^2) = \frac{4(n-1)}{\mu^2} \sum_{k=1}^{\infty} \delta(\mu^2 - m_k^2)$$

Three-point functions

Toda equations

We can gain some intuitions about interactions in this one-dimensional effective description by analysing three-point functions. These are related to the propagators by

$$C_3 = \frac{G^{(3)}_{\vec{\alpha},\vec{n}}}{\sqrt{G^{(2)}_{\alpha_1,n_1} G^{(2)}_{\alpha_2,n_2} G^{(2)}_{\alpha_2,n_2}}} = \frac{1}{\sqrt{LN}} \prod_{i=1}^{3} n_i \mathcal{V}_{\alpha_i,n_i} \qquad \mathcal{V}_{\alpha,n} = \sqrt{1 + \frac{1}{2n} \hat{\lambda} \partial_{\hat{\lambda}} \log G^{(2)}_{\alpha,n}}$$

Fredholm determinants of these integrable Bessel operators satisfy a Toda-like equation

$$\hat{\lambda} \partial_{\hat{\lambda}} \left(\mathcal{F}_{\ell+1} - \mathcal{F}_{\ell-1} \right) = 2\ell \left(\mathrm{e}^{2\mathcal{F}_\ell - \mathcal{F}_{\ell-1} - \mathcal{F}_{\ell+1}} - 1 \right) , \qquad \mathcal{F}_\ell = \log \det(1 - \mathbf{K}_\ell)$$

As a result, also the three-point functions are expressible as ratios of Fredholm determinants

$$G^{(3)}_{\vec{\alpha},\vec{n}} \propto \prod_{i=1}^{3} \mathrm{e}^{\mathcal{F}_{n_i} - \mathcal{F}_{n_i - 1}}$$

where the proportionality constant is independent of $\hat{\lambda}$

Three-point functions in the long-quiver limit at strong coupling

For $\hat{\lambda}, L \to \infty$ and fixed $L/\hat{\lambda}$ we replace the Fredholm determinants with their leading behaviour

$$G^{(3)}_{\vec{\alpha},\vec{n}} \propto \prod_{i=1}^{3} \frac{I_{n_i-1}(2p_{\alpha_i}\hat{\lambda})}{p_{\alpha_i} I_{n_i-1}(2p_{\alpha_i}\hat{\lambda})}$$

We expect the emergence of a fifth dimension in the regime $L/\hat{\lambda} \gg 1$. We analyse again the Fourier transform

$$f_{\mathbf{n}}(y_1, y_2, y_3) = \frac{1}{L^2} \sum_{\alpha_1,\alpha_2=0}^{L-1} \mathrm{e}^{\alpha_1 Y_{13} + \alpha_2 Y_{23}} \prod_{i=1}^{3} \frac{I_{n_i-1}(2p_{\alpha_i}\hat{\lambda})}{p_{\alpha_i} I_{n_i-1}(2p_{\alpha_i}\hat{\lambda})}$$

where $Y_{ij} = (l_i - l_j) = 2\hat{\lambda} y_{ij}$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$ (mod L). In this regime, quasimomenta becomes continuos and the Fourier transform takes the form a three-point vertex

$$f_{\mathbf{n}}(y_1, y_2, y_3) = \int_{-\infty}^{\infty} \mathrm{d}\omega \prod_{i=1}^{3} f_{n_i}(\omega - y_i) \simeq \mathrm{e}^{-\nu y_1} \qquad \nu = \min(m_{1,n_1-2}, m_{1,n_2-2} + m_{1,n_3-2})$$

where $f_n(x)$ is the (two-point) propagator in the interacting theory and we assumed $y_1 \gg y_2, y_3$.

Conclusions and future perspectives

- We studied quiver theories in the limit of large number of nodes and showed deconstructions effects
- The fifth dimension emerges when $L/\hat{\lambda} \gg 1$. In the opposite regime, we remain with a four-dimensional strongly coupled model
- These results hold in the planar limit. Do they hold when analysing $1/N^2$ terms ?
- What about higher-point functions ?

Thank you for your attention